

Universal Polynomial Interpolation Need Not Be Hereditary

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21 October 2026

Abstract

A finite family of affine algebraic sets may allow polynomial interpolation of every pointwise-compatible tuple although a subfamily does not. For every integer $m \geq 2$, we construct an irredundant family of three proper, reduced, pure curves in $\mathbb{C}^3$ with this property, while the sum of two vanishing ideals has quotient $\mathbb{C}[x]/(x^m)$. An explicit formula interpolates all compatible data on the whole family. The construction disproves the necessity direction of the radical-chain criterion in Theorem 2.5 of McKinley–Shekhtman–Tuesink, arXiv:2208.12406v1, even after correcting its indexing. We give the precise hereditary replacement and prove that the full-family criterion is restored when all pairwise intersections are finite. The argument uses classical two-set interpolation and Chinese remainder methods; no priority claim is made.

1 The quantifiers in universal interpolation

Let $R = \mathbb{C}[x_1, \dots, x_d]$ and let $\mathcal{I}(V)$ denote the vanishing ideal of an affine algebraic set V . Algebraic sets need not be irreducible. For a finite family $\mathcal{V} = (V_1, \dots, V_n)$, write $\mathcal{U}(\mathcal{V})$ for the following property: for every tuple $p_1, \dots, p_n \in R$ satisfying

$$p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j} \quad (1 \leq i, j \leq n), \quad (1)$$

there exists $f \in R$ with $f|_{V_i} = p_i|_{V_i}$ for all i . We call this property hereditary for $\mathcal{V}$ if every nonempty subfamily has it.

McKinley, Shekhtman and Tuesink [1] study interpolation on arbitrary algebraic sets. Their valid two-set criterion is the following.

Lemma 1 ([1, Lemma 2.4]). *For two algebraic sets A, B , property $\mathcal{U}(A, B)$ holds if and only if $\mathcal{I}(A) + \mathcal{I}(B)$ is radical.*

Proof. If the sum is radical, the Nullstellensatz turns pointwise agreement into $p_A - p_B \in \mathcal{I}(A) + \mathcal{I}(B)$. Write this difference as $a + b$ with $a \in \mathcal{I}(A)$ and $b \in \mathcal{I}(B)$. Then $p_A - a = p_B + b$ interpolates. Conversely, if the sum is not radical, choose $g \notin \mathcal{I}(A) + \mathcal{I}(B)$ with $g^r \in \mathcal{I}(A) + \mathcal{I}(B)$ for some $r > 1$. The data $g, 0$ agree on $A \cap B$. An interpolant would express g as a sum of an element of $\mathcal{I}(A)$ and an element of $\mathcal{I}(B)$, a contradiction. $\square$

Theorem 2.5 of [1] asserts a generalization to several sets; the same assertion is Theorem 4 of the workshop contribution [2, pp. 411–412]. As printed, its condition (b) adds $\mathcal{I}(V_k)$ to the ideal

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of a union that already contains V_k . That sum is simply $\mathcal{I}(V_k)$. The induction instead requires the corrected condition

$$\mathcal{I}(V_1 \cup \cdots \cup V_{k-1}) + \mathcal{I}(V_k) \text{ is radical} \quad (2 \leq k \leq n). \quad (2)$$

Condition (2) is sufficient: glue successively and use Lemma 1. We show that it is not necessary for $\mathcal{U}(\mathcal{V})$, so repairing the index alone does not repair the equivalence. The issue is that compatible data on a subfamily need not extend to compatible data on the whole family.

2 An irredundant family of curves

Theorem 2. *For every integer $m \geq 2$, there is an irredundant family $\mathcal{V}_m = (V_1, V_2, V_3)$ of proper reduced pure one-dimensional algebraic sets in $\mathbb{C}^3$ such that $\mathcal{U}(\mathcal{V}_m)$ holds but*

$$R/(\mathcal{I}(V_1) + \mathcal{I}(V_2)) \simeq \mathbb{C}[x]/(x^m).$$

In particular, the corrected condition (2) fails for the displayed order, with a nilpotent of exact order m .

Proof. In coordinates x, y, z , set

$$P_m = \{(t, t^m, 0) : t \in \mathbb{C}\}, \quad L = \{(t, 0, 0) : t \in \mathbb{C}\}, \quad E_j = \{(t, 0, j) : t \in \mathbb{C}\} \quad (j = 1, 2, 3),$$

and define

$$V_1 = P_m \cup E_1, \quad V_2 = L \cup E_2, \quad V_3 = P_m \cup L \cup E_3. \quad (3)$$

All the components are irreducible affine curves. Each V_j has the private component E_j , so no V_j is contained in the union of the other two. The family is therefore irredundant and pairwise incomparable.

Let p_1, p_2, p_3 satisfy (1). In particular, $p_1 = p_3$ on P_m and $p_2 = p_3$ on L . For $j = 0, 1, 2, 3$ put

$$e_j(z) = \prod_{\substack{0 \leq k \leq 3 \\ k \neq j}} \frac{z - k}{j - k}.$$

Then

$$f = (e_0 + e_3)p_3 + e_1p_1 + e_2p_2 \quad (4)$$

equals p_3 on the slices $z = 0, 3$, equals p_1 on $z = 1$, and equals p_2 on $z = 2$. Consequently it has the prescribed restriction on each set in (3). This proves $\mathcal{U}(\mathcal{V}_m)$ for arbitrary compatible polynomial data.

Write

$$A = (z, y - x^m), \quad B = (z, y), \quad C_j = (z - j, y).$$

These are the vanishing ideals of P_m, L, E_j , respectively. The ideals A and C_1 are comaximal, so

$$\begin{aligned} I_1 &:= \mathcal{I}(V_1) = AC_1 \\ &= (z(z-1), zy, (y-x^m)(z-1), y(y-x^m)). \end{aligned}$$

Similarly,

$$I_2 := \mathcal{I}(V_2) = BC_2 = (y, z(z-2)).$$

For the latter equality, pass to $R/(y) = \mathbb{C}[x, z]$ and intersect the ideals (z) and $(z-2)$.

Every displayed generator of $I_1 + I_2$ belongs to (x^m, y, z) . Conversely, $y \in I_2$ and

$$z = z(z-1) - z(z-2) \in I_1 + I_2.$$

The identity

$$x^m = (y - x^m)(z - 1) - yz + y + x^m z$$

therefore gives $x^m \in I_1 + I_2$. Thus $I_1 + I_2 = (x^m, y, z)$. The class of x in its quotient has exact nilpotence order m , which proves the theorem. $\square$

Remark 3. The obstruction on two tangent curves is already present in [1, Example 1.1], for $m = 2$. It is not new here. What changes is its relation to the data on the third set. Indeed, $p_1 = (1 - z)x$ and $p_2 = 0$ agree on $V_1 \cap V_2$, which is the origin. They cannot be extended to a compatible p_3 : such a polynomial would equal x on P_m and vanish on L , forcing $x \in (z, y, x^m)$. This is false.

The converse proof of [1, Theorem 2.5] constructs obstructed prefix data but does not extend them to the later sets. The example exhibits precisely why that omitted step fails. The target of this correction is the specified preprint version and the workshop report, not an unidentified later version.

3 The hereditary replacement

Theorem 4. *For a fixed finite family $(V_1, \dots, V_n)$, the following are equivalent:*

1. $\mathcal{U}$ holds for every nonempty subfamily.
2. For every nonempty $M \subset \{1, \dots, n\}$ and $k \notin M$,

$$\mathcal{I}\left(\bigcup_{j \in M} V_j\right) + \mathcal{I}(V_k) \text{ is radical.} \quad (5)$$

Proof. Assume (2). In any subfamily, construct an interpolant successively. The polynomial already constructed and the next datum agree on the intersection of the previous union with the next set. Apply (5) and Lemma 1 at each step.

Conversely, assume (1) and fix M, k . Let g, h be arbitrary polynomials agreeing on $(\bigcup_{j \in M} V_j) \cap V_k$. Prescribe g on every V_j with $j \in M$ and h on V_k . These data are compatible. By (1) they admit an interpolant, so the pair $(\bigcup_{j \in M} V_j, V_k)$ has property $\mathcal{U}$. Lemma 1 gives (5). $\square$

Remark 2 following Theorem 2.5 in [1] proposes (5) as an equivalent condition for whole-family interpolation. Theorem 4 identifies its correct quantifiers. Taking $M = \{1\}$ and $k = 2$ in Theorem 2 shows that whole-family interpolation alone does not imply it.

Theorem 5. *Suppose $V_i \cap V_j$ is finite whenever $i \neq j$. Then $\mathcal{U}(V_1, \dots, V_n)$ holds if and only if $\mathcal{U}$ holds for every nonempty subfamily. Equivalently, (5) holds. These conditions are also equivalent to (2) in any fixed ordering.*

Proof. Only the implication from whole-family interpolation to heredity requires proof. Prescribe compatible data on a subfamily indexed by S . Let

$$T = \bigcup_{i < j} (V_i \cap V_j),$$

which is finite. At a point of T lying in a set indexed by S , assign the value of its prescribed polynomial. Compatibility makes this assignment well defined. Assign zero at the remaining points of T .

For every $j \notin S$, choose an ambient polynomial taking these values on $T \cap V_j$. To do so, select a linear form separating the finitely many points and compose it with the corresponding univariate Lagrange interpolant. Such a form exists over $\mathbb{C}$, since the prohibited forms lie in finitely many proper linear hyperplanes. If $T \cap V_j$ is empty, use zero. The resulting tuple on the full family is compatible, and whole-family $\mathcal{U}$ supplies an interpolant of the original data.

Theorem 4 now gives (5), and hence every ordered condition (2). Conversely, one such condition is sufficient by successive applications of Lemma 1. $\square$

Corollary 6. *The corrected radical-chain criterion is valid for a finite family of pairwise distinct irreducible affine curves.*

Proof. Two distinct irreducible affine curves have finite intersection. Indeed, their intersection is a proper closed subset of each curve and hence is zero-dimensional and finite. Apply Theorem 5. $\square$

The sets in (3) have positive-dimensional shared components. They therefore do not contradict this corollary.

4 Dependence on the order

The example also makes the ordering issue explicit. Put

$$J = \mathcal{I}(P_m \cup L) = (z, y(y - x^m)).$$

To verify this equality, pass modulo z and use that y and $y - x^m$ have greatest common divisor one in the UFD $\mathbb{C}[x, y]$. Their principal ideals intersect in their product; they need not be comaximal. As an intersection of prime ideals, J is radical. We have $I_3 = JC_3$.

For the first two members in an ordering, the relevant sums are

First pair	Sum of their ideals	Ideal of their union plus the last ideal
1, 2	(x^m, y, z)	J
1, 3	A	B
2, 3	B	A

For $I_1 + I_3$, the difference of $z(z - 1)$ and $z(z - 3)$ puts z in the sum; reduction modulo z then gives A . The calculation for $I_2 + I_3 = B$ is the same, using $z(z - 2)$ and $z(z - 3)$.

The ideals of the three possible prefix unions are JC_1C_2 , JC_1C_3 and JC_2C_3 , respectively: the distinct z -slices are comaximal. In each final sum, the univariate z -polynomials supplied by the two ideals have greatest common divisor z . Modulo z , the resulting ideals are $(y(y - x^m))$, (y) and $(y - x^m)$, respectively. This proves the table.

Thus four of the six orderings satisfy the corrected radical chain, and two do not. This calculation does not assert that an arbitrary family having $\mathcal{U}$ must possess some ordering satisfying the chain condition.

5 Context and reproducibility

The preceding arguments distinguish pointwise compatibility from compatibility modulo the ordinary sums of ideals. The generalized Chinese remainder theorem and its equalizer formulation are classical;

see, for example, Maszczyk [3, Corollary 5], which assumes a distributive lattice of subgroups. That hypothesis must not be silently omitted. The earlier interpolation theory on flats is due to de Boor, Dyn and Ron [4].

For given data, a standard iterative ideal-membership test is available independently of universal interpolation. If the solutions on the first $k - 1$ sets form $f_{k-1} + K_{k-1}$, where $K_{k-1} = \bigcap_{j < k} I_j$, then extension to the next set is possible exactly when

$$p_k - f_{k-1} \in K_{k-1} + I_k.$$

Writing the difference as $a + b$, with $a \in K_{k-1}$ and $b \in I_k$, gives the new interpolant $f_{k-1} + a$. This is the ordinary two-ideal congruence calculation, not a new algorithm claimed in this note. It does not impose the additional PDE constraints considered in the second part of [1].

A supplementary Python checker uses exact rational arithmetic. It checks the ideal equalities in both directions with generator representations, verifies the resulting Buchberger S-pairs, and checks polynomial restrictions by a separate univariate substitution routine. It also detects a deliberately corrupted interpolation formula. Normal and optimized runs each pass 2,119 explicit checks, including 613 S-pairs and 80 restriction fixtures. The nilpotence indices $m = 2, \dots, 7$ are checked exactly. The proofs above, rather than these finite tests, establish the claims for all data and every $m \geq 2$.

AI tools assisted with derivations, source searches, exact-check code and manuscript preparation. The note is unrefereed; the computations are not formal verification or external review. The classical tangency obstruction and interpolation machinery are credited to their sources, and no absolute priority claim is made.

References

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