

Gaussian Cubature of Even Degree for the Product Chebyshev Weight in Three Variables: An Explicit Ten-Node Rule and No Twenty-Node Rule with a Sign Symmetry

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Abstract

In Oberwolfach Report 7/2022 Y. Xu asked whether Gaussian cubature rules of degree $2n - 2$, with $\dim \Pi_{n-1}^3 = \binom{n+2}{3}$ real nodes, exist for the product Chebyshev weight of the second kind on the cube $[-1, 1]^3$; existence is known only for $n = 2$ and $n = 3$. (The source prints the formula of the weight of the first kind under the name of the second kind; we follow the name.) We give partial results. From Xu's characterisation of Gaussian rules of even degree we derive that, for this weight in d variables, such a rule exists if and only if an explicit quadratic system has a real solution: for a function h on the multi-indices of length $2n - 1$, the matrix $(\sum_{\beta} h(\alpha + \beta)h(\alpha' + \beta) - \delta_{\alpha\alpha'})$ must depend only on $\alpha + \alpha'$; the rules of Morrow and Patterson in two variables correspond to $h = \delta_{(n,n-1)}$. For $n = 3$ we give an explicit ten-node rule with nodes and weights in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. For $n = 4$ we prove that no Gaussian rule of degree 6 has a node set invariant under a coordinate reflection, a half-turn about a coordinate axis, or $x \mapsto -x$, hence under any non-trivial group of coordinate sign changes. The cases of one reflection and of a half-turn are computer-assisted, by exact Nullstellensatz certificates with rational coefficients, checked by independently written programs; two reflections and the central symmetry have short direct proofs. The case $n = 4$ remains open for node sets whose symmetry group contains no non-trivial sign change (for example a group generated by $(x, y, z) \mapsto (-y, -x, z)$ or by a rotation of order three, or the trivial group), and so do all $n \geq 5$; numerical searches for $n = 4$ and $n = 5$ found no rule, which is evidence only. The question and the theory of Gaussian cubature of even degree are due to Xu, building on Morrow, Patterson and Schmid; new, to our knowledge, are the exclusion of coordinate reflections and half-turns for $n = 4$ and the explicit rule for $n = 3$. This is an unrefereed note.

1 Introduction

1.1 The question

Let $w \geq 0$ be a weight function on a domain $\Omega \subset \mathbb{R}^d$, and let Π_m^d denote the space of real polynomials of total degree at most m in d variables. Following Xu's abstract [Xu22, p. 415], a *cubature rule of degree m* is a sum $C(f) = \sum_{k=1}^N \lambda_k f(x_k)$ with nodes $x_k \in \mathbb{R}^d$ and weights $\lambda_k \in \mathbb{R}$ such that $\int_{\Omega} f w = C(f)$ for all $f \in \Pi_m^d$, while the identity fails for at least one polynomial of degree $m + 1$. For $m = 2n - 1$ and for $m = 2n - 2$ the number of nodes satisfies $N \geq \dim \Pi_{n-1}^d = \binom{n+d-1}{d}$, and the rule is called *Gaussian* if equality holds. For $m = 2n - 2$ the nodes of a Gaussian rule are common zeros of a basis of quasi-orthogonal polynomials of degree n , and Xu notes that the existence of such rules is characterised by a non-linear system of equations in the coefficient matrices of the three-term

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relations [Xu22, p. 416]. For the product Chebyshev weight of the second kind on $[-1, 1]^2$ these rules exist for every n ; this exceptional case goes back to Morrow and Patterson [MP78] (see [Xu18, Theorem 5]). Xu then asks [Xu22, p. 416] “if such a rule exists for the product Chebyshev weight function in three variables”, and adds that existence is known only for $n = 2$ and $n = 3$. This question is record OWR-9790358-011 of the UnsolvedMath collection [ula26], where it is listed as open.

Three conventions are needed to state the question precisely.

The weight. The source calls the weight “of the second kind” but prints the formula $1/(\sqrt{1-x_1^2}\sqrt{1-x_2^2})$, which is the product Chebyshev weight of the first kind [Xu22, p. 416]. The two-variable theorem to which the question refers holds for the weight of the second kind [Xu18, Theorem 5], and we follow the name. Thus, normalised to total mass 1,

$$w(t) = \frac{2}{\pi} \sqrt{1-t^2} \quad (-1 \leq t \leq 1), \quad W(x) = w(x_1) \cdots w(x_d) \quad (x \in [-1, 1]^d). \quad (1)$$

The normalisation plays no role. For the weight of the first kind our exact results do not apply; see Section 7.

Nodes and weights. As in the source, nodes may lie anywhere in $\mathbb{R}^d$ and weights may have any sign. The weights of a Gaussian rule of even degree are automatically positive (Lemma 2.1). Our non-existence results hold even for complex nodes and complex weights.

Degree. We call a rule with $N = \dim \Pi_{n-1}^d$ distinct nodes which is exact on Π_{2n-2}^d an n -rule. For our weight every n -rule has degree exactly $2n - 2$ (Theorem 1.2(b)), so the n -rules are precisely the Gaussian rules of degree $2n - 2$.

Problem 1.1 (Xu [Xu22]). *For which $n \geq 2$ does an n -rule exist for the weight W of (1) on $[-1, 1]^3$?*

The cases $n = 2, 3, 4, 5$ have 4, 10, 20 and 35 nodes and degrees 2, 4, 6 and 8.

1.2 Results

Let $\Lambda_k = \{\alpha \in \mathbb{N}_0^d : |\alpha| = k\}$ and let $e_1, \dots, e_d$ be the unit vectors. For a function $h : \Lambda_{2n-1} \rightarrow \mathbb{C}$ consider the system

$$\sum_{\beta \in \Lambda_{n-1}} h(\alpha + \beta) h(\alpha' + \beta) - \delta_{\alpha\alpha'} \quad \text{depends only on} \quad \alpha + \alpha' \quad (\alpha, \alpha' \in \Lambda_n), \quad (\star)$$

that is, there is $g : \Lambda_{2n} \rightarrow \mathbb{C}$ with $\sum_{\beta} h(\alpha + \beta) h(\alpha' + \beta) = \delta_{\alpha\alpha'} + g(\alpha + \alpha')$ for all $\alpha, \alpha' \in \Lambda_n$. Let U_k be the Chebyshev polynomials of the second kind, $P_\alpha(x) = U_{\alpha_1}(x_1) \cdots U_{\alpha_d}(x_d)$, and for a solution h put

$$q_\alpha = P_\alpha + \sum_{\beta \in \Lambda_{n-1}} h(\alpha + \beta) P_\beta \quad (\alpha \in \Lambda_n). \quad (2)$$

Theorem 1.2. *Let $d \geq 2$ and $n \geq 2$, and let W be the weight (1) on $[-1, 1]^d$.*

- (a) *An n -rule exists if and only if $(\star)$ has a real solution. More precisely, the node sets of n -rules correspond bijectively to the real solutions h of $(\star)$: the node set belonging to h is the set of common zeros of the polynomials q_α , $\alpha \in \Lambda_n$, and the weights are $\lambda_k = 1/K_{n-1}(x_k, x_k) > 0$, where K_{n-1} is the reproducing kernel of Π_{n-1}^d .*
- (b) *Every n -rule has degree exactly $2n - 2$.*
- (c) *If $N = \dim \Pi_{n-1}^d$ distinct points of $\mathbb{C}^d$ and complex weights give a rule which is exact on Π_{2n-2}^d , then $(\star)$ has a complex solution h , and the points are common zeros of the q_α .*

The condition $(\star)$ is the form which Xu's matrix equations [Xu94], [Xu18, Theorem 12] take for this weight; see Section 3. For $d = 2$ the function $h = \delta_{(n,n-1)}$ is a solution for every n , and it gives the rules of Morrow and Patterson (Remark 3.5).

Proposition 1.3. *The ten points*

$$(0, 0, \frac{\sqrt{6}-\sqrt{2}}{4}), (0, 0, -\frac{\sqrt{6}+\sqrt{2}}{4}), (\pm\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{2}}{2}), (0, \pm\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}), (\pm\frac{\sqrt{6}}{4}, \pm\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4})$$

with the weights $\frac{3+\sqrt{3}}{18}$ and $\frac{3-\sqrt{3}}{18}$ for the first two, $\frac{1}{18}$ for the next four and $\frac{1}{9}$ for the last four form a Gaussian cubature rule of degree 4 for W on $[-1, 1]^3$. All nodes lie in the open cube.

This is Proposition 5.1 below. The source states that the case $n = 3$ is known; we did not find an explicit rule in the sources we read. Numerically, the 3-rules form a family of dimension five whose generic members have no symmetry (Remark 5.2).

A *coordinate sign change* is a map $D_\varepsilon(x) = (\varepsilon_1 x_1, \dots, \varepsilon_d x_d)$ with $\varepsilon \in \{\pm 1\}^d$; these maps form the group $\mathbb{Z}_2^d$. For $d = 3$ its non-trivial elements are the three reflections $x_j \mapsto -x_j$ in the coordinate planes, the three half-turns about the coordinate axes, such as $(x, y, z) \mapsto (x, -y, -z)$, and the central symmetry $x \mapsto -x$.

Theorem 1.4. *Let $X \subset \mathbb{C}^3$ be the node set of a 4-rule for W on $[-1, 1]^3$, with real or complex nodes and weights. Then X is invariant neither under a reflection in a coordinate plane, nor under a half-turn about a coordinate axis, nor under $x \mapsto -x$. Consequently no Gaussian cubature rule of degree 6 for W has a node set invariant under a non-trivial subgroup of $\mathbb{Z}_2^3$.*

The three cases are Theorems 6.2 and 6.3 and Proposition 4.2. The cases of one reflection and of a half-turn are computer-assisted: in each case a finite system of quadratic equations with integer coefficients is shown to have no complex solution by an explicit identity $\sum_k q_k E_k = 1$ with polynomials q_k of degree at most 2 with rational coefficients. The special case of two reflections, which is implied by either of them, has a short direct proof (Theorem 6.1), and so has the central symmetry, for every n .

What is not covered. The square $[-1, 1]^3$ and W are invariant under the group B_3 of signed permutations of the coordinates, of order 48, which contains $\mathbb{Z}_2^3$ as a normal subgroup with quotient S_3 . Theorem 1.4 says nothing about node sets whose symmetry group meets $\mathbb{Z}_2^3$ only in the identity. Within B_3 such a group maps injectively to S_3 , so it has order 1, 2, 3 or 6; examples are the groups generated by the permutation $(x, y, z) \mapsto (y, x, z)$, by the reflection $(x, y, z) \mapsto (-y, -x, z)$, by the half-turn $(x, y, z) \mapsto (y, x, -z)$ about the axis through the midpoints of two opposite edges, or by the rotation $(x, y, z) \mapsto (y, z, x)$ of order three. A rotation of order four about a coordinate axis is covered, since its square is a half-turn about that axis.

1.3 What is new, what is not, and what stays open

The question is Xu's [Xu22]. Theorem 1.2 is a reformulation of the theory of Gaussian cubature of even degree: the description of the nodes as common zeros of quasi-orthogonal polynomials goes back to Morrow and Patterson [MP78] and Schmid [Sch78] in two variables (see [Xu18, Theorem 2]), and the characterisation by the joint eigenvalues of block Jacobi matrices and by matrix equations for the matrix Γ in $\mathbb{P}_n + \Gamma \mathbb{P}_{n-1}$ is due to Xu [Xu94] (in two variables [Xu18, Theorem 12]). That these equations force Hankel structure is also in the literature: Xu writes Γ in terms of a Hankel matrix for the constant weight on the square [Xu18, (21)], and Sauer and Xu use a lemma of [Xu94] to the effect that matrices satisfying commutation conditions of this kind are Hankel matrices [SX23, Section 4]. We include complete proofs, but we claim no priority for Theorem 1.2. The improved lower bound for rules of odd degree and centrally symmetric weights quoted in the source is Möller's [Mö76], and cubature rules on the cube with more nodes than a Gaussian rule were constructed by Li,

Sun and Xu [LSX09] and by De Marchi, Vianello and Xu [DVX09]. To our knowledge, the exclusion of reflections and half-turns in Theorem 1.4 and the explicit rule of Proposition 1.3 are not in print; see the paragraph “Scope and priority” at the end.

Problem 1.1 remains open for every $n \geq 4$: for $n = 4$ we exclude only node sets with a symmetry in $\mathbb{Z}_2^3$, and for $n \geq 5$ we prove nothing. Numerical searches for $n = 4$ and $n = 5$ found no rule, and the near-solutions they produce run off to infinity (Section 7); this is evidence, not proof.

2 Preliminaries

Throughout, $d \geq 2$, $n \geq 2$ and W is the weight (1) on $[-1, 1]^d$. The Chebyshev polynomials of the second kind, $U_0 = 1$, $U_1(t) = 2t$, $U_{k+1} = 2tU_k - U_{k-1}$, are orthonormal for w , and with $U_{-1} = 0$

$$tU_k(t) = \frac{1}{2}U_{k+1}(t) + \frac{1}{2}U_{k-1}(t), \quad U_k(-t) = (-1)^k U_k(t) \quad (k \geq 0). \quad (3)$$

Hence the polynomials $P_\alpha(x) = \prod_i U_{\alpha_i}(x_i)$ form an orthonormal basis of $\Pi^d = \bigcup_m \Pi_m^d$ for the inner product $\langle p, q \rangle = \int pqW$. We write $\Pi_m = \Pi_m^d$, $r_k = |\Lambda_k| = \binom{k+d-1}{d-1}$ and $N = \dim \Pi_{n-1}$, fix an order of each Λ_k , and let $\mathbb{P}_k$ be the column vector $(P_\alpha)_{\alpha \in \Lambda_k}$. Then Π_m is the orthogonal sum of the spaces V_k spanned by the entries of $\mathbb{P}_k$, $k \leq m$. By (3), $x_i P_\alpha = \frac{1}{2}P_{\alpha+e_i} + \frac{1}{2}P_{\alpha-e_i}$ (with $P_{\alpha-e_i} = 0$ if $\alpha_i = 0$), that is,

$$x_i \mathbb{P}_k = A_{k,i} \mathbb{P}_{k+1} + A_{k-1,i}^\top \mathbb{P}_{k-1}, \quad (A_{k,i})_{\alpha, \alpha'} = \frac{1}{2} [\alpha' = \alpha + e_i] \quad (\alpha \in \Lambda_k, \alpha' \in \Lambda_{k+1}), \quad (4)$$

where $[\cdot]$ is 1 if the condition holds and 0 otherwise. The $r_k \times r_{k+1}$ matrices $A_{k,i}$ satisfy, for all i, j ,

$$A_{k,i} A_{k+1,j} = A_{k,j} A_{k+1,i}, \quad A_{k,i} A_{k,j}^\top + A_{k-1,i}^\top A_{k-1,j} = A_{k,j} A_{k,i}^\top + A_{k-1,j}^\top A_{k-1,i}. \quad (5)$$

Indeed, the (α, α') entry of the first product is $\frac{1}{4}[\alpha' = \alpha + e_i + e_j]$, and for $i \neq j$ the (α, α') entries of $A_{k,i} A_{k,j}^\top$ and $A_{k-1,i}^\top A_{k-1,j}$ are $\frac{1}{4}[\alpha' = \alpha + e_i - e_j]$ and $\frac{1}{4}[\alpha' = \alpha - e_i + e_j]$, which are exchanged by $i \leftrightarrow j$. Finally, the stacked matrix $(A_{k,1}^\top, \dots, A_{k,d}^\top)^\top$ has full column rank r_{k+1} : the column of $\alpha' \in \Lambda_{k+1}$ has the entry $\frac{1}{2}$ in the rows $(i, \alpha' - e_i)$ with $\alpha'_i \geq 1$ and zeros elsewhere, and distinct columns have disjoint non-empty supports.

The reproducing kernel of Π_{n-1} is $K_{n-1}(x, y) = \sum_{|\alpha| \leq n-1} P_\alpha(x) P_\alpha(y)$; it satisfies $\langle p, K_{n-1}(\cdot, y) \rangle = p(y)$ for $p \in \Pi_{n-1}$, and $K_{n-1}(x, x) \geq P_0(x)^2 = 1$ for real x . A set X of N points is *unisolvent* for Π_{n-1} if the only polynomial in Π_{n-1} that vanishes on X is 0; then every function on X is interpolated by a unique element of Π_{n-1} . For complex points the same definitions apply to polynomials with complex coefficients.

Lemma 2.1. (a) Let $x_1, \dots, x_N \in \mathbb{C}^d$ be distinct and $\lambda_k \in \mathbb{C}$, and let $\sum_k \lambda_k f(x_k) = \int fW$ for all $f \in \Pi_{2n-2}$. Then $X = \{x_1, \dots, x_N\}$ is unisolvent for Π_{n-1} .

(b) If, in addition, the nodes and weights are real, then $\lambda_k = 1/K_{n-1}(x_k, x_k) > 0$ and $K_{n-1}(x_k, x_l) = 0$ for $k \neq l$.

(c) Conversely, if $x_1, \dots, x_N \in \mathbb{R}^d$ satisfy $K_{n-1}(x_k, x_l) = 0$ for $k \neq l$, then they are distinct, and with the weights $1/K_{n-1}(x_k, x_k)$ they form an n -rule.

Proof. (a) Let $p \in \Pi_{n-1}$ vanish on X , and let $\bar{p}$ be the polynomial with the complex conjugate coefficients. Since $p\bar{p} \in \Pi_{2n-2}$ and $p(x_k) = 0$, we get $\int |p|^2 W = \int p\bar{p}W = \sum_k \lambda_k p(x_k) \bar{p}(x_k) = 0$, so $p = 0$.

(b) By (a) there are Lagrange polynomials $\ell_k \in \Pi_{n-1}$ with $\ell_k(x_l) = \delta_{kl}$. As $\ell_k \ell_l \in \Pi_{2n-2}$, we have $\langle \ell_k, \ell_l \rangle = \sum_m \lambda_m \ell_k(x_m) \ell_l(x_m) = \lambda_k \delta_{kl}$. Hence $\lambda_k = \langle \ell_k, \ell_k \rangle > 0$, the polynomials $\ell_k / \sqrt{\lambda_k}$ form an orthonormal basis of Π_{n-1} , and $K_{n-1}(x, y) = \sum_k \ell_k(x) \ell_k(y) / \lambda_k$, which gives $K_{n-1}(x_k, x_l) = \delta_{kl} / \lambda_k$.

(c) The polynomials $K_{n-1}(\cdot, x_k)$ are pairwise orthogonal, since $\langle K_{n-1}(\cdot, x_k), K_{n-1}(\cdot, x_l) \rangle = K_{n-1}(x_k, x_l)$, and non-zero; so the points are distinct and these polynomials form a basis of Π_{n-1} . For $p = \sum_k c_k K_{n-1}(\cdot, x_k)$ we have $p(x_l) = c_l K_{n-1}(x_l, x_l)$. Hence for $p, q \in \Pi_{n-1}$, $\int pqW = \sum_k c_k c'_k K_{n-1}(x_k, x_k) = \sum_k p(x_k)q(x_k)/K_{n-1}(x_k, x_k)$, where $q = \sum_k c'_k K_{n-1}(\cdot, x_k)$. The products pq span Π_{2n-2} . $\square$

3 The quadratic system

We write $A_i = A_{n-1,i}$. For an $r_n \times r_{n-1}$ matrix Γ consider the conditions

$$A_i \Gamma \text{ is symmetric} \quad (1 \leq i \leq d), \quad (\text{S})$$

$$A_i(\Gamma \Gamma^\top - I)A_j^\top \text{ is symmetric} \quad (1 \leq i, j \leq d), \quad (\text{C})$$

and the $N \times N$ block matrices $J_i = J_i(\Gamma)$, with blocks indexed by $0 \leq k, l \leq n-1$, whose non-zero blocks are $(J_i)_{k,k+1} = A_{k,i}$ and $(J_i)_{k+1,k} = A_{k,i}^\top$ for $k \leq n-2$, and $(J_i)_{n-1,n-1} = -A_i \Gamma$. Using (S), condition (C) is equivalent to $\Gamma^\top (A_i^\top A_j - A_j^\top A_i) \Gamma = A_i A_j^\top - A_j A_i^\top$, since $A_i \Gamma \Gamma^\top A_j^\top = (A_i \Gamma)(A_j \Gamma)^\top = \Gamma^\top A_i^\top A_j \Gamma$; for $d = 2$ these are the equations of [Xu18, Theorem 12], which are taken there from [Xu94].

Lemma 3.1 (Xu). *(a) Let X be the node set of a rule with N distinct nodes in $\mathbb{C}^d$ and complex weights which is exact on Π_{2n-2} . Then there is a unique complex matrix Γ such that the entries of $\mathbb{P}_n + \Gamma \mathbb{P}_{n-1}$ form a basis of the space of polynomials in Π_n that vanish on X , and Γ satisfies (S) and (C). If the nodes and weights are real, then Γ is real.*

(b) Conversely, let Γ be real and satisfy (S) and (C). Then the entries of $\mathbb{P}_n + \Gamma \mathbb{P}_{n-1}$ have exactly N common zeros in $\mathbb{C}^d$. They are real, and they are the nodes of an n -rule whose weights are $1/K_{n-1}(x_k, x_k)$.

Proof. (a) By Lemma 2.1(a), X is unisolvent for Π_{n-1} . Let $\mathcal{I}$ be the space of polynomials in Π_n (with complex coefficients) that vanish on X . Evaluation on X maps Π_n onto $\mathbb{C}^X$, so $\dim \mathcal{I} = \dim \Pi_n - N = r_n$, and $\mathcal{I} \cap \Pi_{n-1} = 0$. For $q \in \mathcal{I}$ and $p \in \Pi_{n-2}$ we have $qp \in \Pi_{2n-2}$ and hence $\int qpW = \sum_k \lambda_k q(x_k)p(x_k) = 0$; so q lies in $V_n \oplus V_{n-1}$. The projection of $\mathcal{I}$ onto V_n is injective, because its kernel lies in $\mathcal{I} \cap \Pi_{n-1}$, and therefore bijective. This gives the unique Γ . For real nodes and weights everything can be done over $\mathbb{R}$.

For $p \in \Pi_{n-1}$ let $T_i p \in \Pi_{n-1}$ interpolate $x_i p$ on X . Write $\mathbb{P} = (\mathbb{P}_0^\top, \dots, \mathbb{P}_{n-1}^\top)^\top$. On X we have $\mathbb{P}_n = -\Gamma \mathbb{P}_{n-1}$, so (4) gives $x_i \mathbb{P} = J_i \mathbb{P}$ on X , and hence $T_i \mathbb{P} = J_i \mathbb{P}$ (applied entrywise). The bilinear form $\langle p, q \rangle = \int pqW$ is non-degenerate on Π_{n-1} , the entries of $\mathbb{P}$ are orthonormal for it, and

$$\langle T_i p, q \rangle = \sum_k \lambda_k x_{k,i} p(x_k) q(x_k) = \langle p, T_i q \rangle \quad (p, q \in \Pi_{n-1}),$$

because $(T_i p)q$ and $p(T_i q)$ lie in Π_{2n-2} and agree with $x_i pq$ on X . Since J_i is the matrix of T_i in the orthonormal basis $\mathbb{P}$, it is symmetric; its last diagonal block gives (S). Moreover $T_i T_j = T_j T_i$, since both interpolate $x_i x_j p$; hence $J_i J_j = J_j J_i$.

Given (S), the commutation $J_i J_j = J_j J_i$ is equivalent to (C). Indeed, every block of $J_i J_j$ in a position (k, l) with $k, l \leq n-2$ is a block of the product of the infinite block Jacobi matrices of the multiplication operators, and is symmetric in i, j by (5). The blocks in the positions $(n-2, n-1)$ and $(n-1, n-2)$ are $-A_{n-2,i} A_{n-1,j} \Gamma$ and $-A_{n-1,i} \Gamma A_{n-2,j}^\top = -\Gamma^\top (A_{n-2,j} A_{n-1,i})^\top$, which are symmetric in i, j by (5) and (S); those in the positions $(n-3, n-1)$ and $(n-1, n-3)$ are $A_{n-3,i} A_{n-2,j}$ and its analogue, also symmetric. The last diagonal block is

$$A_{n-2,i}^\top A_{n-2,j} + A_i \Gamma A_j \Gamma = A_{n-2,i}^\top A_{n-2,j} + A_i A_j^\top + A_i (\Gamma \Gamma^\top - I) A_j^\top,$$

using $A_j \Gamma = \Gamma^\top A_j^\top$. By the second identity of (5) with $k = n - 1$, this block is symmetric in i, j if and only if $A_i(\Gamma \Gamma^\top - I)A_j^\top = A_j(\Gamma \Gamma^\top - I)A_i^\top$, which is the transpose of the left-hand side; this is (C).

(b) Now $J_1, \dots, J_d$ are commuting real symmetric matrices, so $\mathbb{R}^N$ has an orthonormal basis $v_1, \dots, v_N$ of common eigenvectors, $J_i v_k = x_{k,i} v_k$ with $x_k \in \mathbb{R}^d$. Write $v = v_k$, $x = x_k$ and $v = (v^{(0)}, \dots, v^{(n-1)})$ in blocks. The block rows $m \leq n - 2$ of $J_i v = x_i v$ read $A_{m-1,i}^\top v^{(m-1)} + A_{m,i} v^{(m+1)} = x_i v^{(m)}$. If $v^{(l)} = v^{(0)} \mathbb{P}_l(x)$ for $l \leq m$, then by (4) $A_{m,i} v^{(m+1)} = v^{(0)} A_{m,i} \mathbb{P}_{m+1}(x)$ for all i , and the full column rank of the stacked matrices gives $v^{(m+1)} = v^{(0)} \mathbb{P}_{m+1}(x)$. So $v = v^{(0)} \mathbb{P}(x)$ with $v^{(0)} \neq 0$. The last block row and (4) for $k = n - 1$ give $A_{n-1,i}(\mathbb{P}_n(x) + \Gamma \mathbb{P}_{n-1}(x)) = 0$ for all i , hence $\mathbb{P}_n(x) + \Gamma \mathbb{P}_{n-1}(x) = 0$. The orthogonality of the v_k means $K_{n-1}(x_k, x_l) = 0$ for $k \neq l$, and Lemma 2.1(c) shows that the x_k are distinct and carry an n -rule with weights $1/K_{n-1}(x_k, x_k)$. Conversely, if $x \in \mathbb{C}^d$ is a common zero, then (4) gives $J_i \mathbb{P}(x) = x_i \mathbb{P}(x)$ with $\mathbb{P}(x) \neq 0$; writing $\mathbb{P}(x) = \sum_k c_k v_k$ we get $\sum_k c_k(x_{k,i} - x_i)v_k = 0$, so $x = x_k$ whenever $c_k \neq 0$. $\square$

Lemma 3.2. *A matrix Γ satisfies (S) if and only if there is a function $h : \Lambda_{2n-1} \rightarrow \mathbb{C}$ with $\Gamma(\alpha, \beta) = h(\alpha + \beta)$ for all $\alpha \in \Lambda_n$, $\beta \in \Lambda_{n-1}$. The function h is unique.*

Proof. By (4), $(A_i \Gamma)(\beta, \beta') = \frac{1}{2} \Gamma(\beta + e_i, \beta')$ for $\beta, \beta' \in \Lambda_{n-1}$, so (S) says $\Gamma(\beta + e_i, \beta') = \Gamma(\beta' + e_i, \beta)$ for all i and β, β' . This clearly holds if $\Gamma(\alpha, \beta) = h(\alpha + \beta)$. Conversely, let $\alpha \in \Lambda_n$, $\beta \in \Lambda_{n-1}$, $\alpha_i \geq 1$, $\beta_j \geq 1$, $i \neq j$. Applying (S) for i and then for j gives $\Gamma(\alpha, \beta) = \Gamma(\beta + e_i, \alpha - e_i) = \Gamma(\alpha - e_i + e_j, \beta + e_i - e_j)$. Fix $\gamma \in \Lambda_{2n-1}$; the set $D = \{\alpha \in \Lambda_n : \alpha \leq \gamma\}$ (componentwise) is non-empty, and $\Gamma(\alpha, \gamma - \alpha)$ does not change under the moves $\alpha \mapsto \alpha - e_i + e_j$ with $\alpha_i \geq 1$ and $\gamma_j - \alpha_j \geq 1$, which stay in D . These moves connect D : if $\alpha \neq \alpha'$ lie in D , choose i with $\alpha_i > \alpha'_i$ and j with $\alpha_j < \alpha'_j \leq \gamma_j$; the move decreases $\sum_l |\alpha_l - \alpha'_l|$. So $\Gamma(\alpha, \gamma - \alpha)$ depends only on γ . $\square$

For $h : \Lambda_{2n-1} \rightarrow \mathbb{C}$ let $\Gamma_h(\alpha, \beta) = h(\alpha + \beta)$ and $G_h = \Gamma_h \Gamma_h^\top - I$, so that $G_h(\alpha, \alpha') = \sum_{\beta \in \Lambda_{n-1}} h(\alpha + \beta)h(\alpha' + \beta) - \delta_{\alpha\alpha'}$ is the left-hand side of $(\star)$. For $i \neq j$ and $b, b' \in \Lambda_{n-1}$ put

$$E_{ij}(b, b') = G_h(b + e_i, b' + e_j) - G_h(b' + e_i, b + e_j). \quad (6)$$

Explicitly, $E_{ij}(b, b') = \sum_{\beta} (h(b + \beta + e_i)h(b' + \beta + e_j) - h(b' + \beta + e_i)h(b + \beta + e_j)) - [b' = b + e_i - e_j] + [b' = b + e_j - e_i]$, a quadratic polynomial in the values of h with integer coefficients, and $E_{ij}(b', b) = -E_{ij}(b, b') = E_{ji}(b, b')$.

Lemma 3.3. *For $\Gamma = \Gamma_h$, condition (C) holds if and only if all $E_{ij}(b, b')$ vanish, and if and only if h satisfies $(\star)$.*

Proof. By (4), $(A_i G A_j^\top)(b, b') = \frac{1}{4} G(b + e_i, b' + e_j)$ for every matrix G , so (C) for $G = G_h$ is the vanishing of all $E_{ij}(b, b')$. If $(\star)$ holds, they vanish, since $(b + e_i) + (b' + e_j) = (b' + e_i) + (b + e_j)$. Conversely, let them vanish. Then $G_h(\alpha, \alpha') = G_h(\alpha - e_i + e_j, \alpha' + e_i - e_j)$ whenever $i \neq j$, $\alpha_i \geq 1$ and $\alpha'_j \geq 1$ (take $b = \alpha - e_i$, $b' = \alpha' - e_j$ and use the symmetry of G_h). For $\sigma \in \Lambda_{2n}$ the set $\{\alpha \in \Lambda_n : \alpha \leq \sigma\}$ is connected under these moves, by the argument in the proof of Lemma 3.2; so $G_h(\alpha, \sigma - \alpha)$ depends only on σ . $\square$

Lemma 3.4. *The function $h = 0$ does not satisfy $(\star)$.*

Proof. For $h = 0$ the left-hand side of $(\star)$ is $-\delta_{\alpha\alpha'}$. Let $\alpha'' = e_1 + (n - 1)e_2 \in \Lambda_n$; then $2\alpha'' = \alpha'' + \alpha'' = (2e_1 + (n - 2)e_2) + ne_2$, with value -1 for the first decomposition and 0 for the second. $\square$

Proof of Theorem 1.2. (a) Let X be the node set of an n -rule. By Lemma 3.1(a) and Lemmas 3.2 and 3.3, its matrix Γ equals Γ_h for a real solution h of $(\star)$, the q_α of (2) are the entries of $\mathbb{P}_n + \Gamma \mathbb{P}_{n-1}$, and they vanish on X . Conversely, for a real solution h the matrix Γ_h satisfies (S) and (C), and

Lemma 3.1(b) shows that the common zeros of the q_α are N real points carrying an n -rule. The two maps are inverse to each other: the n -rule built from h has the matrix Γ_h by the uniqueness in Lemma 3.1(a), and the node set X of an n -rule is contained in, hence by Lemma 3.1(b) equal to, the set of the N common zeros of its q_α . The weights are given by Lemma 2.1(b).

(b) If an n -rule were exact on Π_{2n-1} , then every $q \in \mathcal{I}$ would be orthogonal to Π_{n-1} , so $\Gamma = 0$ and $h = 0$, contrary to Lemma 3.4.

(c) This is Lemma 3.1(a) together with Lemmas 3.2 and 3.3. $\square$

Remark 3.5 (Two variables). For $d = 2$ and $h = \delta_{(n,n-1)}$ we have $\sum_\beta h(\alpha + \beta)h(\alpha' + \beta) = [\alpha = \alpha'] [\alpha \leq (n, n-1)]$, and among $\alpha \in \Lambda_n$ only $(0, n)$ violates $\alpha \leq (n, n-1)$. So the left-hand side of $(\star)$ is $-[\alpha = \alpha' = (0, n)]$, which depends only on $\alpha + \alpha'$ because $(0, 2n)$ has the single decomposition $(0, n) + (0, n)$. The polynomials (2) are $q_{(0,n)} = U_n(y)$ and $q_{(a,n-a)} = U_a(x)U_{n-a}(y) + U_{n-a}(x)U_{a-1}(y)$ for $1 \leq a \leq n$. With $x = \cos \theta$, $\theta = i\pi/(n+2)$, $y = \cos \varphi$, $\varphi = j\pi/(n+1)$ and $U_k(\cos t) = \sin((k+1)t)/\sin t$, the numerator of $q_{(a,n-a)}(x, y)$ is $-((-1)^i + (-1)^j) \sin((a+1)\theta) \sin(a\varphi)$. Hence the $n(n+1)/2$ points $(\cos \frac{i\pi}{n+2}, \cos \frac{j\pi}{n+1})$ with $1 \leq i \leq n+1$, $1 \leq j \leq n$ and $i+j$ odd are common zeros, and by Theorem 1.2(a) they are the nodes of the n -rule belonging to h . This is the rule of Morrow and Patterson [MP78], with the nodes as in [Xu18, Theorem 5].¹ Since $x \mapsto -x$ maps i to $n+2-i$ and $y \mapsto -y$ maps j to $n+1-j$, the node set is invariant under $y \mapsto -y$ for odd n and under $x \mapsto -x$ for even n , but not under both; compare Lemma 4.1. The function $\delta_{(n-1,n)}$ gives the mirror image.

Remark 3.6 (Size of the system). For $d = 3$ there are $\binom{2n+1}{2}$ unknowns, and $(\star)$ consists of $\binom{r_n+1}{2} - |\Lambda_{2n}|$ conditions: 21 unknowns and 27 conditions for $n = 3$, 36 and 75 for $n = 4$, 55 and 165 for $n = 5$. For $n = 4$ the $3 \cdot \binom{10}{2} = 135$ equations $E_{ij}(b, b')$ with $i < j$ and b before b' in a fixed order contain 117 distinct ones up to sign, and they span the same space of dimension 75 as the conditions of $(\star)$ (verified by the program of Section 8).

4 Sign symmetries

For $\varepsilon \in \{\pm 1\}^d$ and $\gamma \in \mathbb{N}_0^d$ put $\chi_\varepsilon(\gamma) = \prod_i \varepsilon_i^{\gamma_i}$. By (3), $P_\alpha \circ D_\varepsilon = \chi_\varepsilon(\alpha)P_\alpha$.

Lemma 4.1. *Let $X \subset \mathbb{C}^d$ be the node set of a rule as in Theorem 1.2(c), and let h be the corresponding solution of $(\star)$. If $D_\varepsilon(X) = X$, then $h(\gamma) = 0$ for every $\gamma \in \Lambda_{2n-1}$ with $\chi_\varepsilon(\gamma) = -1$.*

Proof. Let $\mathcal{I}$ be the space of polynomials in Π_n that vanish on X . Since $D_\varepsilon(X) = X$, $q \in \mathcal{I}$ implies $q \circ D_\varepsilon \in \mathcal{I}$, so $\mathcal{I}$ contains

$$q_\alpha \circ D_\varepsilon = \chi_\varepsilon(\alpha)P_\alpha + \sum_{\beta \in \Lambda_{n-1}} \chi_\varepsilon(\beta)h(\alpha + \beta)P_\beta.$$

Its component in V_n is $\chi_\varepsilon(\alpha)P_\alpha$, and by the uniqueness in Lemma 3.1(a) the only element of $\mathcal{I}$ with this component is $\chi_\varepsilon(\alpha)q_\alpha$. Comparing the coefficients of P_β gives $\chi_\varepsilon(\beta)h(\alpha + \beta) = \chi_\varepsilon(\alpha)h(\alpha + \beta)$. As $\chi_\varepsilon(\alpha)\chi_\varepsilon(\beta) = \chi_\varepsilon(\alpha + \beta)$, this says $h(\alpha + \beta) = 0$ whenever $\chi_\varepsilon(\alpha + \beta) = -1$, and every $\gamma \in \Lambda_{2n-1}$ is such a sum. $\square$

Conversely, if a real solution h vanishes wherever $\chi_\varepsilon = -1$, then $q_\alpha \circ D_\varepsilon = \chi_\varepsilon(\alpha)q_\alpha$, and the node set of Theorem 1.2(a) is invariant under D_ε . In two variables the support $\{(n, n-1)\}$ of the Morrow–Patterson solution explains the symmetries found in Remark 3.5.

¹There the polynomials are printed as $U_{n-k}(x)U_k(y) - U_k(x)U_{n-1-k}(y)$. With the nodes printed there the sign must be +: for $n = 2$ the node $(0, \frac{1}{2})$ is a zero of $U_2(x) + U_1(y)$, not of $U_2(x) - U_1(y)$.

Proposition 4.2. *For $d \geq 2$ and $n \geq 2$, the node set of a rule as in Theorem 1.2(c), in particular of a Gaussian cubature rule of degree $2n - 2$ for W , is not invariant under $x \mapsto -x$.*

Proof. For $\varepsilon = (-1, \dots, -1)$ we have $\chi_\varepsilon(\gamma) = (-1)^{2n-1} = -1$ for all $\gamma \in \Lambda_{2n-1}$, so Lemma 4.1 gives $h = 0$, which is impossible by Lemma 3.4. $\square$

The source notes that Gaussian rules do not exist when the weight and the domain are centrally symmetric [Xu22, p. 416]; in context this concerns the degree $2n - 1$, and for our weight it is Theorem 1.2(b). Proposition 4.2 is a statement of a different kind: the weight is centrally symmetric, Gaussian rules of degree $2n - 2$ may exist, but their node sets are not.

Reduction for $d = 3$. Since W is invariant under permutations of the coordinates, the image of the node set of a rule as in Theorem 1.2(c) under such a permutation is again such a node set. Hence it suffices to treat the reflection $(x, y, z) \mapsto (x, y, -z)$, the half-turn $(x, y, z) \mapsto (x, -y, -z)$, and the pair of reflections $y \mapsto -y, z \mapsto -z$. For $n = 4$ every $\gamma \in \Lambda_7$ has an odd coordinate sum, so $\gamma \bmod 2$ is one of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$; these four classes have 10, 10, 10 and 6 elements. By Lemma 4.1, invariance under $z \mapsto -z$ leaves the 20 unknowns of the classes $(1, 0, 0)$ and $(0, 1, 0)$; invariance under the half-turn leaves the 16 unknowns of the classes $(1, 0, 0)$ and $(1, 1, 1)$, where $\gamma_2 + \gamma_3$ is even; and invariance under $y \mapsto -y$ and $z \mapsto -z$ leaves the 10 unknowns of the class $(1, 0, 0)$. We call the system obtained from the equations $E_{ij}(b, b')$ by setting all other values of h to 0 the *restricted system* of the symmetry.

For a permutation π of the coordinates the argument of Lemma 4.1 gives $h(\pi\gamma) = h(\gamma)$ instead of a condition on the support; we do not pursue these symmetries.

5 Ten nodes: the case $n = 3$

Let $d = 3$ and write (x, y, z) for the coordinates. Put

$$\begin{aligned} c_1 &= \frac{\sqrt{6}-\sqrt{2}}{4} = \cos \frac{5\pi}{12}, & c_2 &= -\frac{\sqrt{6}+\sqrt{2}}{4} = \cos \frac{11\pi}{12}, & \lambda_{1,2} &= \frac{3 \pm \sqrt{3}}{18}, \\ a &= \frac{\sqrt{3}}{2}, & c_3 &= \frac{\sqrt{2}}{2}, & b &= \frac{\sqrt{6}}{4}, & c_4 &= -\frac{\sqrt{2}}{4}. \end{aligned}$$

Proposition 5.1. *The rule with the nodes $(0, 0, c_1)$ and $(0, 0, c_2)$, with weights λ_1 and λ_2 , the four nodes $(\pm a, 0, c_3)$, $(0, \pm a, c_3)$ with weight $\frac{1}{18}$ each, and the four nodes $(\pm b, \pm b, c_4)$ with weight $\frac{1}{9}$ each, is a Gaussian cubature rule of degree 4 for W on $[-1, 1]^3$. All nodes lie in the open cube, and $\lambda_k = 1/K_2(x_k, x_k)$ for every node.*

Proof. The rule and W are invariant under the group D_4 of order 8 generated by $(x, y, z) \mapsto (-y, x, z)$ and $(x, y, z) \mapsto (y, x, z)$. For $f \in \Pi_4$ let $\bar{f}$ be the average of $f \circ g$ over $g \in D_4$; then f and $\bar{f}$ have the same integral and the same value under the rule. The group contains $(x, y, z) \mapsto (-x, y, z)$, so a D_4 -invariant polynomial is even in x and in y and symmetric in x, y ; those of degree at most 4 are spanned by the ten polynomials in the first column of Table 1. Their integrals follow from $\int t^2 w = \frac{1}{4}$ and $\int t^4 w = \frac{1}{8}$ (in general $\int t^{2k} w = \binom{2k}{k} / (4^k (k+1))$). The table also lists the contributions of the two nodes on the z -axis, of the four nodes $(\pm a, 0, c_3)$, $(0, \pm a, c_3)$, on which $x^2 + y^2 = \frac{3}{4}$ and $x^2 y^2 = 0$, and of the four nodes $(\pm b, \pm b, c_4)$, on which $x^2 + y^2 = \frac{3}{4}$ and $x^2 y^2 = \frac{9}{64}$. The contributions of the axis nodes are $M_k = \lambda_1 c_1^k + \lambda_2 c_2^k$, which equal $\frac{1}{3}, 0, \frac{1}{12}, -\frac{\sqrt{2}}{24}, \frac{1}{16}$ for $k = 0, \dots, 4$ (from $c_1^2 = \frac{2-\sqrt{3}}{4}$, $c_2^2 = \frac{2+\sqrt{3}}{4}$). In every row the three contributions add up to the integral, so the rule is exact on Π_4 . It is not exact for z^5 , whose integral is 0, while the rule gives $-\sqrt{2}/64$. Hence its degree is 4; it has $10 = \dim \Pi_2^3$ nodes. All coordinates have absolute value at most $|c_2| = \cos \frac{\pi}{12} < 1$, and the weights are positive, as they must be by Lemma 2.1(b), which also gives $\lambda_k = 1/K_2(x_k, x_k)$. $\square$

f	$\int fW$	axis nodes	nodes $(\pm a, 0, c_3), (0, \pm a, c_3)$	nodes $(\pm b, \pm b, c_4)$
1	1	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{4}{9}$
z	0	0	$\frac{\sqrt{2}}{9}$	$-\frac{\sqrt{2}}{9}$
z^2	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{1}{18}$
z^3	0	$-\frac{\sqrt{2}}{24}$	$\frac{\sqrt{2}}{18}$	$-\frac{\sqrt{2}}{72}$
z^4	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{18}$	$\frac{1}{144}$
$x^2 + y^2$	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{3}$
$(x^2 + y^2)z$	0	0	$\frac{\sqrt{2}}{12}$	$-\frac{\sqrt{2}}{12}$
$(x^2 + y^2)z^2$	$\frac{1}{8}$	0	$\frac{1}{12}$	$\frac{1}{24}$
$(x^2 + y^2)^2$	$\frac{3}{8}$	0	$\frac{1}{8}$	$\frac{1}{4}$
x^2y^2	$\frac{1}{16}$	0	0	$\frac{1}{16}$

Table 1: The ten D_4 -invariant polynomials of degree at most 4: their integrals against W and the contributions of the three groups of nodes of the rule of Proposition 5.1.

In the notation of Theorem 1.2, the rule of Proposition 5.1 belongs to the solution

$$h^* = \frac{1}{\sqrt{2}}(\delta_{(0,0,5)} + \delta_{(2,2,1)} - \delta_{(4,0,1)} - \delta_{(2,0,3)} - \delta_{(0,4,1)} - \delta_{(0,2,3)})$$

of $(\star)$ for $n = 3$: the ten polynomials q_α of (2) vanish at the ten nodes, as the program of Section 8 checks in exact arithmetic. Its support consists of the $\gamma \in \Lambda_5$ with γ_1 and γ_2 even, in accordance with Lemma 4.1, since the node set is invariant under $x \mapsto -x$ and $y \mapsto -y$. The rule was found by imposing the symmetry group D_4 with orbits of sizes 1, 1, 4, 4; within this ansatz the rule is unique up to the reflection $z \mapsto -z$. Indeed, an orbit of size four lies either on the coordinate axes, as $(\pm r, 0, t)$, $(0, \pm r, t)$, or on the diagonals, as $(\pm r, \pm r, t)$. Two orbits on the axes are impossible, since x^2y^2 would vanish at all nodes; for two orbits on the diagonals, with nodes $(\pm r_j, \pm r_j, t_j)$ and weights μ_j ($j = 1, 2$), the moments of x^2y^2 and of $(x^2 + y^2)^2$ would give both $\sum_j \mu_j r_j^4 = \frac{1}{64}$ and $\sum_j \mu_j r_j^4 = \frac{3}{128}$. In the remaining case a short computation with **sympy** in the package gives the rule and its mirror image. In the class of node sets invariant under $x \mapsto -x$ and $y \mapsto -y$ the restricted system for $n = 3$ (six unknowns) has exactly the two solutions $\pm h^*$, by Gröbner basis computations of the verification runs; $-h^*$ gives the mirror image under $z \mapsto -z$. So the symmetry that is excluded for $n = 4$ by Theorem 6.1 does occur for $n = 3$.

Remark 5.2 (The 3-rules form a family). Proposition 5.1 is not an isolated rule. A verification run solved $(\star)$ for $n = 3$ by the Levenberg–Marquardt method from 40 random starts; every start converged to a solution (residual below 10^{-10}), at each of them the Jacobian of the system had a kernel of dimension 5, and none of the 40 solutions vanishes on the odd class of any coordinate reflection or half-turn, that is, none of the corresponding node sets has such a symmetry. This suggests a five-dimensional family of 3-rules whose generic member has no symmetry; the count $10 \cdot (3 + 1) - \dim \Pi_4^3 = 40 - 35 = 5$ of unknowns (nodes and weights) minus moment equations gives the same number. Node-space searches also found 3-rules with a node outside the cube. These observations are numerical and are not used anywhere. For $n = 4$ and $n = 5$ the same count gives $80 - 84 = -4$ and $140 - 165 = -25$, which is the heuristic reason to expect that n -rules for $n \geq 4$ are rare or absent.

6 Twenty nodes: the case $n = 4$

Let $d = 3$ and $n = 4$. A 4-rule has 20 nodes and is exact on Π_6^3 ; we allow complex nodes and weights as in Theorem 1.2(c). We write $h_{abc} = h(a, b, c)$ and $G = G_h$.

Theorem 6.1. *No 4-rule for W has a node set invariant under two of the reflections $x \mapsto -x$, $y \mapsto -y$, $z \mapsto -z$ in the coordinate planes.*

Proof. By the reduction in Section 4 we may take the reflections in y and z ; by Lemma 4.1 and Theorem 1.2(c) we obtain a solution h of $(\star)$ with $h(\gamma) = 0$ unless γ_2 and γ_3 are even (and then γ_1 is odd). For $\alpha, \alpha' \in \Lambda_4$ a term $h(\alpha + \beta)h(\alpha' + \beta)$ of $G(\alpha, \alpha')$ can be non-zero only if $\beta \equiv (1, 0, 0) - \alpha \equiv (1, 0, 0) - \alpha' \pmod{2}$. The following table gives, for the α we need, the $\beta \in \Lambda_3$ of the right parity and the vector $(h(\alpha + \beta))_\beta$, where $a = h_{124}$, $b = h_{142}$ and $c = h_{322}$:

β	α and $(h(\alpha + \beta))_\beta$
$(1, 1, 1)$	$(0, 1, 3): a, (0, 3, 1): b, (2, 1, 1): c$
$(1, 0, 2), (1, 2, 0), (3, 0, 0)$	$(0, 2, 2): (a, b, c), (0, 0, 4): (h_{106}, a, h_{304}),$ $(0, 4, 0): (b, h_{160}, h_{340}), (2, 0, 2): (h_{304}, c, h_{502})$
$(0, 1, 2), (0, 3, 0), (2, 1, 0)$	$(1, 1, 2): (a, b, c), (1, 3, 0): (b, h_{160}, h_{340})$
$(0, 0, 3), (0, 2, 1), (2, 0, 1)$	$(1, 2, 1): (a, b, c), (1, 0, 3): (h_{106}, a, h_{304}),$ $(3, 0, 1): (h_{304}, c, h_{502})$

For two α in the same row, $G(\alpha, \alpha')$ is the scalar product of their vectors minus $\delta_{\alpha\alpha'}$. Put $s = a^2 + b^2 + c^2 - 1$, and write $G(013, 031)$ for $G((0, 1, 3), (0, 3, 1))$, and so on. Since $(0, 1, 3) + (0, 3, 1) = (0, 2, 2) + (0, 2, 2)$, $(0, 1, 3) + (2, 1, 1) = (1, 1, 2) + (1, 1, 2)$ and $(0, 3, 1) + (2, 1, 1) = (1, 2, 1) + (1, 2, 1)$, condition $(\star)$ gives

$$\begin{aligned} ab = G(013, 031) = G(022, 022) = s, \quad ac = G(013, 211) = G(112, 112) = s, \\ bc = G(031, 211) = G(121, 121) = s. \end{aligned}$$

Next,

$$\begin{aligned} a^2 - 1 = G(013, 013) = G(004, 022) = h_{106}a + ab + h_{304}c = G(103, 121) = s, \\ b^2 - 1 = G(031, 031) = G(022, 040) = ab + bh_{160} + ch_{340} = G(112, 130) = s, \\ c^2 - 1 = G(211, 211) = G(121, 301) = ah_{304} + bc + ch_{502} = G(022, 202) = s. \end{aligned}$$

In each line the second and the last equality come from $(\star)$, for the sums $(0, 2, 6)$ and $(2, 2, 4)$, $(0, 6, 2)$ and $(2, 4, 2)$, $(4, 2, 2)$ and $(2, 2, 4)$ respectively, together with $G(112, 112) = G(121, 121) = s$; the other equalities are read off from the table. Hence $a^2 = b^2 = c^2 = 1 + s$, so $s = a^2 + b^2 + c^2 - 1 = 3s + 2$ and $s = -1$. Then $a = b = c = 0$, and $s = ab = 0$, a contradiction. $\square$

Theorem 6.1 is also implied by each of the next two theorems, whose proofs need a computer.

Theorem 6.2. *No 4-rule for W has a node set invariant under a reflection in a coordinate plane.*

Theorem 6.3. *No 4-rule for W has a node set invariant under a half-turn about a coordinate axis.*

Proofs (computer-assisted). By the reduction in Section 4 we may take the reflection $z \mapsto -z$, with the 20 unknowns $h(\gamma)$, $\gamma \equiv (1, 0, 0)$ or $(0, 1, 0)$, and the half-turn $(x, y, z) \mapsto (x, -y, -z)$, with the 16 unknowns $h(\gamma)$, $\gamma \equiv (1, 0, 0)$ or $(1, 1, 1)$. By Lemmas 3.3 and 4.1 and Theorem 1.2(c) it suffices to show that the restricted system has no complex solution. In both cases the 135 equations (6) restrict

to 63 quadratic polynomials E_k with integer coefficients that are distinct up to sign; they span a space of dimension 41. We give polynomials q_k with rational coefficients and degree at most 2 such that

$$\sum_k q_k E_k = 1 \quad (7)$$

identically. A common zero of the E_k in $\mathbb{C}^{20}$, resp. $\mathbb{C}^{16}$, would make the left-hand side vanish. For the reflection, all 63 multipliers are non-zero; they have 1575 terms in all, with numerators up to 6623 and denominators up to 144. For the half-turn, 60 multipliers are non-zero, with 394 terms, numerators up to 89 and denominators up to 39. The multipliers are listed in two files of the package; expanding the identities is a finite computation in exact rational arithmetic (Section 8). $\square$

The identities were found numerically, by least squares for the coefficients of the q_k (a linear system with the products mE_k , m a monomial of degree at most 2, as columns), followed by an exact solution over $\mathbb{Q}$ on a smaller support; the final verification is exact, so the floating-point stage cannot produce a false identity. The same method gives for Theorem 6.1 an identity with 11 multipliers with integer coefficients of absolute value at most 8 and 151 terms; it uses only equations $E_{12}(b, b')$. Each identity was expanded by the programs of the three verification runs, written independently of the first computation, and by the program of this note; each of them compared the stored equations with equations generated by its own code (Section 8). Independently of the identities, the reduced Gröbner bases over $\mathbb{Q}$ of the restricted systems of Theorem 6.1 and Theorem 6.3 are $\{1\}$ (computed with `sympy` in the verification runs). For the reflection $z \mapsto -z$ such a computation over $\mathbb{Q}$ did not finish within the time limits of the first two runs; modulo the prime 32003 the Gröbner basis is $\{1\}$, which is consistent but proves nothing over $\mathbb{Q}$, so Theorem 6.2 rests on its identity.

Corollary 6.4. *Let $X \subset \mathbb{C}^3$ be the node set of a 4-rule for W , for example of a Gaussian cubature rule of degree 6. Then $D_\varepsilon(X) \neq X$ for every $\varepsilon \in \{\pm 1\}^3$ other than $(1, 1, 1)$.*

Proof. D_ε is a reflection in a coordinate plane if ε has one entry -1 (Theorem 6.2), a half-turn about a coordinate axis if it has two (Theorem 6.3), and $x \mapsto -x$ if it has three (Proposition 4.2). $\square$

This proves Theorem 1.4. For $n = 3$ the corresponding statement is false (Section 5). For the symmetries not covered by Corollary 6.4 see the end of Section 1; a verification run reports that the analogous reduced systems for node sets invariant under the cyclic permutation $(x, y, z) \mapsto (y, z, x)$ (twelve unknowns) and under all coordinate permutations (eight unknowns) also have the Gröbner basis $\{1\}$, but we have not produced certificates for these cases and do not claim them.

7 Numerical tests

Nothing in this section is a proof. We report the searches made for the open cases, because their outcome is easy to misread.

The case $n = 4$ without symmetry. Levenberg–Marquardt searches were run on (S) and (C) (36 free parameters; 200 random starts), on $(\star)$ in the form of the 75 conditions (145 starts), and directly on the nodes and weights (80 unknowns, 84 moment equations; 56 starts); the last two searches were made by the second verification run. None found a rule. Small residuals do occur ($4.8 \cdot 10^{-4}$, $1.9 \cdot 10^{-5}$ and $6 \cdot 10^{-6}$ were the best values of the three searches), but they are not near-solutions: along the iterations $|h|$ grows without bound while the product of the residual and $|h|$ stays constant (between 2.6 and 3.0 in six runs in which $|h|$ grew from below 1100 to above $4 \cdot 10^4$, so the residual is about $2.7/|h|$); continuing the best run of the second search drove the residual to $1.2 \cdot 10^{-6}$ while the largest value of $|h|$ grew to $1.1 \cdot 10^7$, and the best point of the node search had a node at distance 722 from

the origin with weight 10^{-19} . The mechanism is visible in $(\star)$: the unknown h_{070} enters $G(\alpha, \alpha')$ only for α or α' equal to $(0, 4, 0)$, and its square only in $G((0, 4, 0), (0, 4, 0))$, whose index sum $(0, 8, 0)$ has a single decomposition and is therefore unconstrained; so a large value of h_{070} can be balanced by small values of other unknowns. In the limit the polynomial $U_3(y) = P_{(0,3,0)}$ enters the ideal of the nodes and one node escapes to infinity with vanishing weight, which leaves 19 finite nodes. Searches with the norm of h bounded find no solution: the minimum of the residual over the sphere $|h| = R$ (30 starts for each R) decreases from 2.88 at $R = 0.5$ to 0.42 at $R = 5$ and 0.065 at $R = 40$, roughly like $2.6/R$, and box-constrained searches with $|h_\gamma| \leq 2, 5, 20$ end at residuals 0.54, 0.31, 0.092 on the boundary of the box. A least-squares computation indicates that the full system has no identity of the form (7) with multipliers of degree 2 (residual 0.71).

The case $n = 5$. For the 55 unknowns and 165 conditions of $(\star)$, a verification run minimised the residual over the spheres $|h| = R$: the minimum decreases from 2.03 at $R = 1$ to 0.14 at $R = 40$, again like c/R , and no real solution with $|h| \leq 40$ was found. For the classes of the two-reflection, half-turn and one-reflection symmetries the minimum stays above 1.16 for $1 \leq R \leq 20$. An earlier exploratory node-space search with 117 starts found no rule either (its log is not part of the package). For $n \geq 6$ no computation was made.

The weight of the first kind. For the printed weight $1/(\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2})$, the analogue of Lemma 3.1 holds, with the recurrence coefficients of the Chebyshev polynomials of the first kind, but the solutions of (S) are no longer of Hankel form, so $(\star)$ and our theorems do not apply. A search with 400 starts on (S) and (C) for $n = 4$ found no rule (best residual 0.32). In two variables, the rules of degree $2n - 2$ with $\dim \Pi_{n-1}^2$ nodes found numerically for $n = 3, 4, 5$ and this weight all have a node outside the square (largest coordinates 1.13, 1.07, 1.05 in one search; none of 56 rules found in another search had all nodes inside). For $n = 3$ in three variables such searches find 3-rules for both weights.

Lattice candidates. Subsets of index six, defined by congruences, of the grids $\{\cos \frac{i\pi}{n+1}\} \times \{\cos \frac{j\pi}{n+2}\} \times \{\cos \frac{k\pi}{n+3}\}$ (analogues of the nodes of Remark 3.5) have $\binom{n+2}{3}$ points, but for $n = 3, 4, 5$ none of those tried carries a rule of degree $2n - 2$, whatever the weights (best moment residuals from $5 \cdot 10^{-3}$ to 0.15).

8 Computations and reproducibility

Theorem 1.2, Propositions 4.2 and 5.1 and Theorem 6.1 are proved without computation; Theorems 6.2 and 6.3 rest on the identities (7), which are finite computations in exact rational arithmetic. The reproducibility package has five parts. The program of this note, `check_note.py` (standard library only, about one second), checks in exact arithmetic: Proposition 5.1 (the 35 moments of degree at most 4 in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, the 6 monomials of degree 5 that fail, the signs, Table 1, the weights $1/K_2(x_k, x_k)$, and that h^* solves $(\star)$ and its q_α vanish at the nodes); Lemmas 3.2 and 3.3 as statements of linear algebra for $(d, n) = (2, 4), (3, 3), (3, 4)$, and the 135 equations for $n = 4$, generated by two routes, with the counts of Remark 3.6; the values of G used in the proof of Theorem 6.1; the three identities (7), expanded with equations generated by the program, after the supports have been computed from Lemma 4.1 and every stored equation has been compared with a generated one; and Remark 3.5 for $n \leq 8$, exactly for $(\star)$ and in floating point for the nodes. Negative controls (a changed weight, a changed coefficient of a multiplier, a changed equation, a changed sign of h^* , a wrong solution in two variables and a wrong support) are rejected. All 27 checks pass. The other parts are the programs with which the results were first obtained, with their recorded outputs and the three certificate files; the programs and outputs of the first two verification runs; a log of re-runs made when this note was written, in which the recorded outputs of the quick programs were reproduced; and the programs and outputs of the third verification run, with its re-run of the package.

Verification. The results were first obtained, with the programs of the package, in a first written version. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the first written version, and neither saw the report of the other; the third examined the final text.

The first run re-derived the lemmas of Sections 2 to 4 line by line; regenerated the 135 equations with its own generator, built from the moments of w and the matrices $A_{k,i}$, and found them identical to those of the first computation; expanded the three identities (7) against its own equations; computed the Gröbner bases mentioned in Section 6; verified Proposition 5.1 in its own exact arithmetic and the rules of Remark 3.5 for $n \leq 7$ with 50 digits; and made the computations of Remark 5.2 and the searches on spheres of Section 7. It supplied the argument of Proposition 4.2, which replaced an appeal to Möller’s bound. The second run generated the system by two routes of its own; verified the identities with its own parser and exact arithmetic, and that a changed coefficient is detected; computed Gröbner bases; verified Proposition 5.1 and reconstructed its rule from h^* through the matrices J_i ; made the node-space searches and the analysis of the escape to infinity of Section 7; reproduced the recorded outputs of the original programs byte for byte; and checked the literature and the bibliographic data. It gave a second argument for the central symmetry, through (C) with $\Gamma = 0$. Both runs supplied the justification for complex nodes and corrected the statement on the symmetries in Remark 3.5. Neither run found an error in a lemma, proposition or theorem. They required nineteen corrections, all of which were made. The mathematical ones are the argument for the central symmetry, the justification for complex nodes, the symmetries in Remark 3.5 and the description of the symmetries that are not covered; the others concern the description of the identities (rational coefficients, sizes), the status of the cases $n \geq 5$ and of the weight of the first kind, the reading of small residuals, the remark on the family of 3-rules, and citations. When this note was written, the direct proof of Theorem 6.1 and the proof of Proposition 5.1 by the ten invariants were found, and `check_note.py` was written; it checks the statements as printed.

The third run examined the final text as if nothing had been checked, and these two new proofs first. It read the source, Theorems 2, 5 and 12 of [Xu18] with the printed nodes and polynomials of Theorem 5, the end of Section 4 of [SX23] and the report [CMSX23] again, re-derived every proof line by line, and wrote its own programs before reading those of the package. They build the system from the definitions: the matrices $A_{k,i}$ as integrals of products of the orthonormal polynomials, the solutions of (S) for a general matrix Γ , and all blocks of the commutators of the matrices $J_i(\Gamma_h)$ with symbolic h , which vanish except in the last diagonal block, where they give the equations (6); for $n = 4$ they confirm the counts of Remark 3.6 and of Section 6. With these equations and with the multipliers read by its own parser, the three identities (7) expand to 1 exactly; the stored equations coincide with its own, and the sizes of the multipliers are as stated. It also checked in exact arithmetic the fifteen values of G in the proof of Theorem 6.1, and that the six relations derived there have the Gröbner basis $\{1\}$; Proposition 5.1 and Table 1 in an exact field of its own; the dimension 10 of the space of D_4 -invariant polynomials of degree at most 4; the solutions $\pm h^*$ and the uniqueness of the D_4 ansatz for all placements of the orbits; and Remark 3.5 with its footnote for $n \leq 8$ with 50 digits. It recomputed the Gröbner bases $\{1\}$ for Theorem 6.1 and Theorem 6.3. It re-ran the package from the archive: the program of this note, the quick programs of the first computation and of the first two runs, and one of the searches of Section 7 reproduced their recorded outputs, and the other numbers of Section 7 were compared with the recorded logs. It repeated the literature search. It found no error in a lemma, proposition or theorem and no gap in the proofs. It required five corrections, all made: three sentences of the abstract (the symmetries that are not covered, the range of the numerical searches and the claim of novelty); the justification of the uniqueness of the D_4 ansatz in Section 5; the range of radii for $n = 5$ in Section 7; the account of the first two runs in this paragraph; and the record of this run in the note and in the package. Its programs and outputs are in `independent_run_2/`.

Scope and priority. The note does not answer Problem 1.1: for $n = 4$ it excludes only node sets with a symmetry in $\mathbb{Z}_2^3$, and for $n \geq 5$ it proves nothing; the numerical results of Section 7 are evidence only. The question and the conventions are Xu’s [Xu22]. The theory of Gaussian cubature of even degree on which Theorem 1.2 rests is due to Xu [Xu94] and, in two variables, to Morrow and Patterson [MP78] and Schmid [Sch78]; the rules of Remark 3.5 are those of Morrow and Patterson; the improved lower bound for odd degree quoted in the source is Möller’s [Mö76]; cubature rules on the cube of other kinds are due to Li, Sun and Xu [LSX09] and to De Marchi, Vianello and Xu [DVX09]; Hankel structure in characterisations of Gaussian cubature appears in [Xu18] and in the work of Sauer and Xu [SX23]. We claim no priority for Theorem 1.2 or for the existence of 3-rules, which the source states as known. To our knowledge, the explicit rule of Proposition 5.1 and Theorems 6.1 to 6.3 are not in print; Proposition 4.2 is a simple consequence of the theory and may well be known.

What was read: the source [Xu22], pp. 415–417, in the publisher’s open file; of [Xu18], in the arXiv version (the published version was not compared), Sections 2 and 3 with Theorems 2 and 5, and Section 5.1 with Theorem 12 and equations (19)–(21), which concern the constant weight; of [SX23], in the arXiv version, the abstract and the end of Section 4 with Theorem 4.8; the report [CMSX23] of a later mini-workshop on multivariate orthogonal polynomials, searched in full for cubature and with the abstracts of Xu and of Sauer read, which contains no statement on the question; the abstracts of [LSX09, DVX09]; and the publisher’s public chapter summaries of Xu’s monograph [Xu25], according to which its epilogue discusses “cubature rules of more than two variables” as an open problem; the chapter itself was not accessible to us. We did not read [Xu94, MP78, Sch78, Mö76, SX94]; what we say about them is taken from [Xu22, Xu18, SX23] and, for [Xu94, SX94], from their reviews in zbMATH (Zbl 0809.41028 and Zbl 0812.65013; the entry of [MP78] is Zbl 0402.65013).

The searches were made on 15 October 2026, at the first computation, by the verification runs and when this note was written: queries to the arXiv interface (Gaussian and Gauss cubature, Xu’s work on cubature, cubature with Chebyshev weights in three variables, quasi-orthogonal polynomials and cubature, cubature of even degree), to Crossref, OpenAlex, zbMATH and Semantic Scholar, the lists of works citing [Xu94], [Xu18] and the source in OpenAlex, and four web searches. No publication was found that constructs or excludes a Gaussian cubature rule of degree $2n - 2$ for this weight in three variables for some $n \geq 4$, and no earlier treatment of the symmetric case $n = 4$ was found in these sources. The two families of weights with Gaussian rules of odd degree mentioned in the source (see [SX94]) and the rules of [LSX09, DVX09] concern other degrees or numbers of nodes. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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