

Algebraic Relations Among Mean Value Coordinates: A Complete Answer for Quadrilaterals and Partial Results for Larger Polygons

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Abstract

Mean value coordinates of a planar polygon P with n vertices define a map from P to $\mathbb{P}^{n-1}$. At an Oberwolfach mini-workshop in 2022, F. Sottile asked for the homogeneous equations of the Zariski closure S_P of its image. We answer this question for quadrilaterals and give partial results for larger polygons. For every quadrilateral with no three vertices collinear, the ideal of S_P is generated by one explicit irreducible polynomial F_P of degree 14. It is obtained by clearing the square roots in the relation $\lambda_1|x - v_1| - \lambda_2|x - v_2| + \lambda_3|x - v_3| - \lambda_4|x - v_4| = 0$, which follows at once from Floater’s formula; for the unit square, $F_P/16$ has 116 integer terms. For every $n \geq 4$, S_P is an irreducible surface, and the projection $w \mapsto \sum w_i v_i / \sum w_i$ restricts to a map of degree 2^{n-1} from S_P to the plane. Over a general point x , a point w with $\sum w_i \neq 0$ and $\sum w_i(v_i - x) = 0$ lies on S_P exactly when the closed polygon with edge vectors $w_i J(v_i - x)$, J a quarter turn, is circumscribed about a circle. The passage from coordinates to circumscribed polygons is the classical tangent-length picture behind Floater’s construction; its converse is what produces equations. For $n = 5, 6, 7$, computations modulo primes for sample polygons give relations of lowest degree 10, 6, 7, and $\deg S_P = 52$ for $n = 5$; for a single hexagon they give $\deg S_P = 152$. These values are consistent with a conjectured value $2^{n-3}(6n - 17)$. A generating set for $n \geq 5$ remains open. This is an unrefereed note.

1 Introduction

Let P be a polygon in the plane with vertices $v_1, \dots, v_n$, indices taken modulo n . For a point x off the lines through the edges, let $\alpha_i \in (-\pi, \pi)$ be the signed angle at x of the triangle $[x, v_i, v_{i+1}]$. Floater’s mean value coordinates [Flo03], extended to arbitrary polygons by Hormann and Floater [HF06], are

$$\lambda_i(x) = \frac{w_i(x)}{\sum_j w_j(x)}, \quad w_i(x) = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{|x - v_i|}. \quad (1)$$

They satisfy $\sum_i \lambda_i = 1$ and $\sum_i \lambda_i v_i = x$; see [Flo15] for a survey of generalized barycentric coordinates.

In the problem session of the 2022 Oberwolfach mini-workshop *Interpolation, Approximation, and Algebra* [dBSSS22, p. 424], F. Sottile considered the map $x \mapsto [\lambda_1(x) : \dots : \lambda_n(x)]$ from P to $\mathbb{P}^{n-1}$ and asked: “What are the homogeneous equations for the Zariski closure of the image?” The question makes sense because the coordinates are algebraic functions of x , although (1) involves $\tan$. According to the report, it was inspired by the PhD work of Irving [Irv12], who studied the same questions for Wachspress coordinates; see [GPS10, IS14, KR20] for Wachspress surfaces and varieties. The question is part of record OWR-9790358-016 of the UnsolvedMath dataset [ula26]. That record is titled “What

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about in higher dimensions?”, a question from the item on three-dimensional Wachspress varieties (Sottile, Schenck). Its statement consists of this question, the whole mean value item, and the whole item on moduli spaces of Wachspress varieties (Sottile), all from p. 424 of [dBSSS22]. This note addresses only the mean value item.

Let $S_P \subset \mathbb{P}^{n-1}$ be the Zariski closure of the image; it does not depend on which open part of the polygon is used (Lemma 2.3). For a triangle, mean value coordinates are the barycentric coordinates and $S_P = \mathbb{P}^2$, so we assume $n \geq 4$. For $w \in \mathbb{C}^n$ put $s(w) = \sum_i w_i$ and

$$R_i(w) = \left(\sum_j w_j (v_i - v_j) \right) \cdot \left(\sum_j w_j (v_i - v_j) \right), \quad (2)$$

a quadratic form in w ; if $\sum_j w_j = 1$ and $x = \sum_j w_j v_j$, then $R_i(w) = |x - v_i|^2$. Let

$$\mathcal{N}_4(a, b, c, d) = (a^2 + b^2 + c^2 + d^2 - 2ab - 2ac - 2ad - 2bc - 2bd - 2cd)^2 - 64abcd, \quad (3)$$

so that $\mathcal{N}_4(b_1^2, \dots, b_4^2) = \prod_{\eta} (b_1 + \eta_2 b_2 + \eta_3 b_3 + \eta_4 b_4)$, the product over the eight $\eta \in \{\pm 1\}^3$.

Theorem 1.1 (quadrilaterals). *Let P be a quadrilateral with vertices $v_1, \dots, v_4 \in \mathbb{R}^2$, no three of them collinear; P need not be convex or simple. Then s^2 divides $\mathcal{N}_4(w_1^2 R_1, \dots, w_4^2 R_4)$, and*

$$F_P(w) = \frac{\mathcal{N}_4(w_1^2 R_1(w), w_2^2 R_2(w), w_3^2 R_3(w), w_4^2 R_4(w))}{(w_1 + w_2 + w_3 + w_4)^2}$$

is an irreducible homogeneous polynomial of degree 14 that generates the homogeneous ideal of S_P : $I(S_P) = (F_P)$. So S_P is the surface $F_P = 0$ of degree 14 in $\mathbb{P}^3$, and no nonzero polynomial of smaller degree vanishes on it. The point $p_0 = [k_1 : \dots : k_4]$ with $\sum_i k_i = 0$ and $\sum_i k_i v_i = 0$ lies on S_P with multiplicity 6.

For the mean value coordinates $\lambda = \lambda(x)$ of a real point x ,

$$F_P(\lambda) = \prod_{\eta} (\lambda_1 |x - v_1| + \eta_2 \lambda_2 |x - v_2| + \eta_3 \lambda_3 |x - v_3| + \eta_4 \lambda_4 |x - v_4|),$$

and one factor vanishes identically. Indeed, (1) says $w_i |x - v_i| = \tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)$, and the alternating sum telescopes:

$$\lambda_1 |x - v_1| - \lambda_2 |x - v_2| + \lambda_3 |x - v_3| - \lambda_4 |x - v_4| = 0. \quad (4)$$

Theorem 1.1 says that the polynomial obtained by clearing the square roots in (4) generates all algebraic relations. For the unit square, $F_P/16$ has 116 terms (Example 5.5). Since $\mathcal{N}_4$ is symmetric, the three quadrilaterals with a given set of four vertices have the same surface (Remark 5.4).

For general n we describe S_P geometrically. Let $J(a, b) = (b, -a)$. For $x \in \mathbb{C}^2$ and $w \in \mathbb{C}^n$ with $\sum_i w_i (v_i - x) = 0$, the points $G_0 = 0$ and $G_i = \sum_{j \leq i} w_j J(v_j - x)$ form a closed polygon $G(w)$ ($G_n = G_0$). Its i -th side has direction $J(v_i - x)$ and lies on the line $L_i(w) = \{g : (v_i - x) \cdot g = (v_i - x) \cdot G_i\}$.

Theorem 1.2 (all n). *Let $n \geq 4$ and let P have distinct vertices, not all on a line. Then S_P is an irreducible surface, and the projection $\pi(w) = \sum_i w_i v_i / \sum_i w_i$ restricts to a dominant rational map $S_P \dashrightarrow \mathbb{P}^2$ of degree 2^{n-1} . More precisely, for all x in a dense Zariski-open subset of $\mathbb{C}^2$:*

- (a) *the points of S_P over x are 2^{n-1} distinct points: the weights (9) with r_i replaced by $\varepsilon_i r_i$, where $r_i^2 = (v_i - x) \cdot (v_i - x)$ (so $r_i = |x - v_i|$ for real x) and $\varepsilon \in \{\pm 1\}^n$ modulo ± 1 ;*

- (b) a point $[w]$ with $\sum_i w_i(v_i - x) = 0$ and $\sum_i w_i \neq 0$ lies on S_P if and only if the lines $L_1(w), \dots, L_n(w)$ are tangent to a common circle of nonzero radius;
- (c) if n is even, the rationalized Pitot relation F_P of Section 4 is a nonzero element of $I(S_P)$ of degree $2^n - 2$.

The precise forms are Theorem 3.2 and Proposition 4.2. The condition $\sum_i w_i \neq 0$ in (b) excludes the centre Λ of π , which every fibre closure contains; for $n = 4$, $\Lambda = \{p_0\}$ and p_0 lies on S_P (Remark 3.3).

The map π is the linear projection that Garcia-Puente and Sottile [GPS10] attach to the vertices in their study of linear precision, and Λ is its centre of projection; Irving [Irv12] studies the Wachspress surface $W \subset \mathbb{P}^{n-1}$ through the same projection. The two cases differ sharply. For Wachspress coordinates, which are rational, π restricts to a birational map from W to $\mathbb{P}^2$, and for a quadrilateral W is a quadric in $\mathbb{P}^3$ and the centre is a point of W [Irv12]. For mean value coordinates, π has degree 2^{n-1} on S_P , and for a quadrilateral S_P is a surface of degree 14 on which the centre p_0 has multiplicity 6.

Prior work and what is new. The direction from points of S_P to circumscribed polygons in Theorem 1.2(b) is the classical geometry behind (1): the numbers $\tan(\alpha_i/2)$ are the tangent lengths of a polygon circumscribed about the unit circle centred at x ; compare Floater's derivation [Flo03] and the polar-dual construction of Ju, Schaefer, Warren and Desbrun [JSWD05]. The relation (4), and its analogue for every even n , is immediate from Floater's formula; it is Pitot's theorem for that polygon. The three-point form of the weights is due to Floater, Hormann and Kós [FHK06]. What is new here, as far as we could find, is the converse direction of Theorem 1.2(b) and its use to produce equations: the ideal of S_P for quadrilaterals, the fibre description and the degree 2^{n-1} for all n , and the data for $n \geq 5$.

Computations for $n = 5, 6, 7$. For the sample polygons, computations modulo large primes give relations of lowest degree $d_0 = 10, 6$ and 7 for $n = 5, 6, 7$, which, with one exception for $n = 7$, come from tangency conditions. That no relation of lower degree exists is certified for the polygons tested. The existence of relations of degree 10 for $n = 5$ and of degree 7 for $n = 7$ rests on quotients computed modulo p , whereas $d_0 = 6$ is certified for four hexagons, for which $I(S_P)_6$ is spanned by a Brianchon sextic (Section 6). We do not know a generating set for any $n \geq 5$.

2 Set-up

Notation. Throughout, P has vertices $v_1, \dots, v_n \in \mathbb{R}^2$ ($n \geq 3$, indices modulo n), which are distinct and not all on one line. We work over $\mathbb{C}$. For $u, v \in \mathbb{C}^2$ let $u \cdot v = u_1 v_1 + u_2 v_2$, $\det(u, v) = u_1 v_2 - u_2 v_1$ and $Ju = (u_2, -u_1)$. Then $u \cdot Jv = \det(u, v)$, $Ju \cdot Jv = u \cdot v$ and $u \cdot Ju = 0$. We use the identities

$$(a \cdot b) \det(b, c) + (b \cdot c) \det(a, b) = (b \cdot b) \det(a, c), \quad (5)$$

$$\det(a, b) c - \det(a, c) b + \det(b, c) a = 0, \quad (6)$$

$$(a \cdot a)(b \cdot b) - (a \cdot b)^2 = \det(a, b)^2 \quad (7)$$

for $a, b, c \in \mathbb{C}^2$, which are checked by expanding. For $x \in \mathbb{C}^2$ put

$$d_i = v_i - x, \quad q_i = d_i \cdot d_i, \quad A_i = \det(d_i, d_{i+1}), \quad B_i = \det(d_{i-1}, d_{i+1}), \quad D_i = d_i \cdot d_{i+1},$$

and $\kappa_i = \det(v_{i-1} - v_i, v_i - v_{i+1})$, which does not depend on x . Expanding shows

$$A_{i-1} + A_i - B_i = \kappa_i. \quad (8)$$

Let $U \subset \mathbb{C}^2$ be the complement of the edge lines, i.e. the set where all $A_i \neq 0$. On $\mathbb{C}^n$ we use the linear forms $s(w) = \sum_i w_i$ and $m(w) = \sum_i w_i v_i \in \mathbb{C}^2$, and for $x \in \mathbb{C}^2$ the subspace

$$K(x) = \left\{ w \in \mathbb{C}^n : \sum_i w_i d_i = 0 \right\} = \{ w : m(w) = s(w)x \}.$$

The $2 \times n$ matrix with columns $d_1, \dots, d_n$ has rank 2 for every x , since otherwise all vertices would lie on a line through x . Hence $\dim K(x) = n - 2$, and the orthogonal complement $K(x)^\perp$ for the bilinear form $\sum_i u_i w_i$ is spanned by the two rows $D^1, D^2 \in \mathbb{C}^n$ of this matrix.

Lemma 2.1 (three-point form). *For $x \in U$ and $\rho \in \mathbb{C}^n$ define $M(x)\rho \in \mathbb{C}^n$ by*

$$(M(x)\rho)_i = \frac{\rho_{i-1}}{A_{i-1}} - \frac{\rho_i B_i}{A_{i-1}A_i} + \frac{\rho_{i+1}}{A_i}. \quad (9)$$

- (a) *For real $x \in U$, not a vertex, Floater's weights are $w(x) = M(x)r$, where $r_i = |x - v_i|$.*
- (b) *$M(x)$ is symmetric of rank $n - 2$, with kernel spanned by D^1, D^2 and image $K(x)$. In particular $\sum_i (M(x)\rho)_i d_i = 0$ for every ρ .*
- (c) *$s(M(x)\rho) = \sum_i \kappa_i \rho_i / (A_{i-1}A_i)$.*

Formula (9) with $\rho = r$ is the mean value case of the three-point construction in [FHK06]; we include the short proof.

Proof. (a) Write $r_i = |x - v_i|$. Since $\sin \alpha_i = A_i / (r_i r_{i+1})$ and $\cos \alpha_i = D_i / (r_i r_{i+1})$, we have $\tan(\alpha_i/2) = (1 - \cos \alpha_i) / \sin \alpha_i = (r_i r_{i+1} - D_i) / A_i$. So

$$w_i = \frac{r_{i-1}}{A_{i-1}} + \frac{r_{i+1}}{A_i} - \frac{D_{i-1}A_i + D_i A_{i-1}}{r_i A_{i-1}A_i},$$

and $D_{i-1}A_i + D_i A_{i-1} = q_i B_i = r_i^2 B_i$ by (5) with $(a, b, c) = (d_{i-1}, d_i, d_{i+1})$.

(b) The nonzero entries of $M(x)$ are $M_{i,i} = -B_i / (A_{i-1}A_i)$ and $M_{i,i+1} = M_{i+1,i} = 1/A_i$, so $M(x)$ is symmetric. For the k -th column, $\sum_i M_{ik} d_i = (A_k d_{k-1} - B_k d_k + A_{k-1} d_{k+1}) / (A_{k-1}A_k) = 0$ by (6) with $(a, b, c) = (d_{k-1}, d_k, d_{k+1})$. Hence the image of $M(x)$ lies in $K(x)$, and by symmetry D^1 and D^2 lie in the kernel. The submatrix with rows $1, \dots, n - 2$ and columns $2, \dots, n - 1$ is lower triangular with diagonal entries $1/A_1, \dots, 1/A_{n-2}$, so the rank is at least $n - 2$. Since D^1, D^2 are linearly independent and $\dim K(x) = n - 2$, the claims follow.

(c) The k -th column sum of $M(x)$ is $(A_{k-1} + A_k - B_k) / (A_{k-1}A_k) = \kappa_k / (A_{k-1}A_k)$ by (8). $\square$

The branched cover. Let

$$X = \{(x, \rho) \in \mathbb{C}^2 \times \mathbb{C}^n : \rho_i^2 = q_i(x) \text{ for all } i\}, \quad \mathcal{O} = \mathbb{C}[x_1, x_2, \rho_1, \dots, \rho_n] / (\rho_1^2 - q_1, \dots, \rho_n^2 - q_n).$$

As a $\mathbb{C}[x]$ -module, $\mathcal{O}$ is free with basis $\rho_S = \prod_{i \in S} \rho_i$, $S \subseteq \{1, \dots, n\}$. The group $\{\pm 1\}^n$ acts by $\rho \mapsto \eta \rho = (\eta_1 \rho_1, \dots, \eta_n \rho_n)$.

Lemma 2.2. *$\mathcal{O}$ is a domain, so X is an irreducible surface, finite over $\mathbb{C}^2$. If $h \in \mathbb{C}[x, \rho]$ is nonzero in $\mathcal{O}$, then $N(h) = \prod_{\eta \in \{\pm 1\}^n} h(x, \eta \rho)$ is a nonzero polynomial in x alone, and $N(h)(x) = \prod_{\eta} h(x, \eta \rho)$ whenever $\rho_i^2 = q_i(x)$ for all i .*

Proof. With $v_j = (a_j, b_j)$, $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ we have $q_j = (z - a_j - ib_j)(\bar{z} - a_j + ib_j)$. As the vertices are distinct, for $S \neq \emptyset$ and $j \in S$ the prime $z - a_j - ib_j$ divides $\prod_{k \in S} q_k$ exactly once, so this product is not a square in $\mathbb{C}(x)$. By Kummer theory [Lan02, Ch. VI, §8] the field $\mathbb{C}(x)(\sqrt{q_1}, \dots, \sqrt{q_n})$ has degree 2^n over $\mathbb{C}(x)$. The surjection $\mathcal{O} \otimes_{\mathbb{C}[x]} \mathbb{C}(x) \rightarrow \mathbb{C}(x)(\sqrt{q_1}, \dots, \sqrt{q_n})$, $\rho_i \mapsto \sqrt{q_i}$, is therefore an isomorphism of 2^n -dimensional vector spaces. Since $\mathcal{O}$ is free, it embeds in $\mathcal{O} \otimes \mathbb{C}(x)$, so it is a domain. The element $N(h)$ is invariant under $\{\pm 1\}^n$, and the invariants of $\mathcal{O} = \bigoplus_S \mathbb{C}[x]\rho_S$ form $\mathbb{C}[x]$, because η multiplies ρ_S by $\prod_{i \in S} \eta_i$. Finally $N(h) \neq 0$, since each factor is the image of $h \neq 0$ under an automorphism of the domain $\mathcal{O}$. $\square$

Let $X^\circ = \{(x, \rho) \in X : x \in U, M(x)\rho \neq 0\}$, and let $\Phi : X^\circ \rightarrow \mathbb{P}^{n-1}$, $\Phi(x, \rho) = [M(x)\rho]$. Clearing denominators, $\Phi(x, \rho) = [W(x, \rho)]$ with the polynomial weights $W_i = \prod_j A_j \cdot (M(x)\rho)_i$. We now identify S_P with the Zariski closure of $\Phi(X^\circ)$.

Lemma 2.3. (a) *The Zariski closure $\overline{\Phi(X^\circ)}$ is an irreducible surface, and s does not vanish identically on it.*

(b) *Let $\Omega \subset U \cap \mathbb{R}^2$ be a nonempty open set of points that are not vertices and at which $w(x) \neq 0$. Then $\overline{\Phi(X^\circ)}$ is the Zariski closure of $\{[w(x)] : x \in \Omega\}$. In particular, $S_P = \overline{\Phi(X^\circ)}$.*

Proof. (a) Let $h = \sum_i \kappa_i \rho_i \prod_{j \neq i-1, i} A_j$, so that $h = \prod_j A_j \cdot s(M(x)\rho)$ by Lemma 2.1(c). Not all κ_i vanish: otherwise every three consecutive vertices would be collinear and, the vertices being distinct, all of them would lie on one line. As the ρ_i belong to a basis of $\mathcal{O}$ over $\mathbb{C}[x]$, $h \neq 0$ in $\mathcal{O}$. So $s(M(x)\rho) \neq 0$ on a dense open subset of X ; in particular $X^\circ \neq \emptyset$ and s does not vanish identically on $\Phi(X^\circ)$. At such points $m(M(x)\rho) = s(M(x)\rho)x$ by Lemma 2.1(b), so the map $w \mapsto m(w)/s(w)$ sends $\Phi(x, \rho)$ to x . Hence the closure has dimension at least 2; being the closure of the image of the irreducible surface X° , it is irreducible of dimension 2.

(b) By Lemma 2.1(a), $\tilde{\Omega} = \{(x, r(x)) : x \in \Omega\}$ lies in X° and $\Phi(\tilde{\Omega}) = \{[w(x)] : x \in \Omega\}$. Let Z be the Zariski closure of $\tilde{\Omega}$ in X . The projection $X \rightarrow \mathbb{C}^2$ is finite, hence closed, so it maps Z onto a closed set containing Ω . A polynomial vanishing on a nonempty open subset of $\mathbb{R}^2$ is zero, so Z maps onto $\mathbb{C}^2$, $\dim Z = 2$ and $Z = X$. Thus $\tilde{\Omega}$ is dense in X° , and $\Phi(\tilde{\Omega})$ is dense in $\Phi(X^\circ)$. For the interior of P , removing the edge lines, the vertices and the zeros of w leaves a nonempty open set Ω , since by the same density argument $h(x, r(x)) \neq 0$ there. The removed points with $w(x) \neq 0$ do not change the closure, because w is continuous on the interior of P and a Zariski closure contains the closure in the classical topology. $\square$

The projection and its centre. Since the vertices are not collinear, the forms s, m_1, m_2 are linearly independent. Let $\pi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^2$, $[w] \mapsto [s(w) : m_1(w) : m_2(w)]$, be the linear projection with centre

$$\Lambda = \{s = m_1 = m_2 = 0\} \cong \mathbb{P}^{n-4}.$$

For $n = 4$, Λ is the point $p_0 = [k]$, where $k \neq 0$ satisfies $\sum_i k_i = 0$ and $\sum_i k_i v_i = 0$. We identify $x \in \mathbb{C}^2$ with $[1 : x] \in \mathbb{P}^2$. The fibre of π over x is $\mathbb{P}(K(x)) \setminus \Lambda$. The projective space $\mathbb{P}(K(x)) \cong \mathbb{P}^{n-3}$ contains Λ , and $\pi(\Phi(x, \rho)) = x$ whenever $s(M(x)\rho) \neq 0$. For $n = 3$, $K(x)$ is spanned by the barycentric coordinates of x , so $S_P = \mathbb{P}^2$. From now on $n \geq 4$.

In the terminology of [GPS10, §2], π is the tautological projection $[w] \mapsto \sum_i w_i(1, v_i)$ defined by the vertices as control points, and Λ is its centre of projection; it is also the linear projection of [Irv12]. By [GPS10, Thm. 2.9], a patch has rational linear precision, that is, its reparametrization with linear precision is given by rational functions, exactly when this projection is birational on the Zariski closure of its image; this is the case for Wachspress coordinates. Mean value coordinates have linear precision, $\sum_i \lambda_i v_i = x$, but they are not rational, and by Theorem 3.2 π has degree 2^{n-1} on S_P .

3 Circumscribed polygons and the fibres of π

Lemma 3.1. *Let $x \in U$ and $\rho \in \mathbb{C}^n$. The lines $T_i = \{g \in \mathbb{C}^2 : d_i \cdot g = \rho_i\}$ and T_{i+1} meet in the single point $\sigma_i = (\rho_i J d_{i+1} - \rho_{i+1} J d_i) / A_i$, and*

$$\sigma_i - \sigma_{i-1} = -(M(x)\rho)_i J d_i \quad \text{for all } i. \quad (10)$$

Proof. The equalities $d_i \cdot \sigma_i = \rho_i$ and $d_{i+1} \cdot \sigma_i = \rho_{i+1}$ follow from $u \cdot Jv = \det(u, v)$, and the intersection point is unique because $A_i \neq 0$. Multiplying (10) by $A_{i-1}A_i$ and collecting terms, it becomes $\rho_i J(A_{i-1}d_{i+1} - B_i d_i + A_i d_{i-1}) = 0$, which is (6). $\square$

If $\rho_i^2 = q_i \neq 0$, the line T_i is tangent to the conic $g \cdot g = 1$ at d_i/ρ_i . For real x inside a convex polygon and $\rho = r$, the σ_i form the polygon circumscribed about the unit circle with points of contact $(v_i - x)/r_i$, with tangent lengths $\tan(\alpha_i/2)$ and, by (10), side lengths $w_i r_i$: the geometry behind (1) [Flo03, JSWD05]. Conversely, circumscribed polygons characterize the points of S_P .

For $x \in \mathbb{C}^2$ and $w \in K(x)$ let $G_0 = 0$ and $G_i(w) = \sum_{j \leq i} w_j J d_j$, so that $G_n = J \sum_j w_j d_j = 0$, and let $L_i(w) = \{g : d_i \cdot g = d_i \cdot G_i(w)\}$; this line contains G_{i-1} and G_i . We say that $L_1(w), \dots, L_n(w)$ are *tangent to a common circle* if there are $c \in \mathbb{C}^2$ and $R \in \mathbb{C} \setminus \{0\}$ with

$$(d_i \cdot (G_i(w) - c))^2 = R^2 q_i \quad (i = 1, \dots, n). \quad (11)$$

When $q_i \neq 0$, (11) says exactly that $L_i(w)$ is tangent to the conic $(g - c) \cdot (g - c) = R^2$: parametrize the line as $g_0 + t J d_i$ and use (7) to compute the discriminant of the resulting quadratic equation in t .

Theorem 3.2. *Let $n \geq 4$. There is a dense Zariski-open set $V \subset U$ such that for every $x \in V$ and every $r \in \mathbb{C}^n$ with $r_i^2 = q_i(x)$ the following hold, where $\varepsilon r = (\varepsilon_1 r_1, \dots, \varepsilon_n r_n)$ for $\varepsilon \in \{\pm 1\}^n$.*

- (a) *The point $p_\varepsilon(x) = [M(x)(\varepsilon r)]$ is defined, s does not vanish at it, and $p_\varepsilon(x) = p_{\varepsilon'}(x)$ only if $\varepsilon' = \pm \varepsilon$.*
- (b) *$S_P \cap \pi^{-1}(x) = \{p_\varepsilon(x) : \varepsilon \in \{\pm 1\}^n\}$, a set of 2^{n-1} points.*
- (c) *For $w \in K(x) \setminus \{0\}$, the point $[w]$ is one of the $p_\varepsilon(x)$ if and only if $L_1(w), \dots, L_n(w)$ are tangent to a common circle.*

Consequently π restricts to a dominant rational map $S_P \dashrightarrow \mathbb{P}^2$ of degree 2^{n-1} , and Φ induces a birational map from $X/\{\pm 1\}$ to S_P , where -1 acts by $\rho \mapsto -\rho$.

Proof. The set V . Let $\varepsilon, \varepsilon' \in \{\pm 1\}^n$ with $\varepsilon' \neq \pm \varepsilon$. Choose indices j, k with $\varepsilon_j \varepsilon'_j \neq \varepsilon_k \varepsilon'_k$ and two further indices a, b (possible as $n \geq 4$). Let $\delta_{\varepsilon, \varepsilon'}$ be the determinant of the 4×4 matrix with rows $(\varepsilon_i \rho_i, \varepsilon'_i \rho_i, d_{i,1}, d_{i,2})$ for $i \in \{j, k, a, b\}$. Laplace expansion along the first two columns writes $\delta_{\varepsilon, \varepsilon'}$ as a sum over the pairs $\{p, q\} \subset \{j, k, a, b\}$ of the terms $\pm(\varepsilon_p \varepsilon'_q - \varepsilon_q \varepsilon'_p) \det(d_{p'}, d_{q'}) \rho_p \rho_q$, where $\{p', q'\}$ is the complementary pair. Since $\varepsilon_j \varepsilon'_k - \varepsilon_k \varepsilon'_j = \varepsilon_j \varepsilon_k (\varepsilon_k \varepsilon'_k - \varepsilon_j \varepsilon'_j) = \pm 2$, the coefficient of $\rho_j \rho_k$ is $\pm 2 \det(v_a - x, v_b - x)$. This is a nonzero polynomial in x , since its linear part $\det(v_b - v_a, x)$ is nonzero. The products $\rho_p \rho_q$ belong to a $\mathbb{C}[x]$ -basis of $\mathcal{O}$, so $\delta_{\varepsilon, \varepsilon'} \neq 0$ in $\mathcal{O}$. With h as in the proof of Lemma 2.3, let V be the set of $x \in U$ with $q_i(x) \neq 0$ for all i , $N(h)(x) \neq 0$, and $N(\delta_{\varepsilon, \varepsilon'})(x) \neq 0$ for all such pairs. By Lemma 2.2, V is a dense Zariski-open subset of $\mathbb{C}^2$.

(a) Let $x \in V$. Since $N(h)(x) \neq 0$, $h(x, \varepsilon r) \neq 0$, that is, $s(M(x)(\varepsilon r)) \neq 0$. So $M(x)(\varepsilon r) \neq 0$, $p_\varepsilon(x)$ is defined, and s does not vanish at it. Suppose $p_\varepsilon(x) = p_{\varepsilon'}(x)$ with $\varepsilon' \neq \pm \varepsilon$. Then $M(x)(\alpha \varepsilon r + \beta \varepsilon' r) = 0$ for some $(\alpha, \beta) \neq (0, 0)$, so $\alpha \varepsilon r + \beta \varepsilon' r$ lies in the span of D^1, D^2 by Lemma 2.1(b). Hence the four columns $\varepsilon r, \varepsilon' r, D^1, D^2$ are linearly dependent and $\delta_{\varepsilon, \varepsilon'}(x, r) = 0$, contradicting $N(\delta_{\varepsilon, \varepsilon'})(x) \neq 0$.

(b) The points $p_\varepsilon(x) = \Phi(x, \varepsilon r)$ lie in S_P , and $\pi(p_\varepsilon(x)) = x$ because $s \neq 0$ there; by (a) there are exactly 2^{n-1} of them. Conversely, let $[w] \in S_P$ with $\pi([w]) = x$. By Chevalley's theorem $\Phi(X^\circ)$

is constructible, so its Zariski closure S_P equals its closure in the classical topology [Mum99, Ch. I, §10]. Choose $(x_k, \rho_k) \in X^\circ$ with $\Phi(x_k, \rho_k) \rightarrow [w]$. As $s(w) \neq 0$, s does not vanish at $\Phi(x_k, \rho_k)$ for large k , so $x_k = \pi(\Phi(x_k, \rho_k)) \rightarrow \pi([w]) = x$. The ρ_k are bounded, since $\rho_{k,i}^2 = q_i(x_k)$, so a subsequence converges to some ρ with $\rho_i^2 = q_i(x)$. Then $M(x_k)\rho_k \rightarrow M(x)\rho$, which is nonzero by (a), so $[w] = [M(x)\rho] = p_\varepsilon(x)$ with $\varepsilon_i = \rho_i/r_i$.

(c) Let $[w] = p_\varepsilon(x)$, say $w = \mu M(x)\rho$ with $\rho = \varepsilon r$ and $\mu \neq 0$. By (10), $G_i - G_{i-1} = w_i J d_i = -\mu(\sigma_i - \sigma_{i-1})$, so $G_i + \mu\sigma_i$ is a constant c (indices modulo n , with $G_0 = G_n$). Then $d_i \cdot (G_i - c) = -\mu d_i \cdot \sigma_i = -\mu\rho_i$, and (11) holds with $R = \mu$. Conversely, let $w \in K(x) \setminus \{0\}$ satisfy (11), and put $\rho_i = d_i \cdot (c - G_i)/R$, so that $\rho_i^2 = q_i$. The point $g_i = (c - G_i)/R$ satisfies $d_i \cdot g_i = \rho_i$ and $d_{i+1} \cdot g_i = d_{i+1} \cdot (c - G_{i+1})/R = \rho_{i+1}$, because $G_{i+1} - G_i = w_{i+1} J d_{i+1}$ is orthogonal to d_{i+1} . So $g_i = \sigma_i$ by Lemma 3.1, and $\sigma_i - \sigma_{i-1} = -(G_i - G_{i-1})/R = -(w_i/R) J d_i$. Comparing with (10) gives $w = R M(x)\rho$, so $[w] = p_\varepsilon(x)$ with $\varepsilon_i = \rho_i/r_i$.

Consequences. By (b), the general fibre of the dominant map $\pi|_{S_P}$ has 2^{n-1} points, which is its degree. Since $\Phi(x, -\rho) = \Phi(x, \rho)$, Φ factors through $X/\{\pm 1\}$, and by (a) and (b) the induced map is injective over V . A dominant, generically injective morphism of irreducible complex varieties is birational. $\square$

Remark 3.3 (the centre Λ). Theorem 3.2(b),(c) concern points off Λ , and this restriction is necessary: every $\mathbb{P}(K(x))$ contains Λ , which can meet S_P (for $n \geq 5$ our computations indicate that $S_P \cap \Lambda$ is a curve; for Wachspress surfaces the centre meets W in the image of the adjoint curve [Irv12]). For $n = 4$, $p_0 \in S_P$, and for $x \in V$ the line $\mathbb{P}(K(x))$ meets S_P exactly in p_0 and the eight points $p_\varepsilon(x)$ (proof of Theorem 1.1); since $p_0 = [k]$ is not a $p_\varepsilon(x)$, the lines $L_i(k)$ are not tangent to a common circle. Outside V the description can fail: if P is a convex quadrilateral and x is the intersection of its diagonals, then $d_3 = -td_1$ and $d_4 = -ud_2$ with $t, u > 0$, so $\rho = (r_1, r_2, -r_3, -r_4) = (d_i \cdot \sigma)_i$ for suitable σ , and $M(x)\rho = 0$ by Lemma 2.1(b).

Proposition 3.4 (Pitot relations). *Let $x \in U$, let $\rho_i^2 = q_i(x)$ for all i , let $w = M(x)\rho$ and $t_i = (\rho_i \rho_{i+1} - D_i)/A_i$. Then $w_i \rho_i = t_{i-1} + t_i$ for all i . If n is even, then $\sum_i (-1)^i w_i \rho_i = 0$. In particular, the mean value coordinates of a polygon with an even number of vertices satisfy $\sum_i (-1)^i \lambda_i(x) |x - v_i| = 0$.*

Proof. By (9), $w_i \rho_i = \rho_{i-1} \rho_i / A_{i-1} + \rho_i \rho_{i+1} / A_i - q_i B_i / (A_{i-1} A_i)$, and $q_i B_i = D_{i-1} A_i + D_i A_{i-1}$ by (5); this is $t_{i-1} + t_i$. For even n the sign $(-1)^i$ is well defined modulo n , and $\sum_i (-1)^i (t_{i-1} + t_i) = 0$. For real x and $\rho = r$, w is the vector of Floater's weights by Lemma 2.1(a). $\square$

For $\rho = r$, $t_i = \tan(\alpha_i/2)$ and $w_i r_i = t_{i-1} + t_i$ is Floater's definition (1), so we claim no novelty for Proposition 3.4. In general t_i is a signed tangent length of the polygon (σ_i) , since $\sigma_i = d_i/\rho_i - (t_i/\rho_i) J d_i$, and the relation for even n is Pitot's theorem.

4 The rationalized Pitot relation

For $n \geq 2$ let

$$\mathcal{N}_n(a_1, \dots, a_n) = \prod_{\eta \in \{\pm 1\}^n, \eta_1=1} (\eta_1 \sqrt{a_1} + \dots + \eta_n \sqrt{a_n}).$$

Changing the sign of one square root permutes the 2^{n-1} factors and changes the sign of none or of all of them, so $\mathcal{N}_n$ is a polynomial in a with integer coefficients, homogeneous of degree 2^{n-2} . For $n = 4$ it is (3). Let

$$N_P(w) = \mathcal{N}_n(w_1^2 R_1(w), \dots, w_n^2 R_n(w)),$$

a form of degree 2^n , with $R_i = (sv_i - m) \cdot (sv_i - m)$ as in (2).

Lemma 4.1. s^2 divides N_P .

Proof. Let $H = \{s = 0\} \subset \mathbb{C}^n$. On H we have $sv_i - m = -m$, so $R_i = \mu$ with $\mu = m \cdot m$ for every i ; μ does not vanish identically on H , because m maps H onto $\mathbb{C}^2$. Let $w^0 \in H$ with $\mu(w^0) \neq 0$ and $\eta \cdot w^0 = \sum_i \eta_i w_i^0 \neq 0$ for all $\eta \in \{\pm 1\}^n$ other than $\pm \mathbf{1}$, $\mathbf{1} = (1, \dots, 1)$; such points are dense in H . Near w^0 choose holomorphic square roots $\rho_i(w)$ of $R_i(w)$, all with the same value $\sqrt{\mu(w^0)}$ at w^0 . Then $N_P = \prod_{\eta=1} \ell_\eta$ with $\ell_\eta(w) = \sum_i \eta_i w_i \rho_i(w)$, and $\ell_\eta(w^0) = \sqrt{\mu(w^0)}(\eta \cdot w^0)$. This is nonzero for $\eta \neq \mathbf{1}$ and zero for $\eta = \mathbf{1}$. For $e \in \mathbb{C}^n$, the derivative of ℓ_1 at w^0 in the direction e is $\sum_i e_i \rho_i + \sum_i w_i^0 \partial_e R_i / (2\rho_i) = \sqrt{\mu} s(e) + (2\sqrt{\mu})^{-1} \sum_i w_i^0 \partial_e R_i(w^0)$. Expanding, and using $\sum_i w_i(sv_i - m) = 0$, one finds

$$\sum_i w_i \partial_e R_i = 2 \sum_i w_i (sv_i - m) \cdot (s(e)v_i - m(e)) = 2s(e) \left(s \sum_i w_i v_i \cdot v_i - m \cdot m \right), \quad (12)$$

which equals $-2s(e)\mu$ at w^0 . So ℓ_1 vanishes at w^0 together with its first derivatives, while the other factors do not vanish there. Thus N_P vanishes to order at least 2 at the points of a dense open subset of H . In particular N_P vanishes on H , so $N_P = sN'$. If $s \nmid N'$, then at the points of H where $N' \neq 0$, which form a dense open subset, $\partial_e N_P = s(e)N' \neq 0$ for $s(e) \neq 0$. This is impossible, so s^2 divides N_P . $\square$

Proposition 4.2. (a) For every $x \in \mathbb{C}^2$ and $w \in K(x)$,

$$N_P(w) = s(w)^{2^{n-1}} \mathcal{N}_n(w_1^2 q_1(x), \dots, w_n^2 q_n(x)) = s(w)^{2^{n-1}} \prod_{\eta=1} \ell_{\eta,x}(w), \quad \ell_{\eta,x}(w) = \sum_i \eta_i r_i w_i,$$

for any fixed r_i with $r_i^2 = q_i(x)$.

(b) If n is even, $F_P = N_P/s^2$ is a nonzero form of degree $2^n - 2$ in $I(S_P)$.

Proof. (a) On $K(x)$ we have $sv_i - m = s d_i$, so $R_i = s^2 q_i$, and $\mathcal{N}_n$ is homogeneous of degree 2^{n-2} . (b) By Lemma 4.1, F_P is a form. Let $(x, \rho) \in X^\circ$ with $s(M(x)\rho) \neq 0$, and $w = M(x)\rho \in K(x)$. Taking $r = \rho$ in (a), the factor with $\eta_i = (-1)^{i+1}$ is $-\sum_i (-1)^i w_i \rho_i = 0$ by Proposition 3.4, so $F_P(w) = 0$. Such points $\Phi(x, \rho)$ are dense in S_P by Lemma 2.3(a), so $F_P \in I(S_P)$. For $x \in V$, every $\ell_{\eta,x}$ is nonzero on $K(x)$: otherwise $(\eta_i r_i)_i \in K(x)^\perp = \ker M(x)$, contradicting Theorem 3.2(a). Since also $s \neq 0$ on $K(x)$, $N_P \neq 0$. $\square$

For $n = 4$ this F_P is the polynomial of Theorem 1.1. For $n \geq 6$ it is far from generating $I(S_P)$: for hexagons it has degree 62, while $I(S_P)$ contains a sextic (Section 6).

5 Quadrilaterals

In this section $n = 4$ and no three vertices are collinear.

Lemma 5.1. Let $C(w) = \sum_i w_i \det(v_i, m(w))^2$. Then

$$2F_P \equiv (m \cdot m)^2 C(w) \prod_{\eta=1, \eta \neq \mathbf{1}} (\eta \cdot w) \pmod{s},$$

and the right-hand side is not divisible by s . In particular $s \nmid F_P$.

Proof. Keep the notation of the proof of Lemma 4.1, and choose e with $s(e) = 1$ and $m(e) = 0$ (three independent linear conditions). Along $w(t) = w^0 + te$ we have $sv_i - m = -m^0 + tv_i$, where $m^0 = m(w^0)$, so $R_i(t) = \mu - 2tm^0 \cdot v_i + t^2 v_i \cdot v_i$. Since $N_P(w(t)) = t^2 F_P(w(t))$ and ℓ_1 and its first derivative vanish at $t = 0$,

$$2F_P(w^0) = \frac{d^2}{dt^2} N_P(w(t)) \Big|_{t=0} = \ell_1''(0) \prod_{\eta_1=1, \eta \neq \mathbf{1}} \ell_\eta(w^0).$$

The product equals $\mu^{7/2} \prod_{\eta \neq \mathbf{1}} (\eta \cdot w^0)$. Differentiating $\ell_1(t) = \sum_i (w_i^0 + te_i) \rho_i(t)$ twice, with $\rho_i(0) = \sqrt{\mu}$, $\rho_i'(0) = -m^0 \cdot v_i / \sqrt{\mu}$ and $\rho_i''(0) = v_i \cdot v_i / \sqrt{\mu} - (m^0 \cdot v_i)^2 / \mu^{3/2}$, gives

$$\ell_1''(0) = -\frac{2m^0 \cdot m(e)}{\sqrt{\mu}} + \frac{1}{\mu^{3/2}} \sum_i w_i^0 (\mu v_i \cdot v_i - (m^0 \cdot v_i)^2) = \frac{C(w^0)}{\mu^{3/2}}$$

by (7) and $m(e) = 0$. Hence $2F_P(w^0) = \mu^2 C(w^0) \prod_{\eta \neq \mathbf{1}} (\eta \cdot w^0)$ on a dense subset of H , which proves the congruence.

On H , μ is a nonzero quadratic form and each $\eta \cdot w$ with $\eta \neq \pm \mathbf{1}$ is a nonzero linear form. The restriction of C to H is unchanged by a translation of the plane, because $\sum_i w_i \det(v_i, m) = \det(m, m) = 0$ and $s = 0$ on H . So we may assume $v_1 = 0$. The point $w = (-\alpha - \beta, \alpha, \beta, 0) \in H$ has $m = \alpha v_2 + \beta v_3$ and $C(w) = \alpha\beta(\alpha + \beta) \det(v_2, v_3)^2$, which is nonzero for general α, β because v_1, v_2, v_3 are not collinear. Since $\mathbb{C}[H]$ is a domain, the right-hand side is nonzero on H . $\square$

Lemma 5.2. *Let $E \subset \mathbb{R}^2$ consist of the four vertices and of the intersection points of the lines $v_a v_b$ and $v_c v_d$, for the three partitions $\{a, b\} \cup \{c, d\}$ of $\{1, 2, 3, 4\}$; so $|E| \leq 7$. If $x \in \mathbb{C}^2$, $r_i^2 = q_i(x)$ and $\ell_{\eta, x}$ vanishes identically on $K(x)$ for some η , then $x \in E$.*

Proof. If $\ell_{\eta, x}$ vanishes on $K(x)$, then $(\eta_i r_i)_i \in K(x)^\perp$, the span of D^1, D^2 , so there is $\sigma \in \mathbb{C}^2$ with $d_i \cdot \sigma = \eta_i r_i$ for all i . Then $Q(d_i) = 0$ for the binary quadratic form $Q(u) = (u \cdot \sigma)^2 - u \cdot u$. This form is not zero, since its matrix $\sigma\sigma^\top - I$ is not, so its zeros in $\mathbb{C}^2$ lie on at most two lines through 0. Hence the vertices lie on at most two lines through x . If x is not a vertex, each such line contains at most two vertices, so there are two lines, each through two vertices, and x is their intersection point. $\square$

Proof of Theorem 1.1. By Lemmas 4.1 and 5.1 and Proposition 4.2, F_P is a form of degree 14 in $I(S_P)$, not divisible by s . By Lemma 2.3, $S_P \subset \mathbb{P}^3$ is an irreducible surface, so $I(S_P) = (F_S)$ for an irreducible form F_S (a homogeneous prime ideal of height one in a polynomial ring is principal). Write $F_P = F_S G$ with a form G .

Let $x \in V$, with V as in Theorem 3.2, and let Y be the line $\mathbb{P}(K(x))$. It passes through p_0 , and s vanishes on it only at p_0 . By Proposition 4.2(a), $F_P|_{K(x)} = s^6 \prod_{\eta_1=1} \ell_{\eta, x}$. Each $\ell_{\eta, x}$ is a nonzero linear form on $K(x)$ (proof of Proposition 4.2(b)), and by Proposition 3.4 it vanishes at $p_\varepsilon(x)$ for $\varepsilon_i = (-1)^i \eta_i$. By Theorem 3.2(a) these eight points are distinct and differ from p_0 . Hence $F_P|_Y$ has a zero of order exactly 6 at p_0 , simple zeros at the eight points $p_\varepsilon(x)$, and no other zeros. The points $p_\varepsilon(x)$ lie on S_P , so F_S vanishes at them, and $F_S|_Y \not\equiv 0$ because $F_P|_Y \not\equiv 0$. As these zeros of $F_P|_Y$ are simple, G does not vanish at any $p_\varepsilon(x)$. Thus:

$$\text{for every } x \in V, G \text{ vanishes on } \mathbb{P}(K(x)) \text{ at most at } p_0. \quad (13)$$

Let g be an irreducible factor of G and $Z = \{g = 0\} \subset \mathbb{P}^3$, an irreducible surface. A point $z \in Z \setminus \{p_0\}$ either has $s(z) = 0$, and then $\pi(z)$ lies on the line at infinity, or $\pi(z) = x \in \mathbb{C}^2$ and $z \in \mathbb{P}(K(x))$, and then $x \notin V$ by (13). So $\pi(Z \setminus \{p_0\}) \subset \mathbb{P}^2 \setminus V$, a set of dimension at most 1. The fibres of π on $\mathbb{P}^3 \setminus \{p_0\}$ are the lines through p_0 with p_0 removed, so the closure of $\pi(Z \setminus \{p_0\})$ is an

irreducible curve $\Gamma = \{\gamma = 0\} \subset \mathbb{P}^2 \setminus V$. Then Z lies in the cone $\{\gamma(s, m_1, m_2) = 0\}$ over Γ with vertex p_0 , which is an irreducible surface; so Z is this cone and g is a multiple of $\gamma(s, m_1, m_2)$.

If Γ were the line at infinity, g would be a multiple of s , which is excluded by Lemma 5.1. Otherwise Γ has infinitely many points $x \in \mathbb{C}^2$. For each of them $\mathbb{P}(K(x)) \subset Z$, so F_P vanishes on $K(x)$, and by Proposition 4.2(a) some $\ell_{\eta, x}$ vanishes on $K(x)$, since $s \neq 0$ there. By Lemma 5.2 this happens only for finitely many x , a contradiction. Hence G is constant and $I(S_P) = (F_P)$. In particular F_P is irreducible and $\deg S_P = 14$. The statement about p_0 is part of Corollary 5.3. $\square$

Corollary 5.3. *All k_i are nonzero, and p_0 lies on S_P with multiplicity 6. The tangent cone of S_P at p_0 is the cone with vertex p_0 over the projective plane sextic $T = 0$, where*

$$T(s, m_1, m_2) = \frac{\mathcal{N}_4(k_1^2 R_1, \dots, k_4^2 R_4)}{s^2}, \quad R_i = (sv_i - m) \cdot (sv_i - m) \text{ as in (2)}.$$

In the chart $s = 1$, where $m = x$, T is $\mathcal{N}_4(k_1^2 q_1(x), \dots, k_4^2 q_4(x))$, whose real zeros contain the curve $\sum_i \pm k_i |x - v_i| = 0$. A general line through p_0 meets S_P in 8 further points.

Proof. By Cramer's rule, k_i is, up to sign, the determinant of the 3×3 matrix with rows $(1, a_j, b_j)$, $j \neq i$, which is nonzero since no three vertices are collinear. Each $R_i(w)$ depends only on $s(w)$ and $m(w)$, which vanish at k ; so $R_i(k + u) = R_i(u)$ and $N_P(k + u) = \mathcal{N}_4((k_1 + u_1)^2 R_1(u), \dots)$. Its part of lowest degree is $\mathcal{N}_4(k_1^2 R_1(u), \dots, k_4^2 R_4(u))$, of degree 8, provided this is nonzero. For $u \in K(x)$ we have $R_i(u) = s(u)^2 q_i(x)$, so it equals $s(u)^8 \mathcal{N}_4(k_1^2 q_1(x), \dots, k_4^2 q_4(x)) = s(u)^8 \prod_{\eta=1} \sum_i \eta_i k_i r_i$, and this is nonzero for general x by Lemma 2.2, because $\sum_i \eta_i k_i \rho_i \neq 0$ in $\mathcal{O}$. Since $N_P(k + u) = s(u)^2 F_P(k + u)$, the function F_P vanishes to order exactly 6 at p_0 , and its lowest part is the sextic $\mathcal{N}_4(k_1^2 R_1(u), \dots, k_4^2 R_4(u)) / s(u)^2 = T(s(u), m_1(u), m_2(u))$. In the chart $s = 1$, where $m = x$, T is $\mathcal{N}_4(k_1^2 q_1(x), \dots, k_4^2 q_4(x))$, which for real x equals $\prod_{\eta=1} \sum_i \eta_i k_i |x - v_i|$. The last statement holds because $14 - 6 = 8$; it agrees with Theorem 3.2, as the projection from p_0 is π . $\square$

The sextic T can contain the line $s = 0$. Suppose that $v_a v_b \parallel v_c v_d$ for a partition $\{a, b\} \cup \{c, d\}$ of $\{1, 2, 3, 4\}$, say $v_b - v_a = t(v_d - v_c)$. Then $k_a + k_b = k_c + k_d = 0$: the vector with entries $-1, 1, t, -t$ in the positions a, b, c, d satisfies the conditions $\sum_i k_i = 0$ and $\sum_i k_i v_i = 0$, which determine k up to a factor. Near a general point of the plane $s = 0$, where every R_i equals $m \cdot m$, write $\mathcal{N}_4(k_1^2 R_1, \dots, k_4^2 R_4) = \prod_{\eta=1} \sum_i \eta_i k_i \rho_i$ with holomorphic square roots ρ_i of R_i that agree on $s = 0$. The factor with $\eta = 1$ vanishes there to order 2, because its first derivatives there are combinations of $\sum_i k_i$ and $\sum_i k_i v_i$ (as in the proof of Lemma 4.1), and the factor with $\eta_a = \eta_b = -\eta_c = -\eta_d$ vanishes there as well. So s^3 divides $\mathcal{N}_4(k_i^2 R_i)$, and s divides T . Exact computation gives $T = s^2 \cdot$ (quartic) for the unit square of Example 5.5 and the other parallelograms we tested, and $T = s \cdot$ (quintic) for the trapezoids we tested. In these cases the affine equation $\mathcal{N}_4(k_i^2 q_i(x)) = 0$ has degree less than 6, and the tangent cone contains the plane $s = 0$.

Remark 5.4 (the cyclic order). Since $\mathcal{N}_4$ is symmetric and the R_i depend only on the labelled vertices, F_P does not change if the cyclic order is changed, so S_P depends only on the vertex set. For instance, the three quadrilaterals on four points in convex position (one convex, two self-intersecting) have the same mean value surface, although their mean value coordinates differ; their relations (4) are different factors of F_P .

Example 5.5 (the unit square). Let $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, $v_4 = (0, 1)$ and put $a = w_1 + w_2$, $b = w_2 + w_3$, $c = w_3 + w_4$, $d = w_4 + w_1$. Then $R_1 = b^2 + c^2$, $R_2 = c^2 + d^2$, $R_3 = d^2 + a^2$, $R_4 = a^2 + b^2$, so

$$F_P = \frac{\mathcal{N}_4(w_1^2(b^2 + c^2), w_2^2(c^2 + d^2), w_3^2(d^2 + a^2), w_4^2(a^2 + b^2))}{(w_1 + w_2 + w_3 + w_4)^2}.$$

Expanded, F_P has 116 terms, all of degree 14, and content 16. The polynomial $F_P/16$ has coefficients of absolute value at most 27 and is invariant under the dihedral group of the square, acting on the indices; Table 1 lists its 21 orbits of monomials. Here $p_0 = [1 : -1 : 1 : -1]$.

e	$ O $	c_O	e	$ O $	c_O	e	$ O $	c_O
(6, 4, 4, 0)	8	1	(5, 5, 4, 0)	8	2	(5, 3, 5, 1)	4	8
(6, 4, 2, 2)	8	-6	(5, 5, 3, 1)	8	4	(5, 3, 4, 2)	8	4
(6, 4, 0, 4)	4	1	(5, 5, 2, 2)	4	-4	(5, 3, 3, 3)	4	-24
(6, 3, 4, 1)	8	-4	(5, 4, 5, 0)	4	2	(5, 2, 5, 2)	2	12
(6, 3, 3, 2)	8	-16	(5, 4, 4, 1)	8	10	(4, 4, 4, 2)	4	27
(6, 3, 2, 3)	4	-4	(5, 4, 2, 3)	8	4	(4, 4, 3, 3)	4	-4
(6, 2, 4, 2)	4	-10	(5, 4, 1, 4)	4	6	(4, 3, 4, 3)	2	-4

Table 1: The unit square: $F_P/16 = \sum_O c_O \sum_{e \in O} w^e$, where O runs over the orbits of exponent vectors e under the group generated by $e \mapsto (e_2, e_3, e_4, e_1)$ and $e \mapsto (e_1, e_4, e_3, e_2)$, each represented by its lexicographically largest element.

Remark 5.6 (three collinear vertices; computational). The hypothesis on the vertices in Theorem 1.1 is needed. Suppose that three vertices lie on a line ℓ , and let v_j be the fourth. For $x \in \ell$, every $w \in K(x)$ has $w_j = 0$, and some $\ell_{\eta,x}$ vanishes on $K(x)$. So Lemma 5.2 fails, and F_P vanishes on the plane $w_j = 0$, which is the cone over ℓ . For the quadrilaterals with vertices $(0, 0), (4, 1), (8, 2), (3, 7)$ and $(0, 0), (5, 5), (6, 0), (2, 2)$, an exact computation gives $F_P = w_j^2 K$ with a form K of degree 12. Modulo $p = 2^{31} - 1$, the matrix E_{11} of Section 6 has full rank, and $\dim \ker E_d = 1, 4, 10$ for $d = 12, 13, 14$. The ideal $I(S_P)$ is principal and prime, as in the proof of Theorem 1.1, and does not contain w_j , so it contains K . By Lemma 6.1 it has no nonzero element of degree at most 11, so $I(S_P) = (K)$. Thus for these two quadrilaterals S_P has degree 12, and F_P does not generate $I(S_P)$.

6 Pentagons, hexagons and heptagons: computations

Apart from Lemma 6.1 and the use of Theorem 3.2(c) below, this section reports computations for sample polygons, which are listed in the release. The full-rank statements and the hexagon certificate are certified in the sense of Lemma 6.1; everything else is a computation modulo primes, not a proof. In particular, the existence of relations of degree 10 for $n = 5$ and of degree 7 for $n = 7$ rests on quotients computed modulo p , whereas $d_0 = 6$ is certified for four hexagons. *Method.* Let the vertices be integral and let p be a prime (mostly $p = 2^{31} - 1$ and $p = 67108859$). Points of X over $\mathbb{F}_p$ are found by choosing $x \in \mathbb{F}_p^2$ at random until every $q_i(x)$ is a nonzero square, and then random square roots ρ_i . Let E_d be the matrix of values modulo p of all monomials of degree d at the points $W(x, \rho)$. Its rank is computed by exact Gaussian elimination over $\mathbb{F}_p$.

Lemma 6.1. $\dim_{\mathbb{C}} I(S_P)_d \leq \dim_{\mathbb{F}_p} \ker E_d$. In particular, if E_d has full column rank, then no nonzero form of degree d vanishes on S_P . A nonzero kernel gives only an upper bound; lower bounds need explicit elements of $I(S_P)$.

Proof. A form F vanishes on S_P if and only if $F(W(x, \rho))$ vanishes on X° , hence on X , that is, if and only if $F(W) = 0$ in $\mathcal{O}$. Reducing $F(W)$ modulo the monic polynomials $\rho_i^2 - q_i$ gives its coordinates in the basis ρ_S ; they are polynomials in x whose coefficients are $\mathbb{Z}$ -linear in the coefficients of F . So $I(S_P)_d$ is cut out by linear equations with integer coefficients, and $L = I(S_P)_d \cap \mathbb{Z}[w]_d$ is a saturated lattice of rank $\dim I(S_P)_d$. For $F \in L$ the reduction of $F(W)$ lies in the ideal generated by the $\rho_i^2 - q_i$ over $\mathbb{Z}$, so F modulo p vanishes at all sampled points. As L is saturated, its reduction modulo p has the same rank, and it lies in $\ker E_d$. $\square$

Tangency conditions. By Theorem 3.2(c), every polynomial condition on n lines that holds whenever the lines are tangent to a common circle gives an element of $I(S_P)$ when applied to $L_1(w), \dots, L_n(w)$. To make this polynomial in w , put $\tilde{d}_i = sv_i - m$ and $\tilde{g}_i = \sum_{j \leq i} w_j J \tilde{d}_j$; if $s \neq 0$ and $x = m/s$, then $\tilde{d}_i = s d_i$, $\tilde{g}_i = s G_i(w)$, and $L_i(w)$ has line coordinates $\ell_i(w) = (s \tilde{d}_i, -\tilde{d}_i \cdot \tilde{g}_i)$. The resulting forms contain extraneous factors, powers of s and of $\hat{A}_i = s \det(v_i, v_{i+1}) + \det(v_{i+1} - v_i, m)$, which vanishes on the preimage of the i -th edge line. Neither lies in the prime ideal $I(S_P)$, so exact quotients stay in $I(S_P)$; the exponents were found modulo p on random lines of $\mathbb{P}^{n-1}$.

- $n = 5$. For general w the five lines are tangent to a unique conic, whose dual conic $B = (b_{jk})$ passes through the points $\ell_i(w)$. Dual conics of circles satisfy $b_{12}b_{33} = b_{13}b_{23}$ and $(b_{11} - b_{22})b_{33} = b_{13}^2 - b_{23}^2$. With B given by 5×5 minors, these are forms of degree 46; modulo p each is $s^{26} \prod_i \hat{A}_i^2$ times a form of degree 10, and the two quotients span $\ker E_{10}$, of dimension 2, for four pentagons.
- $n = 6$. For general $w \in S_P$ the hexagon with vertices $G_a(w)/s = \tilde{g}_a/s^2$ is circumscribed about a circle, so by Brianchon's theorem its main diagonals are concurrent: with $\hat{G}_a = (\tilde{g}_a, s^2) \in \mathbb{C}^3$, the form $\text{Br} = \det[\hat{G}_0 \times \hat{G}_3, \hat{G}_1 \times \hat{G}_4, \hat{G}_2 \times \hat{G}_5]$ of degree 12 lies in $I(S_P)$. We expanded Br exactly over $\mathbb{Z}$ for the four hexagons with vertices $(0, 0), (6, -1), (10, 3), (9, 8), (3, 10), (-2, 5); (11, 19), (-14, 16), (-15, 12), (-10, -14), (-7, -22), (12, -16); (2, 0), (1, 1), (-1, 1), (-2, 0), (-1, -1), (1, -1);$ and $(0, 0), (8, 0), (8, 6), (4, 2), (0, 6), (-3, 3)$. In each case it is divisible by s^6 , and $K_6 = \text{Br}/s^6$ is a sextic with 50 terms. Modulo $p = 2^{31} - 1$, E_5 has full rank 252 and E_6 has rank $461 = 462 - 1$. By Lemma 6.1, for these hexagons $I(S_P)$ has no nonzero element of degree at most 5, and $I(S_P)_6 = \mathbb{C} K_6$ (certified).
- $n = 7$. Six of the seven lines touch a conic, so the 6×6 determinant of quadratic monomials in the corresponding $\ell_i(w)$ lies in $I(S_P)$. The seven determinants have degree 28, and degree 7 after removing the extraneous factors modulo p . These seven septics are linearly independent and lie in $\ker E_7$, of dimension 8; we could not explain the eighth.

n	d_0	relations of degree d_0	$\dim \ker E_d$ for $d = d_0, d_0 + 1, \dots$	new generators, $d = d_0, \dots$
4	14	F_P (Theorem 1.1)	1	1, 0, 0, $\dots$ (Theorem 1.1)
5	10	two circle conditions	2, 16, 59, 150, 309	2, 6, 5, 5
6	6	Brianchon sextic	1, 6, 21, 56, 129, 345	1, 0, 0, 0, 3
7	7	seven conic conditions, one more	8, 49	8, 0

Table 2: Low-degree parts of $I(S_P)$; d_0 is the lowest degree of a relation. $I(S_P)_d = 0$ for $d < d_0$ is proved for $n = 4$, and for $n = 5, 6, 7$ it is certified by full rank (Lemma 6.1) for the polygons tested (listed in the release); for $n = 6$ the value 1 in degree 6 is certified for four hexagons. All other entries are computations modulo p for several integer polygons (listed in the release): kernel dimensions, which bound $\dim I(S_P)_d$ from above, and the numbers $\dim \ker E_d - \dim(\mathbb{C}[w]_1 \cdot \ker E_{d-1})$ of new generators, where $\mathbb{C}[w]_1$ is the space of linear forms. For a pentagon with four concyclic vertices the entries for $d = 13, 14$ are 151, 310, with 6 new generators in degree 13.

Degrees. We sampled $\mathbb{F}_p$ -points of a general hyperplane section of S_P as roots of univariate norm polynomials, projected them to $\mathbb{P}^2$, and found the least degree of a plane curve through them. For three pentagons a curve of degree 52, but none of degree 51, passes through the points (in one case the kernels in degrees 51, 52, 53 have dimensions 0, 1, 3, as for a single curve); for two of them the Hilbert function of the section has first differences 52 for $d = 18, \dots, 24$. For one hexagon, 12503 points of a section admit exactly one plane curve of degree 152, hence none of lower degree; this was

not reproduced independently. So $\deg S_P = 1, 14, 52, 152$ for $n = 3, 4, 5, 6$, the first two proved and the last computed for one hexagon. The closure C_L of Φ over a general line L of the plane has computed degree 14, 38, 94 for $n = 4, 5, 6$, as an unproved base-point count predicting $n2^{n-2} - 2$ suggests.

Conjecture 6.2. *For a general polygon with $n \geq 3$ vertices, $\deg S_P = 2^{n-3}(6n - 17)$.*

The two parameters of $2^{n-3}(an + b)$ are fixed by $n = 4, 5$, so $n = 3$ and $n = 6$ are the only checks; the value for $n = 6$ rests on one hexagon.

Open questions. (1) Generators of $I(S_P)$ for $n \geq 5$; the data suggest the saturation of the ideal of tangency conditions with respect to s , the $\hat{A}_i$ and the loci where several $L_i(w)$ are concurrent. (2) Conjecture 6.2, and the curve $S_P \cap \Lambda$ for $n \geq 5$. (3) The eighth septic for $n = 7$ and the three new generators of degree 10 for $n = 6$.

Verification. The programs are in the folder `reproducibility` of the release.

1. Programs by the author (folder `lead`). `lead_identities.py` expands over $\mathbb{Z}$ the identities (5)–(8), (10), (12), $w_i \rho_i = t_{i-1} + t_i$, the product formula for $\mathcal{N}_4$, Lemma 2.1(b),(c) for $n = 4, 5$, and the Laplace expansion in the proof of Theorem 3.2 for all 28 pairs of sign classes when $n = 4$. `lead_quad_instances.py` checks exactly, for 31 quadrilaterals (convex, non-convex, self-intersecting and 24 random ones), $s^2 \mid N_P$, Lemma 5.1, the order and tangent cone at p_0 , Proposition 4.2(a), and $F_P(W(x, \rho)) = 0$ in $\mathbb{Z}[\rho]/(\rho_i^2 - q_i(x))$ (all sixteen branches) at two integer points x . `lead_square_orbits.py` produces Table 1; `lead_modp_certificate.py` finds $\text{rank } E_{13} = 560$ (full) and $\text{rank } E_{14} = 679$ for two quadrilaterals; `lead_low_degree.py` recomputes the ranks at $d_0 - 1$ and d_0 in Table 2 for six polygons; `lead_hexagon_brianchon.py` gives the hexagon certificate; `lead_quad_extras.py` computes exactly the power of s dividing T (Corollary 5.3) for twelve quadrilaterals, and the factorization $F_P = w_j^2 K$ and the ranks of Remark 5.6.
2. The programs with which the results were found (folder `claimant`) compute F_P exactly for two quadrilaterals, verify Lemma 3.1 and Proposition 3.4 and Theorem 3.2(c) modulo p for 13 polygons with $4 \leq n \leq 7$ and every sign vector, and carry out all computations of Section 6.
3. Independent verification runs, which were AI-assisted, checked the proofs and used separate code written before the programs in `claimant` were read (folder `verifier`). They confirmed the three-point form, the Pitot relation and (10) in floating point; recomputed F_P for the unit square exactly; found $I(S_P)_{13} = 0$ and $\dim \ker E_{14} = 1$ for five quadrilaterals; reproduced d_0 and $\dim \ker E_{d_0}$ in Table 2, $\dim \ker E_{11} = 16$ for $n = 5$, the circle and Brianchon conditions, $\deg C_L = 14, 38, 94$ and $\deg S_P = 52$ for a further pentagon; and found the collision determinants of Theorem 3.2 nonzero at random points. The degree 152 and the generator counts beyond degree 11 were not reproduced independently.
4. A second independent verification run, also AI-assisted (folder `independent_run_2`), checked the proofs step by step and wrote new code, which takes the points of S_P over $\mathbb{F}_p$ from Floater's definition rather than from (9). It confirmed exactly Lemma 2.1(b),(c) for sample polygons with $4 \leq n \leq 8$, Lemma 2.1(a) and the Pitot relation for $n \leq 6$, and Lemmas 4.1 and 5.1, Corollary 5.3 and Table 1 for 16 quadrilaterals, with $F_P(w) = 0$ on all sixteen branches. It confirmed $I(S_P) = (F_P)$ for 12 quadrilaterals through full rank of E_{13} , Theorem 3.2(a),(c) modulo p for $4 \leq n \leq 8$, the hexagon certificate for six hexagons, and the kernel dimensions of Table 2 up to $d = 14$ for $n = 5$, $d = 11$ for $n = 6$ and $d = 8$ for $n = 7$, on polygons of its own. It also found the examples of Remark 5.6. It did not reproduce $\deg S_P = 152$ or the generator counts.

Scope and priority. Theorem 1.1 answers Sottile’s question [dBSSS22] completely for quadrilaterals with no three vertices collinear. For $n \geq 5$ we give a structural description and computations but no generating set, so for general n the question remains open. The record OWR-9790358-016 of [ula26] also contains the question “What about in higher dimensions?” from the item on three-dimensional Wachspress varieties (Sottile, Schenck) and the item on moduli spaces of Wachspress varieties (Sottile), both from the same page of [dBSSS22]; we do not address them and suggest treating them as separate problems. Searches of arXiv, Crossref, OpenAlex (partly rate-limited), zbMATH and the web, and separate searches in the second verification run, found no equations, degrees or algebraic relations of mean value coordinates. The closest work concerns the geometry and formulas of the coordinates [Flo03, JSWD05, FHK06, HF06, Flo15], the injectivity of mean value mappings [FK10, FM25] (for quadrilaterals, [FM25] uses identities between half-angle tangents, but no implicit equation), and Wachspress surfaces and varieties [GPS10, Irv12, IS14, KR20]. Irving’s thesis [Irv12], which the problem names as its inspiration, treats the Wachspress case through the same projection π (Section 2); we read it in the copy cited in the bibliography. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments and carry out the computations, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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