

The Free Orthogonal Quantum Group O_N^+ Is Topologically Generated by SO_N and the Dual of a Free Group Exactly When $N \geq 4$: On a Question of Freslon and the Nilpotent Residual of $\mathcal{O}(O_N^+)$

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Abstract

In the problem session of the Oberwolfach workshop “Quantum Groups – Algebra, Analysis and Category Theory” (Report 44/2021), Freslon, reporting on joint work with Franz and Skalski, asked three questions about the nilpotent residual $I^\infty = \bigcap_n I^n$ of the augmentation ideal I of a Hopf algebra A : for which A it vanishes; what it is for the free orthogonal quantum group, $A = \mathcal{O}(O_N^+)$; and, as the final problem, whether there is a discrete group Γ , residually torsion-free nilpotent, such that O_N^+ is topologically generated by SO_N and the dual $\hat{\Gamma}$. We answer the final question for every $N \geq 2$. The answer is no for $N = 2$ and $N = 3$: no torsion-free group has this property, for any embedding of its dual (for $N = 2$ the negative answer was known from the work of Franz, Freslon and Skalski). The answer is yes for $N \geq 4$, with the free group $\mathbb{F}_2$, whose dual is embedded by two rotation blocks. As a consequence we determine the nilpotent residual of $\mathcal{O}(O_N^+)$ for every $N \geq 2$: it is zero for $N \geq 4$, so that O_N^+ is strongly connected, and it is the kernel of the map onto $\mathcal{O}(SO_N)$ for $N = 2$ (known) and $N = 3$. For $N = 4$ the proof uses the model $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$, a lowest-weight argument for $\mathfrak{sl}_2$ and words in a free group, and needs only standard facts. For $N \geq 5$ it is an induction which relies on one published theorem, due to Brannan, Collins and Vergnioux; its proof is recalled in an appendix. Two auxiliary statements are cases of generation theorems of Chirvasitu, for which we give new proofs. The general question, for which Hopf algebras the nilpotent residual vanishes, is a programme and stays open; the Gaussian part of O_N^+ is not determined for $N \geq 4$; and we do not know which torsion-free quotients of $\mathbb{F}_2$ could replace $\mathbb{F}_2$. This is an unrefereed note.

1 Introduction

1.1 The problem

Let A be a Hopf algebra with counit ε . Its *augmentation ideal* is $I = \ker \varepsilon$, and $I^\infty = \bigcap_{n \geq 1} I^n$ is called the *nilpotent residual* of I in the problem contribution [Fre21] of Freslon, which reports on joint work with Franz and Skalski and is the source of this note. It contains three questions.

1. The general question: for which A is $I^\infty = \{0\}$? Equivalently, what is A/I^∞ in terms of A ?
2. The main question: what is I^∞ for $A = \mathcal{O}(O_N^+)$, the Hopf $*$ -algebra of the free orthogonal quantum group?

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3. The final problem: is there a discrete group Γ , residually torsion-free nilpotent, such that O_N^+ “is topologically generated by SO_N and $\widehat{\Gamma}$ ” [Fre21, p. 2452]?

The source records what was known: for a classical compact group G one has $I^\infty = \{0\}$ in $\mathcal{O}(G)$ if and only if G is connected; in a group algebra $\mathbb{C}[\Gamma]$ one has $I^\infty = \{0\}$ if and only if Γ is residually torsion-free nilpotent; a non-trivial idempotent in A forces $I^\infty \neq \{0\}$; and $I^\infty = \{0\}$ for the free unitary quantum group U_N^+ , because U_N^+ is topologically generated by U_N and the dual of the free group $\mathbb{F}_N$ [Chi20]. The third question asks for the analogue of the last statement for O_N^+ . Proofs of the recorded facts are in the paper [FFS23] of Franz, Freslon and Skalski, where a compact quantum group with $I^\infty = \{0\}$ is called *strongly connected*. The problem is record OWR-8415347-019 of the UnsolvedMath collection [ula26], where it is listed as partially solved; the text of that record is a damaged extraction of the printed page, and we follow the source.

The terms are used as follows (details are in Section 2). A compact quantum group G is given by its CQG algebra $\mathcal{O}(G)$, and a *quantum subgroup* $H \leq G$ is a surjective Hopf $*$ -homomorphism $\mathcal{O}(G) \rightarrow \mathcal{O}(H)$. The algebra $\mathcal{O}(O_N^+)$ of Wang [Wan95] and Van Daele and Wang [VDW96] is the universal $*$ -algebra generated by the entries of a matrix $u = (u_{ij})_{i,j=1}^N$ with $u_{ij}^* = u_{ij}$ and $uu^t = u^t u = 1$; its abelianisation is $\mathcal{O}(O_N)$, and $SO_N \leq O_N^+$ is the composition $\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(O_N) \rightarrow \mathcal{O}(SO_N)$. The *dual* $\widehat{\Gamma}$ of a discrete group Γ is the compact quantum group with $\mathcal{O}(\widehat{\Gamma}) = \mathbb{C}[\Gamma]$. A group Γ is *residually torsion-free nilpotent* if every $g \neq e$ lies outside some normal subgroup with torsion-free nilpotent quotient [FFS23, Proposition 4.5]; such a group is torsion-free. Quantum subgroups $G_i \leq G$ *topologically generate* G , written $G = \langle G_i \rangle$, if the maps $\mathcal{O}(G) \rightarrow \mathcal{O}(G_i)$ do not all factor through one proper quotient Hopf $*$ -algebra of $\mathcal{O}(G)$ [Chi20, Definition 0.1]; the notion goes back to Brannan, Collins and Vergnoux [BCV17].

The source does not quantify N and does not prescribe how $\widehat{\Gamma}$ sits in O_N^+ . We treat every $N \geq 2$, and for the negative answers every quantum subgroup of O_N^+ which is the dual of Γ . (For $N = 1$ the quantum group O_1^+ is the group with two elements and the answer is trivially negative.)

1.2 Results

For an invertible element g of an algebra put

$$C(g) = \frac{g + g^{-1}}{2}, \quad S(g) = \frac{g - g^{-1}}{2i}, \quad R(g) = \begin{pmatrix} C(g) & S(g) \\ -S(g) & C(g) \end{pmatrix}.$$

If g is a unitary element of a $*$ -algebra, then $C(g)$ and $S(g)$ are self-adjoint, commute and satisfy $C(g)^2 + S(g)^2 = 1$; we call $R(g)$ a *rotation block*. Let $\mathbb{F}_2 = \langle a, b \rangle$ be the free group on two generators. For $N \geq 4$ there is a unique surjective Hopf $*$ -homomorphism

$$\pi_N: \mathcal{O}(O_N^+) \rightarrow \mathbb{C}[\mathbb{F}_2], \quad (\pi_N(u_{ij}))_{i,j} = \text{diag}(R(a), R(b), 1, \dots, 1) \quad (1)$$

(Proposition 3.1), and $\widehat{\mathbb{F}}_2 \leq O_N^+$ always refers to it. This copy of $\widehat{\mathbb{F}}_2$ is not new: for $N = 4$ it consists of the circles $\mathbb{T} \leq O_2^+$ in the two factors of $O_2^+ \hat{*} O_2^+ \leq O_4^+$, and it is implicit in [FFS23, proof of Proposition 6.5], where the dual of the Heisenberg group, a quantum subgroup of this copy, is found inside O_4^+ through $O_2^+ \hat{*} O_2^+$. What is new is the generation statement. By $\text{Fix}_H(u^{\otimes m})$ we denote the space of vectors of $(\mathbb{C}^N)^{\otimes m}$ fixed by a quantum subgroup $H \leq O_N^+$ in the m -th tensor power of the fundamental representation.

Theorem 1.1 ($N = 2$ and $N = 3$). *Let $N \in \{2, 3\}$, let Γ be a discrete group without elements of order two, for instance a torsion-free group, and let $\widehat{\Gamma} \leq O_N^+$ be a quantum subgroup, given by an arbitrary surjective Hopf $*$ -homomorphism $\pi: \mathcal{O}(O_N^+) \rightarrow \mathbb{C}[\Gamma]$. Then Γ is cyclic, π factors through*

$\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(SO_N)$, so that $\widehat{\Gamma}$ is a classical subgroup of SO_N , and the quantum subgroup of O_N^+ generated by SO_N and $\widehat{\Gamma}$ is SO_N . In particular O_N^+ is not topologically generated by SO_N and $\widehat{\Gamma}$.

Theorem 1.2 ($N = 4$). For every $m \geq 0$,

$$\text{Fix}_{SO_4}(u^{\otimes m}) \cap \text{Fix}_{\widehat{\mathbb{F}}_2}(u^{\otimes m}) = \text{Fix}_{O_4^+}(u^{\otimes m}).$$

Consequently $O_4^+ = \langle SO_4, \widehat{\mathbb{F}}_2 \rangle$.

Theorem 1.3 ($N \geq 4$). For every $N \geq 4$ and every $m \geq 0$, $\text{Fix}_{SO_N}(u^{\otimes m}) \cap \text{Fix}_{\widehat{\mathbb{F}}_2}(u^{\otimes m}) = \text{Fix}_{O_N^+}(u^{\otimes m})$, and $O_N^+ = \langle SO_N, \widehat{\mathbb{F}}_2 \rangle$. The same holds for the dual of the free group $\mathbb{F}_k$, $2 \leq k \leq N/2$, embedded by k rotation blocks.

Corollary 1.4 (the final problem of [Fre21]). A residually torsion-free nilpotent group Γ with a quantum subgroup $\widehat{\Gamma} \leq O_N^+$ such that $O_N^+ = \langle SO_N, \widehat{\Gamma} \rangle$ exists if and only if $N \geq 4$. For $N \geq 4$ one can take $\Gamma = \mathbb{F}_2$.

Corollary 1.5 (the main question of [Fre21]). Let I be the augmentation ideal of $\mathcal{O}(O_N^+)$.

- (a) If $N \geq 4$, then $I^\infty = \{0\}$: the quantum group O_N^+ is strongly connected.
- (b) If $N = 2$ or $N = 3$, then I^∞ is the kernel of $\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(SO_N)$: the strongly connected component of the identity of O_N^+ is SO_N .

Theorem 1.1 is proved in Section 3, Theorem 1.2 in Section 4, Theorem 1.3 and Corollary 1.4 in Section 5, and Corollary 1.5 in Section 6.

1.3 What was known, what is new, and what stays open

Known. Franz, Freslon and Skalski [FFS23] proved the facts recorded in the source: [FFS23, Lemma 4.2] for classical groups, [FFS23, Proposition 4.5] for group duals (through a theorem of Passi [Pas79]) and for quantum groups topologically generated by strongly connected quantum subgroups, and [FFS23, Proposition 4.6] for U_N^+ , through Chirvasitu's theorem $U_N^+ = \langle U_N, \widehat{\mathbb{F}}_N \rangle$ [Chi20] and the theorem of Magnus [Mag35] that free groups are residually torsion-free nilpotent. They observe that the same strategy does not apply directly to O_N^+ , whose diagonal torus is the dual of a free product of groups of order two, and they treat $N = 2$: the strongly connected component of O_2^+ is the circle [FFS23, Lemma 4.7]. This is Corollary 1.5(b) for $N = 2$, and with [FFS23, Proposition 4.5] it gives the negative answer to the final problem for $N = 2$. Chirvasitu also proved that O_N^+ is topologically generated by O_N and the dual of $\mathbb{Z}_2^{*N}$, for all $N \geq 2$ [Chi20, Theorem 3.3]; the whole point of the final problem is therefore the absence of torsion.

New. We found the following neither stated nor answered in the literature (the searches and their limits are described in the paragraph ‘‘Scope and priority’’ at the end): the generation statements of Theorems 1.2 and 1.3, hence the positive part of Corollary 1.4; the negative answer for $N = 3$, which is an easy consequence of results of Banica and Patri [BP17]; the strong connectedness of O_N^+ for $N \geq 4$; and the nilpotent residual for $N = 3$. Proposition 3.1 on group duals in O_N^+ follows from known results, as explained there. Two auxiliary statements, Lemmas 4.8 and 4.9, are the fixed-vector forms of cases of generation theorems of Chirvasitu; the statements are his, and the proofs given here are new (Remarks 4.11 and 4.12). A search that finds nothing is not a proof of novelty, and we claim no priority.

Scope. This note answers the final problem of [Fre21] for every $N \geq 2$, and determines the nilpotent residual of $\mathcal{O}(O_N^+)$, that is, answers its main question, for every $N \geq 2$. It does not address

the general question, which is a programme and stays open. The case $N \geq 5$ of Theorem 1.3 relies on one published theorem, [BCV17, Theorem 4.1], whose proof we recall in Appendix A. The case $N = 2$ of the negative answer and of the residual was known [FFS23]. The Gaussian part of O_N^+ , which is contained in the strongly connected component [FFS23, Theorem A], is not determined here for $N \geq 4$; see Section 7.

2 Preliminaries

2.1 Compact quantum groups, representations, quantum subgroups

A *CQG algebra* is a Hopf $*$ -algebra which is spanned by the coefficients of its finite-dimensional unitary corepresentations [DK94]; we write it as $\mathcal{O}(G)$ and say that it defines the compact quantum group G [Wor87, Wor98]. A *representation* of G on $\mathbb{C}^n$ is, in this note, a unitary matrix $v = (v_{ij}) \in M_n(\mathcal{O}(G))$ with $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$; then $\varepsilon(v_{ij}) = \delta_{ij}$ and $S(v_{ij}) = v_{ji}^*$. For $\xi \in \mathbb{C}^n$ we write $v(\xi \otimes 1) = \sum_{i,j} e_i \otimes v_{ij} \xi_j \in \mathbb{C}^n \otimes \mathcal{O}(G)$. The space of *fixed vectors* and the spaces of intertwiners are

$$\text{Fix}_G(v) = \{\xi : v(\xi \otimes 1) = \xi \otimes 1\}, \quad \text{Hom}_G(v, w) = \{T : (T \otimes 1)v = w(T \otimes 1)\},$$

so that $\text{Fix}_G(v) = \text{Hom}_G(1, v)$. Direct sums are formed blockwise, and the tensor product has the entries $(v \otimes w)_{(i,k),(j,l)} = v_{ij} w_{kl}$. Compositions and tensor products of intertwiners are intertwiners, and so are adjoints, the representations being unitary. We use the following standard facts [Wor87, DK94]: every representation is a direct sum of irreducible ones; the coefficients of pairwise inequivalent irreducible representations are linearly independent, and an irreducible representation of dimension d has d^2 linearly independent coefficients; a commutative CQG algebra is the algebra $\mathcal{O}(K)$ of representative functions on the compact group K of its characters (the $*$ -homomorphisms to $\mathbb{C}$). The quantum group G is of *Kac type* if $S^2 = \text{id}$; then $S(a^*) = S(a)^*$ for all a .

A *quantum subgroup* $H \leq G$ is a surjective Hopf $*$ -homomorphism $\pi_H : \mathcal{O}(G) \rightarrow \mathcal{O}(H)$ onto a CQG algebra [Chi20, FFS23]; quantum subgroups correspond to the Hopf $*$ -ideals $\ker \pi_H$ of $\mathcal{O}(G)$. We say that H is contained in a quantum subgroup H' if $\ker \pi_{H'} \subset \ker \pi_H$. A representation v of G restricts to the representation $(\pi_H(v_{ij}))$ of H , and $\text{Fix}_H(v)$, $\text{Hom}_H(v, w)$ refer to the restrictions; thus $\text{Fix}_G(v) \subset \text{Fix}_{H'}(v) \subset \text{Fix}_H(v)$ if H is contained in H' .

2.2 The quantum group O_N^+ and the two kinds of subgroups

The algebra $\mathcal{O}(O_N^+)$ defined in the introduction is a CQG algebra with $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, $\varepsilon(u_{ij}) = \delta_{ij}$ and $S(u_{ij}) = u_{ji}$ [Wan95, VDW96]; it is of Kac type. The fundamental representation u acts on $\mathbb{C}^N$ with its standard basis $e_1, \dots, e_N$, and $u^{\otimes m}$ on $(\mathbb{C}^N)^{\otimes m}$. The vector $\sum_i e_i \otimes e_i$ is fixed by $u^{\otimes 2}$, because $uu^t = 1$. Every element of $\mathcal{O}(O_N^+)$ is a linear combination of products of the generators, and a product of k generators is a coefficient of $u^{\otimes k}$.

The characters of $\mathcal{O}(O_N^+)$ are the real orthogonal matrices, so its abelianisation is $\mathcal{O}(O_N)$, and the quantum subgroups with commutative algebra are the closed subgroups $K \subset O_N$. For such a subgroup,

$$\text{Fix}_K(u^{\otimes m}) = \{x \in (\mathbb{C}^N)^{\otimes m} : g^{\otimes m} x = x \text{ for all } g \in K\}. \quad (2)$$

The kernel of the restriction $\mathcal{O}(O_N) \rightarrow \mathcal{O}(SO_N)$ is generated by $\det -1$: since $\det^2 = 1$ on O_N , the function $\frac{1}{2}(1 - \det)$ is the indicator function of $O_N \setminus SO_N$, and a function f vanishing on SO_N satisfies $f = \frac{1}{2}f(1 - \det)$.

In the group algebra $\mathbb{C}[\Gamma]$ of a discrete group, with $\Delta(g) = g \otimes g$ and $g^* = g^{-1}$, the elements of Γ are linearly independent. Let $\pi : \mathcal{O}(O_N^+) \rightarrow \mathbb{C}[\Gamma]$ be a quantum subgroup, put $w = (\pi(u_{ij}))$, and

suppose that $f_1, \dots, f_N$ is a basis of $\mathbb{C}^N$ with $w(f_j \otimes 1) = f_j \otimes g_j$ for elements $g_j \in \Gamma$. Then $w^{\otimes m}$ maps $f_{j_1} \otimes \dots \otimes f_{j_m} \otimes 1$ to $f_{j_1} \otimes \dots \otimes f_{j_m} \otimes g_{j_1} \cdots g_{j_m}$, and therefore

$$\text{Fix}_{\widehat{\Gamma}}(u^{\otimes m}) = \text{span}\{f_{j_1} \otimes \dots \otimes f_{j_m} : g_{j_1} g_{j_2} \cdots g_{j_m} = e\}. \quad (3)$$

2.3 Topological generation and fixed vectors

Let G be a compact quantum group with quantum subgroups $\pi_i: \mathcal{O}(G) \rightarrow \mathcal{O}(G_i)$, $i \in I$. The sum of all Hopf $*$ -ideals contained in $\bigcap_i \ker \pi_i$ is a Hopf $*$ -ideal; the corresponding quantum subgroup is the quantum subgroup *generated* by the G_i (see [FFS23, Section 3.2]). Consider the conditions

- (C) the only Hopf $*$ -ideal contained in $\bigcap_i \ker \pi_i$ is $\{0\}$; equivalently, a surjective Hopf $*$ -homomorphism $\rho: \mathcal{O}(G) \rightarrow \mathcal{O}(H)$ through which all π_i factor is injective;
- (F) $\text{Fix}_G(v) = \bigcap_i \text{Fix}_{G_i}(v)$ for all representations v in a class containing every irreducible representation as a subrepresentation.

Condition (C) is the definition of topological generation of [Chi20, Definition 0.1], which is used in the source and in [FFS23, Proposition 4.5]. Brannan, Collins and Vergnioux define topological generation by the equality $\text{Hom}_G(v, w) = \bigcap_i \text{Hom}_{G_i}(v, w)$ for all representations [BCV17, Definition 4], prove that this is equivalent to (F) [BCV17, Proposition 3.5], and relate it to (C) through a theorem of Banica and Bichon on Hopf images [BCV17, Remark 3], [BB10]. (For their definition of quantum subgroups see Remark 2.4; all numbers of [BCV17] are those of the arXiv version.) For O_N^+ both implications between (C) and (F) have short direct proofs, which we include. Proposition 2.2 is the direction used in this note: it turns the equalities of fixed vectors in Theorems 1.2 and 1.3 into generation statements, and it enters Corollary 1.5(a). Proposition 2.3 is not used in the proofs; it shows that nothing is lost in passing to fixed vectors. The negative result, Theorem 1.1, needs neither, since its proof exhibits a proper quotient.

Lemma 2.1. *Let v be a representation of G on $\mathbb{C}^n$ and $C(v) \subset \mathcal{O}(G)$ the span of its coefficients. Then $\dim C(v)$ is the dimension of the commutant of $\text{End}_G(v) = \text{Hom}_G(v, v)$ in $M_n(\mathbb{C})$.*

Proof. In a suitable orthonormal basis v is a direct sum of pairwise inequivalent irreducible representations u^α of dimensions d_α , with multiplicities $n_\alpha \geq 1$. Then $\dim C(v) = \sum_\alpha d_\alpha^2$ by the facts recalled above, and $\text{End}_G(v) = \bigoplus_\alpha 1_{d_\alpha} \otimes M_{n_\alpha}(\mathbb{C})$ by Schur's lemma, with commutant $\bigoplus_\alpha M_{d_\alpha}(\mathbb{C}) \otimes 1$. $\square$

Proposition 2.2 ((F) for the tensor powers implies (C)). *Let $G_i \leq O_N^+$ be quantum subgroups with $\text{Fix}_{O_N^+}(u^{\otimes m}) = \bigcap_i \text{Fix}_{G_i}(u^{\otimes m})$ for all $m \geq 0$. Then $O_N^+ = \langle G_i \rangle$ in the sense of (C).*

Proof. Write $G = O_N^+$ and let $\rho: \mathcal{O}(G) \rightarrow \mathcal{O}(H)$ be a quantum subgroup through which all π_i factor. Then $\text{Fix}_G(u^{\otimes m}) \subset \text{Fix}_H(u^{\otimes m}) \subset \bigcap_i \text{Fix}_{G_i}(u^{\otimes m})$, so $\text{Fix}_H(u^{\otimes m}) = \text{Fix}_G(u^{\otimes m})$ for all m . For a multi-index $J = (j_1, \dots, j_k)$ let $e_J = e_{j_1} \otimes \dots \otimes e_{j_k}$ and $J^r = (j_k, \dots, j_1)$. The vector $R_k = \sum_J e_J \otimes e_{J^r}$ consists of nested copies of $\sum_i e_i \otimes e_i$, so $R_k \in \text{Fix}_G(u^{\otimes 2k})$, and the functional $R_k^* = \langle R_k, \cdot \rangle$ lies in $\text{Hom}_G(u^{\otimes 2k}, 1)$. For a linear map $T: (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$ and a vector $\xi \in (\mathbb{C}^N)^{\otimes (l+k)}$ put

$$\Phi(T) = (T \otimes \text{id})(R_k) = \sum_J T(e_J) \otimes e_{J^r}, \quad \Psi(\xi) = (\text{id} \otimes R_k^*)(\xi \otimes \text{id}).$$

Since $\langle R_k, e_{J^r} \otimes e_L \rangle = \delta_{J,L}$, one finds $\Psi(\Phi(T)) = T$ and $\Phi(\Psi(\xi)) = \xi$. For $K = G$ and for $K = H$ the map Φ sends $\text{Hom}_K(u^{\otimes k}, u^{\otimes l})$ into $\text{Fix}_K(u^{\otimes (l+k)})$, and Ψ sends this space back, because R_k and R_k^* are intertwiners for G and hence for H . Therefore $\text{Hom}_H(u^{\otimes k}, u^{\otimes l}) = \text{Hom}_G(u^{\otimes k}, u^{\otimes l})$ for all k, l , and $\text{End}_H(v_m) = \text{End}_G(v_m)$ for $v_m = \bigoplus_{k \leq m} u^{\otimes k}$. By Lemma 2.1, applied to G and to H , $\dim \rho(C(v_m)) = \dim C(v_m)$, so ρ is injective on $C(v_m)$. The spaces $C(v_m)$ increase with m and their union is $\mathcal{O}(G)$. Hence ρ is injective. $\square$

Proposition 2.3 ((C) implies (F)). *Let G be of Kac type, for instance $G = O_N^+$, and let (C) hold. Then $\text{Fix}_G(v) = \bigcap_i \text{Fix}_{G_i}(v)$ for every representation v of G .*

Proof. Let $\xi \in \bigcap_i \text{Fix}_{G_i}(v)$ and $b_k = \sum_l v_{kl}\xi_l - \xi_k 1$, so that $\pi_i(b_k) = 0$ for all i and k . Let J be the two-sided ideal generated by the set X of all $b_k, b_k^*, S(b_k), S(b_k)^*$. Then $J \subset \ker \pi_i$, since the π_i commute with the involution and the antipode. The set X is invariant under the involution and, since $S^2 = \text{id}$ and S commutes with the involution, under S ; so $J^* = J$ and $S(J) \subset J$. Moreover $\varepsilon(b_k) = 0$ and

$$\Delta(b_k) = \sum_j v_{kj} \otimes \left(\sum_l v_{jl}\xi_l \right) - \xi_k 1 \otimes 1 = \sum_j v_{kj} \otimes b_j + b_k \otimes 1.$$

With $\Delta(a^*) = \Delta(a)^*$ and $\Delta(S(a)) = (S \otimes S)\Delta^{\text{op}}(a)$ this gives $\Delta(x) \in J \otimes \mathcal{O}(G) + \mathcal{O}(G) \otimes J$ for all $x \in X$, hence for all $x \in J$. So J is a Hopf $*$ -ideal contained in all $\ker \pi_i$, and $J = \{0\}$ by (C). Thus $b_k = 0$ for all k , that is, $\xi \in \text{Fix}_G(v)$. $\square$

Remark 2.4 (C^* -algebraic formulations). In [BCV17] a quantum subgroup is a surjective $*$ -homomorphism $C^u(G) \rightarrow C^u(H)$ of the universal C^* -algebras which intertwines the coproducts. This is the same notion. Such a map restricts to a Hopf $*$ -homomorphism $\mathcal{O}(G) \rightarrow \mathcal{O}(H)$ with dense image; the image is spanned by the coefficients of some irreducible representations of H , the coefficients of the other ones are orthogonal to it for the Haar state, which is continuous and faithful on $\mathcal{O}(H)$; so the image is $\mathcal{O}(H)$. Conversely a surjective Hopf $*$ -homomorphism $\mathcal{O}(G) \rightarrow \mathcal{O}(H)$ extends to the universal C^* -algebras. The same argument applies to the dual of a discrete group Γ with any C^* -completion of $\mathbb{C}[\Gamma]$ to which the coproduct extends (the full, the reduced or an intermediate one), the Haar state being the canonical trace: all of them lead to the surjective Hopf $*$ -homomorphisms $\mathcal{O}(O_N^+) \rightarrow \mathbb{C}[\Gamma]$. Representations, intertwiners and fixed vectors are the same objects in both pictures.

Remark 2.5 (there is only one copy of SO_N). The final problem refers to SO_N without specifying how it sits in O_N^+ ; we use the standard map, which is no restriction. A surjective Hopf $*$ -homomorphism $\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(SO_N)$ is given by a continuous homomorphism $\rho: SO_N \rightarrow O_N$ whose matrix coefficients generate $\mathcal{O}(SO_N)$; so ρ is injective, its image is a connected closed subgroup of the dimension of SO_N , hence SO_N , and ρ is an automorphism of SO_N . It is a standard fact, which we quote without proof, that every automorphism of the Lie group SO_N is the conjugation by an element of O_N (for the automorphisms of compact connected Lie groups see [Bou82, Chapitre 9]). This is so for every $N \geq 2$: for $N = 2$ the only non-trivial automorphism is the inversion, which is the conjugation by a reflection; and it includes $N = 8$, where the triality automorphisms of the covering group Spin_8 do not descend to SO_8 . Conjugation by $O \in O_N$, $u \mapsto O^t u O$, is a Hopf $*$ -automorphism of $\mathcal{O}(O_N^+)$ which preserves the kernel of $\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(SO_N)$, because SO_N is normal in O_N . So every such homomorphism has the same kernel as the standard one.

3 Group duals in O_N^+ and the cases $N = 2, 3$

Proposition 3.1 (group duals in O_N^+). *Let Γ be a discrete group.*

- (a) *Let $g_1, \dots, g_k \in \Gamma$ with $2k \leq N$ and $O \in O_N$. There is a unique Hopf $*$ -homomorphism $\pi: \mathcal{O}(O_N^+) \rightarrow \mathbb{C}[\Gamma]$ with*

$$O^t (\pi(u_{ij}))_{i,j} O = \text{diag}(R(g_1), \dots, R(g_k), 1, \dots, 1), \quad (4)$$

and its image is the group algebra of the subgroup generated by $g_1, \dots, g_k$.

- (b) Suppose that Γ has no element of order two, and let $\pi: \mathcal{O}(O_N^+) \rightarrow \mathbb{C}[\Gamma]$ be a surjective Hopf $*$ -homomorphism. Then there are $O \in O_N$, an integer k with $0 \leq 2k \leq N$ and elements $g_1, \dots, g_k$ generating Γ such that (4) holds.

Thus the quantum subgroups of O_N^+ which are duals of groups without elements of order two are the duals of such groups with at most $N/2$ generators, embedded by rotation blocks in a real orthonormal basis.

Proposition 3.1 follows from known results. It is the specialisation to O_N^+ and to groups without involutions of the following: by a theorem of Woronowicz [Wor87], finite-dimensional unitary corepresentations of group duals are direct sums of one-dimensional ones; the resulting group dual subgroups of a quantum group $G \subset U_N^+$ are studied by Banica, Bhowmick and De Commer [BBD12] and by Banica and Patri [BP17]: each of them is contained in a standard torus $\widehat{\Gamma}_Q$, $Q \in U_N$ ([BP17, Theorem 1.5], attributed there to Woronowicz), and the relations of the groups Γ_Q for easy quantum groups are described in [BP17, Theorem 6.5] (numbers of the arXiv version). We include the short direct proof.

Proof. (a) Let w be the matrix on the right of (4). Its entries are self-adjoint, and $R(g)R(g)^t = R(g)^t R(g) = 1$ because $C(g)$ and $S(g)$ commute and $C(g)^2 + S(g)^2 = 1$. So $u \mapsto OwO^t$ defines a $*$ -homomorphism π . From $C(g) + iS(g) = g$ and $C(g) - iS(g) = g^{-1}$ one gets $R(g)\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}g$ and $R(g)\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}g^{-1}$, so w is conjugate by a scalar matrix to $\text{diag}(g_1, g_1^{-1}, \dots, g_k, g_k^{-1}, 1, \dots, 1)$. Hence $\Delta(w_{ij}) = \sum_l w_{il} \otimes w_{lj}$ and $\varepsilon(w_{ij}) = \delta_{ij}$, the same holds for OwO^t , and π is a homomorphism of Hopf $*$ -algebras. Its image is generated by the entries of w , hence by the elements $C(g_j) \pm iS(g_j) = g_j^{\pm 1}$.

(b) Write $w = (\pi(u_{ij})) = \sum_g P_g \otimes g$ with matrices $P_g \in M_N(\mathbb{C})$, almost all zero. Comparing coefficients in $\Delta(w_{ij}) = \sum_l w_{il} \otimes w_{lj}$ and in $\varepsilon(w_{ij}) = \delta_{ij}$ gives $P_g P_h = \delta_{g,h} P_g$ and $\sum_g P_g = 1$. The relation $\sum_l w_{il}^* w_{lj} = \delta_{ij}$ gives, in the coefficient of e , $\sum_g P_g^* P_g = 1$; multiplying by P_h from the right we get $P_h = P_h^* P_h$, so P_h is self-adjoint. Thus the P_g are the orthogonal projections onto mutually orthogonal subspaces E_g with sum $\mathbb{C}^N$, and $w(\eta \otimes 1) = \eta \otimes g$ for $\eta \in E_g$. The non-zero P_g are linearly independent, so each g with $E_g \neq 0$ lies in the image of π ; the image is the group algebra of the subgroup generated by these g , which is therefore Γ . The relations $w_{ij}^* = w_{ij}$ give $P_{g^{-1}} = \overline{P_g}$, that is, $E_{g^{-1}} = \overline{E_g}$. In particular E_e is invariant under complex conjugation and has a real orthonormal basis. Let $g \neq e$ with $E_g \neq 0$. Then $g^{-1} \neq g$, so $\overline{E_g}$ is orthogonal to E_g . Choose an orthonormal basis $\eta_1, \dots, \eta_d$ of E_g and write $\eta_j = (\alpha_j + i\beta_j)/\sqrt{2}$ with real vectors α_j, β_j . The equations $\langle \eta_j, \eta_l \rangle = \delta_{jl}$ and $\langle \eta_j, \overline{\eta_l} \rangle = 0$ read

$$\begin{aligned} \langle \alpha_j, \alpha_l \rangle + \langle \beta_j, \beta_l \rangle + i(\langle \alpha_j, \beta_l \rangle - \langle \beta_j, \alpha_l \rangle) &= 2\delta_{jl}, \\ \langle \alpha_j, \alpha_l \rangle - \langle \beta_j, \beta_l \rangle - i(\langle \alpha_j, \beta_l \rangle + \langle \beta_j, \alpha_l \rangle) &= 0, \end{aligned}$$

and all brackets are real; hence $\langle \alpha_j, \alpha_l \rangle = \langle \beta_j, \beta_l \rangle = \delta_{jl}$ and $\langle \alpha_j, \beta_l \rangle = 0$. So $\alpha_1, \beta_1, \dots, \alpha_d, \beta_d$ is a real orthonormal basis of $E_g \oplus E_{g^{-1}}$. From $w(\eta_j \otimes 1) = \eta_j \otimes g$ and $w(\overline{\eta_j} \otimes 1) = \overline{\eta_j} \otimes g^{-1}$ we obtain

$$w(\alpha_j \otimes 1) = \alpha_j \otimes C(g) - \beta_j \otimes S(g), \quad w(\beta_j \otimes 1) = \alpha_j \otimes S(g) + \beta_j \otimes C(g),$$

which is the block $R(g)$ in the basis α_j, β_j . Doing this for one element of each pair $\{g, g^{-1}\}$ with $g \neq e$ and $E_g \neq 0$, and adding a real orthonormal basis of E_e , we obtain a real orthonormal basis of $\mathbb{C}^N$, the columns of a matrix $O \in O_N$, in which w has the form (4); the elements $g_1, \dots, g_k$ are the chosen g , repeated $\dim E_g$ times, and they generate Γ . $\square$

Proof of Theorem 1.1. By Proposition 3.1(b) with $2k \leq N \leq 3$, the group Γ is generated by at most one element g , and $O^t(\pi(u_{ij}))O$ is the block $R(g)$, completed by an entry 1 if $N = 3$, or the identity

matrix. So $\mathbb{C}[\Gamma]$ is commutative and π factors through the abelianisation $\mathcal{O}(O_N)$. In the commutative algebra $\mathbb{C}[\Gamma]$ the determinant of $(\pi(u_{ij}))$ is $C(g)^2 + S(g)^2 = 1$; so π maps $\det -1 \in \mathcal{O}(O_N)$ to 0 and factors through $\mathcal{O}(O_N)/(\det -1) = \mathcal{O}(SO_N)$. Hence $\ker(\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(SO_N))$ is a Hopf $*$ -ideal contained in $\ker \pi$, and it is the largest Hopf $*$ -ideal contained in both kernels: the generated quantum subgroup is SO_N . This ideal is not zero, because $\mathcal{O}(O_N^+)$ is not commutative for $N \geq 2$ (it has the non-commutative quotient $\mathbb{C}[\mathbb{Z}_2 * \mathbb{Z}_2]$). So (C) fails. $\square$

For $N = 2$ and a residually torsion-free nilpotent group Γ the last statement of Theorem 1.1 also follows from [FFS23]: by [FFS23, Proposition 4.5] a quantum group generated by the strongly connected quantum subgroups SO_2 and $\hat{\Gamma}$ is strongly connected, and O_2^+ is not [FFS23, Lemma 4.7].

4 The case $N = 4$

4.1 Conventions for $\mathfrak{sl}_2$

Let $V = \mathbb{C}^2$ with basis e_+, e_- , and let E, F, H be the operators with $Ee_- = e_+$, $Ee_+ = 0$, $Fe_+ = e_-$, $Fe_- = 0$, $He_{\pm} = \pm e_{\pm}$. They span $\mathfrak{sl}_2$ and act on tensor products by the Leibniz rule. For $t \in \{\pm 1\}^m$ put $e_t = e_{t_1} \otimes \dots \otimes e_{t_m}$. These vectors form a basis of $V^{\otimes m}$, which we take to be orthonormal; then $E^* = F$ and $H^* = H$. We have $He_t = (\sum_i t_i) e_t$, and y_t denotes the coefficient of e_t in $y \in V^{\otimes m}$. Let

$$I_m = \{y \in V^{\otimes m} : Ey = Hy = 0\}$$

be the space of invariants. The vector $\omega = e_+ \otimes e_- - e_- \otimes e_+$ is invariant. A *pairing* of $\{1, \dots, m\}$ is a partition into blocks of two elements; it is *noncrossing* if there are no $a < b < c < d$ with $\{a, c\}$ and $\{b, d\}$ among its pairs, and $\text{NC}_2(m)$ is the set of noncrossing pairings. For a pairing p let $T_p \in V^{\otimes m}$ be the tensor product of copies of ω , one in the legs i, j for each pair $\{i < j\}$ of p . Then $T_p \in I_m$ and

$$(T_p)_t = \prod_{\{i < j\} \in p} [t_i = -t_j] t_i, \quad (5)$$

where $[t_i = -t_j]$ is 1 if $t_i = -t_j$ and 0 otherwise. We call p *compatible* with t , and write $p \sim t$, if $t_i = -t_j$ for every pair $\{i, j\}$ of p ; so T_p is supported on $\{t : p \sim t\}$, with coefficients ± 1 . For $d \geq 1$ let $V_d \subset V^{\otimes d}$ be the space of symmetric tensors, the irreducible $\mathfrak{sl}_2$ -module of dimension $d+1$. It has the basis

$$|\tau\rangle = \sum \{e_t : t \in \{\pm 1\}^d, \sum_i t_i = \tau\}, \quad \tau \in \{d, d-2, \dots, -d\},$$

with $H|\tau\rangle = \tau|\tau\rangle$ and $F|\tau\rangle = (\frac{d-\tau}{2} + 1)|\tau-2\rangle$ for $\tau > -d$, $F|-d\rangle = 0$.

Lemma 4.1 (classical facts). *Let $m \geq 0$.*

- (a) *For $q \in \text{NC}_2(m)$ let $t^q \in \{\pm 1\}^m$ be +1 at the smaller and -1 at the larger element of each pair of q , and let $h_q(j)$ be the number of pairs $\{a < b\}$ of q with $a \leq j < b$. If $q' \in \text{NC}_2(m)$ is compatible with t^q and $q' \neq q$, then $\sum_j h_q(j) < \sum_j h_{q'}(j)$.*
- (b) *The vectors T_q , $q \in \text{NC}_2(m)$, form a basis of I_m .*
- (c) *If M is a finite-dimensional $\mathfrak{sl}_2$ -module and $u \in M$ is a non-zero vector with $Fu = 0$ and $Hu = \lambda u$, then $\lambda \leq 0$; if $\lambda = 0$, then $Eu = 0$.*
- (d) *For $d \geq 1$ the module V_d has no non-zero invariant vector.*

Proof. (a) For every j we have $\sum_{i \leq j} t_i^q = h_q(j)$: a pair of q inside $\{1, \dots, j\}$ contributes 0, and a pair $\{a \leq j < b\}$ contributes 1. Summing the same numbers along the pairs of q' gives at most $h_{q'}(j)$: a pair of q' inside $\{1, \dots, j\}$ contributes 0 by compatibility, and a pair $\{a \leq j < b\}$ contributes $t_a^q \leq 1$. So $h_q \leq h_{q'}$. If $h_q = h_{q'}$, then q and q' have the same set $\{j : h(j) - h(j-1) = 1\}$ of smaller elements, and $q' = q$, because a noncrossing pairing is determined by this set: every larger element is paired with the last smaller element before it that is still unpaired.

(b) The T_q are invariant. By (5), $(T_q)_{t^q} = 1$, and $(T_{q'})_{t^q} \neq 0$ only if $q' \sim t^q$. By (a) the matrix $((T_{q'})_{t^q})_{q, q'}$ is triangular with ones on the diagonal when $\text{NC}_2(m)$ is ordered by increasing $\sum_j h_q(j)$. So the T_q are linearly independent. By the Clebsch–Gordan rule $V_d \otimes V_1 \cong V_{d+1} \oplus V_{d-1}$ for $d \geq 1$, the dimension of I_m is the number of sequences $0 = d_0, d_1, \dots, d_m = 0$ of non-negative integers with $|d_j - d_{j-1}| = 1$. These sequences are the functions h_q , $q \in \text{NC}_2(m)$, so $\dim I_m = |\text{NC}_2(m)|$.

(c), (d) The module M is a direct sum of modules V_d , $d \geq 0$, where $V_0 = \mathbb{C}$ is the trivial module, and in V_d the kernel of F is spanned by the vector $|-d\rangle$ of weight $-d$. So $\lambda = -d$ for every summand V_d in which u has a non-zero component. Hence $\lambda \leq 0$, and if $\lambda = 0$, then u lies in a sum of trivial modules. Part (d) holds because V_d is irreducible and not trivial. $\square$

4.2 The model $\mathbb{C}^4 = V \otimes V'$

Let V' be a second copy of V , with basis $e'_\pm$, operators E', F', H' , invariants I'_m and vectors T'_p . In $\mathbb{C}^4$ we use the basis

$$f_a = \frac{1}{\sqrt{2}}(e_1 + ie_2), \quad f_{a^{-1}} = \frac{1}{\sqrt{2}}(e_1 - ie_2), \quad f_b = \frac{1}{\sqrt{2}}(e_3 + ie_4), \quad f_{b^{-1}} = \frac{1}{\sqrt{2}}(e_3 - ie_4),$$

indexed by the four letters $x \in \{a, a^{-1}, b, b^{-1}\}$. By the computation in the proof of Proposition 3.1(a), the matrix $w = (\pi_4(u_{ij})) = \text{diag}(R(a), R(b))$ satisfies $w(f_x \otimes 1) = f_x \otimes x$ for each letter x . By (3),

$$\text{Fix}_{\mathbb{F}_2}(u^{\otimes m}) = \text{span}\{f_{x_1} \otimes \dots \otimes f_{x_m} : x_1 x_2 \dots x_m = e \text{ in } \mathbb{F}_2\}. \quad (6)$$

Define a linear isomorphism $\Theta: V \otimes V' \rightarrow \mathbb{C}^4$ and letters $x(\sigma, \tau)$ for $\sigma, \tau \in \{\pm 1\}$ by

$$\Theta(e_+ \otimes e'_+) = f_a, \quad \Theta(e_- \otimes e'_-) = f_{a^{-1}}, \quad \Theta(e_+ \otimes e'_-) = f_b, \quad \Theta(e_- \otimes e'_+) = -f_{b^{-1}},$$

$$x(+, +) = a, \quad x(-, -) = a^{-1}, \quad x(+, -) = b, \quad x(-, +) = b^{-1},$$

so that $x(-\sigma, -\tau) = x(\sigma, \tau)^{-1}$. Let $\Theta_m: V^{\otimes m} \otimes V'^{\otimes m} \rightarrow (\mathbb{C}^4)^{\otimes m}$ be the isomorphism with $\Theta_m(e_s \otimes e'_t) = \Theta(e_{s_1} \otimes e'_{t_1}) \otimes \dots \otimes \Theta(e_{s_m} \otimes e'_{t_m})$, and for $s, t \in \{\pm 1\}^m$ let

$$W(s, t) = x(s_1, t_1) x(s_2, t_2) \dots x(s_m, t_m) \in \mathbb{F}_2.$$

Thus $\Theta_m(e_s \otimes e'_t) = \pm f_{x_1} \otimes \dots \otimes f_{x_m}$ with $x_i = x(s_i, t_i)$. For $N \geq 2$ and a pairing p of $\{1, \dots, m\}$ let $T_p^{(N)} \in (\mathbb{C}^N)^{\otimes m}$ be the tensor product of copies of $\sum_{i=1}^N e_i \otimes e_i$, one in the legs of each pair of p , and

$$\text{TL}_N(m) = \text{span}\{T_q^{(N)} : q \in \text{NC}_2(m)\}.$$

A noncrossing pairing has a pair of adjacent elements, and removing it leaves a noncrossing pairing. Since $\sum_i e_i \otimes e_i$ is fixed by $u^{\otimes 2}$, and since inserting a fixed vector into adjacent legs of a fixed vector gives a fixed vector, induction on m shows

$$\text{TL}_N(m) \subset \text{Fix}_{O_N^+}(u^{\otimes m}) \quad (N \geq 2, m \geq 0). \quad (7)$$

Lemma 4.2 (the model). *For every $m \geq 0$:*

- (a) $\Theta_m(I_m \otimes I'_m) = \text{Fix}_{SO_4}(u^{\otimes m})$;
- (b) $\Theta_m(T_p \otimes T'_p) = T_p^{(4)}$ for every pairing p ; hence $\Theta_m(D_m) = \text{TL}_4(m)$ for $D_m = \text{span}\{T_q \otimes T'_q : q \in \text{NC}_2(m)\}$;
- (c) $\Theta_m(\text{span}\{e_s \otimes e'_t : W(s, t) = e\}) = \text{Fix}_{\widehat{\mathbb{F}}_2}(u^{\otimes m})$.

Proof. (a) We abbreviate $e_\sigma \otimes e'_\tau$ to $e_\sigma e'_\tau$. Let β be the bilinear form on V with $\beta(e_+, e_-) = 1 = -\beta(e_-, e_+)$ and $\beta(e_\pm, e_\pm) = 0$; it is invariant under $\mathfrak{sl}_2$. Let B be the standard symmetric bilinear form of $\mathbb{C}^4$; then $B(f_x, f_{x^{-1}}) = 1$, and B vanishes on all other pairs of the vectors f_x . The form $\beta \otimes \beta'$ on $V \otimes V'$ is symmetric, and on the basis vectors its only non-zero values are

$$(\beta \otimes \beta')(e_+ e'_+, e_- e'_-) = 1 = B(f_a, f_{a^{-1}}), \quad (\beta \otimes \beta')(e_+ e'_-, e_- e'_+) = -1 = B(f_b, -f_{b^{-1}})$$

and the two values obtained by exchanging the arguments. So Θ is an isometry from $(V \otimes V', \beta \otimes \beta')$ onto $(\mathbb{C}^4, B)$. The Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ acts on $V \otimes V'$ by $X \otimes 1 + 1 \otimes Y$, faithfully, and preserves $\beta \otimes \beta'$. Hence $\Theta(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)\Theta^{-1}$ is a six-dimensional subalgebra of the Lie algebra $\mathfrak{so}_4(\mathbb{C})$ of B , which has dimension six, and the two coincide. The group SO_4 is connected, and the complexification of its Lie algebra is $\mathfrak{so}_4(\mathbb{C})$. By (2) a vector $x \in (\mathbb{C}^4)^{\otimes m}$ lies in $\text{Fix}_{SO_4}(u^{\otimes m})$ if and only if it is annihilated by $\mathfrak{so}_4(\mathbb{C})$, acting by the Leibniz rule, that is, if and only if $\Theta_m^{-1}(x)$ is annihilated by $X \otimes 1$ and $1 \otimes Y$ for all $X, Y \in \mathfrak{sl}_2$. The vectors with this property form the space $I_m \otimes I'_m$.

(b) For $m = 2$,

$$\begin{aligned} \Theta_2(\omega \otimes \omega') &= \Theta(e_+ e'_+) \otimes \Theta(e_- e'_-) - \Theta(e_+ e'_-) \otimes \Theta(e_- e'_+) \\ &\quad - \Theta(e_- e'_+) \otimes \Theta(e_+ e'_-) + \Theta(e_- e'_-) \otimes \Theta(e_+ e'_+) \\ &= f_a \otimes f_{a^{-1}} + f_b \otimes f_{b^{-1}} + f_{b^{-1}} \otimes f_b + f_{a^{-1}} \otimes f_a = \sum_{i=1}^4 e_i \otimes e_i, \end{aligned}$$

since $e_1 \otimes e_1 + e_2 \otimes e_2 = f_a \otimes f_{a^{-1}} + f_{a^{-1}} \otimes f_a$ and similarly for e_3, e_4 . The general case follows, because Θ_m acts leg by leg.

(c) This is (6), since $\Theta_m(e_s \otimes e'_t) = \pm f_{x_1} \otimes \dots \otimes f_{x_m}$ with $x_1 \dots x_m = W(s, t)$. $\square$

Proposition 4.3. *For every $m \geq 0$,*

$$X_m := (I_m \otimes I'_m) \cap \text{span}\{e_s \otimes e'_t : W(s, t) = e\} = D_m.$$

The proof of Proposition 4.3 occupies Sections 4.3 to 4.8.

Proof of Theorem 1.2, assuming Proposition 4.3. By (7), the inclusions between spaces of fixed vectors, Lemma 4.2 and Proposition 4.3,

$$\text{TL}_4(m) \subset \text{Fix}_{O_4^+}(u^{\otimes m}) \subset \text{Fix}_{SO_4}(u^{\otimes m}) \cap \text{Fix}_{\widehat{\mathbb{F}}_2}(u^{\otimes m}) = \Theta_m(X_m) = \Theta_m(D_m) = \text{TL}_4(m).$$

So all these spaces are equal, for every m , and Proposition 2.2 gives $O_4^+ = \langle SO_4, \widehat{\mathbb{F}}_2 \rangle$. $\square$

Corollary 4.4. $\text{Fix}_{O_4^+}(u^{\otimes m}) = \text{TL}_4(m)$ for all $m \geq 0$.

This is the case $N = 4$ of a theorem of Banica [Ban96] (see [BCV17, Theorem 2.1]): $\text{Fix}_{O_N^+}(u^{\otimes m}) = \text{TL}_N(m)$ for all $N \geq 2$. The proof of Theorem 1.2 uses only the elementary inclusion (7); Banica's theorem for $N = 4$ is not an ingredient but a by-product.

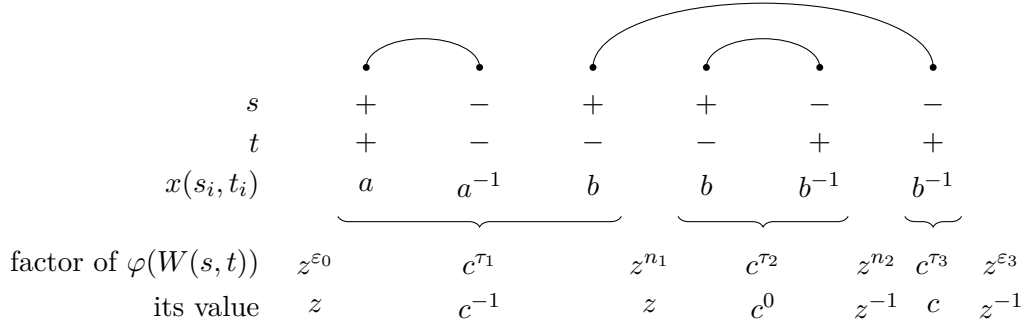


Figure 1: The sign vector $s = (+, -, +, +, -, -)$ with its blocks $B_1 = \{1, 2, 3\}$, $B_2 = \{4, 5\}$, $B_3 = \{6\}$, a vector t with $W(s, t) = e$, and the factors of $\varphi(W(s, t))$ according to Lemma 4.6; here $n_3 = \varepsilon_0 + \varepsilon_3 = 0$. The arcs show one of the two noncrossing pairings compatible with s ; it is also compatible with t . The other one is $\{1, 6\}, \{2, 3\}, \{4, 5\}$.

4.3 Invariants supported on one pairing

Lemma 4.5. *Let $q \in \text{NC}_2(m)$, and let $y \in I_m$ be such that $y_s = 0$ for every $s \in \{\pm 1\}^m$ which is not compatible with q . Then y is a multiple of T_q .*

Proof. Induction on m ; the case $m = 0$ is trivial. Let $\{i, i+1\}$ be a pair of q , and q° the noncrossing pairing of the remaining $m-2$ points. In the legs $i, i+1$ we have the decomposition $V \otimes V = \mathbb{C}\omega \oplus V_2$ into submodules. Accordingly $y = y^1 + y^2$ with invariant vectors $y^1 = \omega_{i,i+1} \otimes y'$, where $y' \in I_{m-2}$ and ω is placed in the legs $i, i+1$, and y^2 , which is symmetric in the legs $i, i+1$. Neither y nor y^1 has a component along a vector e_s with $s_i = s_{i+1}$, so neither has y^2 ; thus $y^2 = (e_+ \otimes e_- + e_- \otimes e_+)_{i,i+1} \otimes z$ for some $z \in V^{\otimes(m-2)}$. The part of Ey^2 with $e_+ \otimes e_+$ in the legs $i, i+1$ is $2(e_+ \otimes e_+)_{i,i+1} \otimes z$, and $Ey^2 = 0$; so $z = 0$ and $y = \omega_{i,i+1} \otimes y'$. The coefficient $y'_{s'}$ equals y_s for the vector s obtained from s' by inserting $(+, -)$ at the places $i, i+1$, and s is compatible with q only if s' is compatible with q° . By induction y' is a multiple of T_{q° , and $\omega_{i,i+1} \otimes T_{q^\circ} = T_q$. $\square$

4.4 The word $W(s, t)$ in terms of block weights

Let $F(c, z)$ be the free group on two generators c, z , and $\varphi: \mathbb{F}_2 \rightarrow F(c, z)$ the homomorphism with $\varphi(a) = zc$ and $\varphi(b) = zc^{-1}$. Let $m \geq 1$ and $s \in \{\pm 1\}^m$. The blocks $B_1 < \dots < B_r$ of s are the maximal intervals of $\{1, \dots, m\}$ on which consecutive entries of s differ. For $t \in \{\pm 1\}^m$ the block weights are $\tau_j(t) = \sum_{i \in B_j} t_i$. For $1 \leq j < r$ let $n_j = s_i$, where $i = \max B_j$, so that $s_i = s_{i+1} = n_j$. Finally let $\varepsilon_0 = 1$ if $s_1 = 1$ and $\varepsilon_0 = 0$ otherwise, $\varepsilon_r = -1$ if $s_m = -1$ and $\varepsilon_r = 0$ otherwise, and $n_r = \varepsilon_0 + \varepsilon_r \in \{-1, 0, 1\}$. See Figure 1.

Lemma 4.6. *Let $m \geq 1$ and $s, t \in \{\pm 1\}^m$, with the notation just introduced.*

- (a) *With $\tau_j = \tau_j(t)$, $\varphi(W(s, t)) = z^{\varepsilon_0} c^{\tau_1} z^{n_1} c^{\tau_2} z^{n_2} \dots z^{n_{r-1}} c^{\tau_r} z^{\varepsilon_r}$.*
- (b) *The homomorphism φ is injective.*
- (c) *$W(s, t) = e$ in $\mathbb{F}_2$ if and only if $c^{\tau_1} z^{n_1} c^{\tau_2} z^{n_2} \dots c^{\tau_r} z^{n_r} = e$ in $F(c, z)$. In particular, whether $W(s, t) = e$ depends on t only through the block weights $\tau_1(t), \dots, \tau_r(t)$.*

Proof. (a) From $\varphi(a) = zc$, $\varphi(b) = zc^{-1}$, $\varphi(a^{-1}) = c^{-1}z^{-1}$ and $\varphi(b^{-1}) = cz^{-1}$ we read off $\varphi(x(s_i, t_i)) = zc^{t_i}$ if $s_i = 1$ and $\varphi(x(s_i, t_i)) = c^{t_i}z^{-1}$ if $s_i = -1$. In the product, the power of z between c^{t_i} and $c^{t_{i+1}}$ is $z^{(s_i + s_{i+1})/2}$. It is trivial inside a block, where the powers of c therefore add up to c^{τ_j} , and it is z^{n_j} between B_j and B_{j+1} . The first factor contributes z^{ε_0} on the left, the last one z^{ε_r} on the right.

(b) Let $x_1 \cdots x_m$ be a non-empty reduced word in the letters $a^{\pm 1}, b^{\pm 1}$, and write $x_i = x(s_i, t_i)$. Since $x(-\sigma, -\tau) = x(\sigma, \tau)^{-1}$, the word being reduced means that $s_{i+1} = -s_i$ implies $t_{i+1} = t_i$. So t is constant on the blocks of s , all block weights are different from zero, and the right-hand side of (a) is a non-empty reduced word in c and z . Hence $\varphi(x_1 \cdots x_m) \neq e$.

(c) By (b), $W(s, t) = e$ if and only if $\varphi(W(s, t)) = e$; conjugate the word in (a) by z^{ε_0} . $\square$

4.5 Trivial words have trivial prefixes

Lemma 4.7. *Let $k \geq 1$, let $\tau_1, \dots, \tau_k$ be integers and $n_1, \dots, n_k$ non-zero integers with*

$$c^{\tau_1} z^{n_1} c^{\tau_2} z^{n_2} \cdots c^{\tau_k} z^{n_k} = e \quad \text{in } F(c, z).$$

Then $k \geq 2$, and there is an index i with $1 \leq i \leq k-1$ such that the prefix $P_i = c^{\tau_1} z^{n_1} \cdots z^{n_{i-1}} c^{\tau_i}$ is trivial; in particular $\tau_1 + \cdots + \tau_i = 0$.

Proof. Induction on k . If all τ_j are different from zero, the word is non-empty and reduced, hence not trivial. For $k = 1$ the word is not trivial either, since $n_1 \neq 0$. So $k \geq 2$ and some τ_j vanishes. If $\tau_1 = 0$, take $i = 1$. Otherwise choose $j \geq 2$ with $\tau_j = 0$, so that $z^{n_{j-1}} c^{\tau_j} z^{n_j} = z^{n_{j-1} + n_j}$.

Case $n_{j-1} + n_j \neq 0$. The word equals the word of the same type with the $k-1$ pairs $(\tau_1, n_1), \dots, (\tau_{j-2}, n_{j-2}), (\tau_{j-1}, n_{j-1} + n_j), (\tau_{j+1}, n_{j+1}), \dots, (\tau_k, n_k)$. By induction it has a trivial prefix $P_{i'}$ with $1 \leq i' \leq k-2$. If $i' \leq j-1$, then $P_{i'} = P_i$. If $i' \geq j$, then $P_{i'} = P_{i'+1}$ as elements of $F(c, z)$, and $i' + 1 \leq k-1$.

Case $n_{j-1} + n_j = 0$. If $j = k$, the word equals P_{k-1} , so $P_{k-1} = e$. If $j < k$, then $k \geq 3$ and $c^{\tau_{j-1}} z^{n_{j-1}} c^{\tau_j} z^{n_j} c^{\tau_{j+1}} = c^{\tau_{j-1} + \tau_{j+1}}$, so the word equals the word of the same type with the $k-2$ pairs $(\tau_1, n_1), \dots, (\tau_{j-2}, n_{j-2}), (\tau_{j-1} + \tau_{j+1}, n_{j+1}), (\tau_{j+2}, n_{j+2}), \dots, (\tau_k, n_k)$. By induction it has a trivial prefix $P_{i'}$ with $1 \leq i' \leq k-3$. If $i' \leq j-2$, then $P_{i'} = P_i$. If $i' \geq j-1$, then $P_{i'} = P_{i'+2}$ as elements of $F(c, z)$, and $i' + 2 \leq k-1$.

The last statement follows by counting the exponents of c . $\square$

4.6 The vanishing lemma

For integers $d_1, \dots, d_k \geq 1$ the module $V_{d_1} \otimes \cdots \otimes V_{d_k}$ has the basis $|\tau\rangle = |\tau_1\rangle \otimes \cdots \otimes |\tau_k\rangle$, $\tau_j \in \{d_j, d_j - 2, \dots, -d_j\}$. The *support* of a vector is the set of all τ whose basis vector has a non-zero coefficient.

Lemma 4.8 (vanishing lemma). *Let $k \geq 1$ and $d_1, \dots, d_k \geq 1$, let $n_1, \dots, n_{k-1}$ be non-zero integers and n_k an arbitrary integer. Let η be an invariant vector of $V_{d_1} \otimes \cdots \otimes V_{d_k}$ such that every τ in its support satisfies*

$$c^{\tau_1} z^{n_1} c^{\tau_2} z^{n_2} \cdots c^{\tau_k} z^{n_k} = e \quad \text{in } F(c, z). \quad (8)$$

Then $\eta = 0$.

Proof. Induction on k . For $k = 1$ this is Lemma 4.1(d). Let $k \geq 2$ and suppose that $\eta \neq 0$.

A general remark. Fix an ordering $o_1, \dots, o_k$ of the factors, and let τ^* be the largest element of the support of η for the lexicographic order which compares τ_{o_1} first, then τ_{o_2} , and so on. Let $1 \leq i \leq k-1$, and let u be the *partial coefficient* of η at $\tau_{o_1}^*, \dots, \tau_{o_i}^*$: the vector of the tensor product of the remaining factors $o_{i+1}, \dots, o_k$ obtained from η by keeping the terms $|\tau\rangle$ with $\tau_{o_p} = \tau_{o_p}^*$ for all $p \leq i$ and deleting the factors $o_1, \dots, o_i$ from them. Then $u \neq 0$, since τ^* contributes; u is a weight vector of weight $-(\tau_{o_1}^* + \cdots + \tau_{o_i}^*)$, since η has weight zero; and $Fu = 0$. To see this, take the same partial coefficient of the equation $F\eta = 0$. The part of F acting on the remaining factors contributes Fu . The part acting on a factor o_p , $p \leq i$, contributes a multiple of the partial coefficient

of η at $\tau_{o_1}^*, \dots, \tau_{o_{p-1}}^*, \tau_{o_p}^* + 2, \tau_{o_{p+1}}^*, \dots, \tau_{o_i}^*$, which is zero, because every weight with these entries is lexicographically larger than τ^* . Consequently, by Lemma 4.1(c): if $\tau_{o_1}^* + \dots + \tau_{o_i}^* = 0$, then u is a non-zero invariant vector of the tensor product of the factors $o_{i+1}, \dots, o_k$.

Case $n_k \neq 0$. Use the ordering $1, 2, \dots, k$. The word (8) of τ^* is trivial and all its exponents of z are different from zero, so by Lemma 4.7 there is i with $1 \leq i \leq k-1$, $P_i = c^{\tau_1^*} z^{n_1} \dots c^{\tau_i^*} = e$ and $\tau_1^* + \dots + \tau_i^* = 0$. Let u be the partial coefficient at $\tau_1^*, \dots, \tau_i^*$, a non-zero invariant vector of $V_{d_{i+1}} \otimes \dots \otimes V_{d_k}$. If τ'' is in the support of u , then $(\tau_1^*, \dots, \tau_i^*, \tau'')$ is in the support of η , so

$$e = P_i z^{n_i} c^{\tau''_{i+1}} z^{n_{i+1}} \dots c^{\tau''_k} z^{n_k} = z^{n_i} c^{\tau''_{i+1}} z^{n_{i+1}} \dots c^{\tau''_k} z^{n_k},$$

and after conjugation by z^{n_i} ,

$$c^{\tau''_{i+1}} z^{n_{i+1}} \dots c^{\tau''_{k-1}} z^{n_{k-1}} c^{\tau''_k} z^{n_k + n_i} = e.$$

So u satisfies the hypothesis of the lemma with the $k-i$ factors $d_{i+1}, \dots, d_k$, the non-zero exponents $n_{i+1}, \dots, n_{k-1}$ and the last exponent $n_k + n_i$. By induction $u = 0$, a contradiction.

Case $n_k = 0$. After conjugation by c^{τ_k} , condition (8) reads

$$c^{\tau_k + \tau_1} z^{n_1} c^{\tau_2} z^{n_2} \dots c^{\tau_{k-1}} z^{n_{k-1}} = e, \quad (9)$$

a word with $k-1$ pairs, all of whose exponents of z are different from zero. If $k=2$, no τ satisfies (9), by Lemma 4.7; so the support of η is empty, a contradiction. Let $k \geq 3$ and use the ordering $k, 1, 2, \dots, k-1$. By Lemma 4.7, applied to the word (9) of τ^* , there is i with $1 \leq i \leq k-2$, $c^{\tau_k^* + \tau_1^*} z^{n_1} \dots c^{\tau_i^*} = e$ and $\tau_k^* + \tau_1^* + \dots + \tau_i^* = 0$. Let u be the partial coefficient at $\tau_k^*, \tau_1^*, \dots, \tau_i^*$, a non-zero invariant vector of $V_{d_{i+1}} \otimes \dots \otimes V_{d_{k-1}}$, which has $k-1-i \geq 1$ factors. For τ'' in its support the word (9) of $(\tau_1^*, \dots, \tau_i^*, \tau'', \tau_k^*)$ is trivial, which gives as before

$$c^{\tau''_{i+1}} z^{n_{i+1}} \dots c^{\tau''_{k-2}} z^{n_{k-2}} c^{\tau''_{k-1}} z^{n_{k-1} + n_i} = e.$$

This is the hypothesis of the lemma with $k-1-i$ factors, the non-zero exponents $n_{i+1}, \dots, n_{k-2}$ and the last exponent $n_{k-1} + n_i$. By induction $u = 0$, a contradiction. $\square$

4.7 The slice lemma

For $s \in \{\pm 1\}^m$ let

$$\begin{aligned} \text{NC}(s) &= \{\sigma \in \text{NC}_2(m) : \sigma \sim s\}, & R_s &= \text{span}\{T_\sigma : \sigma \in \text{NC}(s)\}, \\ Y_s &= \{y \in I_m : y_t = 0 \text{ for every } t \text{ with } W(s, t) \neq e\}. \end{aligned}$$

Lemma 4.9 (slice lemma). $Y_s = R_s$ for every $m \geq 0$ and every $s \in \{\pm 1\}^m$.

Proof. $R_s \subset Y_s$. Let $\sigma \in \text{NC}(s)$ and $t \sim \sigma$. At the two ends of a pair $\{i, j\}$ of σ the letters of $W(s, t)$ are $x(s_i, t_i)$ and $x(-s_i, -t_i) = x(s_i, t_i)^{-1}$. Since σ is noncrossing, cancelling the letters of a pair of adjacent elements and repeating shows $W(s, t) = e$. So T_σ , which is supported on $\{t : t \sim \sigma\}$, lies in Y_s .

$Y_s \subset R_s$. We use induction on m . For $m=0$ both spaces are $\mathbb{C}$, and for odd m we have $I_m = 0$. Since $R_s \subset Y_s$, it suffices to show that a vector $y \in Y_s$ which is orthogonal to R_s vanishes. Let y be such a vector.

Step 1: caps. Let $1 \leq i \leq m-1$ with $s_i \neq s_{i+1}$. Let $\iota_i : V^{\otimes(m-2)} \rightarrow V^{\otimes m}$ be the map inserting ω in the legs $i, i+1$, and ι_i^* its adjoint. Both commute with E, F and H . For $t' \in \{\pm 1\}^{m-2}$ and $\sigma, \tau \in \{\pm 1\}$ let $t'[\sigma\tau]$ be t' with (σ, τ) inserted at the places $i, i+1$; then $(\iota_i^* y)_{t'} = y_{t'[\sigma\tau]} - y_{t'[-\sigma\tau]}$. Let s' be s with

the entries $i, i+1$ deleted. The letters of $W(s, t' [+ -])$ at the places $i, i+1$ are adjacent and mutually inverse, because $s_{i+1} = -s_i$; the same holds for $t' [- +]$. Hence $W(s, t' [+ -]) = W(s, t' [- +]) = W(s', t')$, and $\iota_i^* y \in Y_{s'}$. By induction $Y_{s'} = R_{s'}$. For $\sigma' \in \text{NC}(s')$ the vector $\iota_i T_{\sigma'}$ is T_σ for the pairing σ consisting of σ' and the pair $\{i, i+1\}$, and $\sigma \in \text{NC}(s)$ because $s_i \neq s_{i+1}$. So $\langle T_{\sigma'}, \iota_i^* y \rangle = \langle T_\sigma, y \rangle = 0$. Thus $\iota_i^* y$ lies in $R_{s'}$ and is orthogonal to it: $\iota_i^* y = 0$.

Step 2: symmetry. In $V \otimes V$ the orthogonal complement of ω is V_2 . So $\iota_i^* y = 0$ means that y is symmetric in the legs $i, i+1$. This holds whenever $s_i \neq s_{i+1}$, that is, whenever i and $i+1$ lie in the same block of s , and these transpositions generate the product of the symmetric groups of the blocks $B_1, \dots, B_r$ of s . Hence y is an invariant vector of $V_{d_1} \otimes \dots \otimes V_{d_r}$, $d_j = |B_j|$, and $y = \sum_\tau y_\tau |\tau_1\rangle \otimes \dots \otimes |\tau_r\rangle$ with $y_t = y_{\tau(t)}$ for all t , where $\tau(t) = (\tau_1(t), \dots, \tau_r(t))$.

Step 3. If $y_\tau \neq 0$, choose t with $\tau(t) = \tau$. Then $y_t \neq 0$, so $W(s, t) = e$, and by Lemma 4.6(c) $c^{\tau_1} z^{n_1} \dots c^{\tau_r} z^{n_r} = e$ with $n_1, \dots, n_{r-1} \in \{\pm 1\}$ and $n_r \in \{-1, 0, 1\}$. So Lemma 4.8 applies with $k = r$ and gives $y = 0$. $\square$

4.8 Proof of Proposition 4.3

By Lemma 4.2(b),(c) and (7), or directly from the first part of the proof of Lemma 4.9, we have $D_m \subset X_m$. Conversely let $x \in X_m$. Since the vectors T'_q , $q \in \text{NC}_2(m)$, form a basis of I'_m , we can write $x = \sum_q y_q \otimes T'_q$ with unique vectors $y_q \in I_m$. Fix $s \in \{\pm 1\}^m$. The coefficient of e_s in x is the vector $x_s = \sum_q (y_q)_s T'_q \in I'_m$, and its coefficient at e'_t is $x_{s,t}$, which vanishes unless $W(s, t) = e$. By Lemma 4.9, applied to the copy V' , the vector x_s lies in $\text{span}\{T'_\sigma : \sigma \in \text{NC}(s)\}$. By the linear independence of the T'_q , $(y_q)_s = 0$ whenever q is not compatible with s . This holds for all s , so Lemma 4.5 gives $y_q = \lambda_q T_q$ with scalars λ_q , and $x = \sum_q \lambda_q T_q \otimes T'_q \in D_m$. $\square$

4.9 Remarks on Theorem 1.2

Example 4.10 ($m = 4$). By Lemma 4.2(a), $\text{Fix}_{SO_4}(u^{\otimes 4})$ has dimension $(\dim I_4)^2 = 4$. It contains the vectors $T_p^{(4)}$ for the three pairings $12|34$, $14|23$, $13|24$, and the determinant tensor $D = \sum_{\sigma \in S_4} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(4)}$, which is -1 times the signed sum of the 24 permutations of the factors of $f_a \otimes f_{a-1} \otimes f_b \otimes f_{b-1}$. At the two basis vectors $f_a \otimes f_b \otimes f_{a-1} \otimes f_{b-1}$ and $f_a \otimes f_{b-1} \otimes f_{a-1} \otimes f_b$, whose words are not trivial, the coefficients are $1, 1$ for $T_{13|24}^{(4)}$, $1, -1$ for D , and $0, 0$ for the other two vectors. By (6) a linear combination of the four vectors which is fixed by $\widehat{\mathbb{F}}_2$ involves neither $T_{13|24}^{(4)}$ nor D . So the four vectors form a basis of $\text{Fix}_{SO_4}(u^{\otimes 4})$, and its intersection with $\text{Fix}_{\widehat{\mathbb{F}}_2}(u^{\otimes 4})$ is $\text{TL}_4(4)$, in accordance with Theorem 1.2. The tensor D is fixed by SO_4 and by the classical version of $\widehat{\mathbb{F}}_2$, a torus in SO_4 , but not by $\widehat{\mathbb{F}}_2$. This is how SO_4 and $\widehat{\mathbb{F}}_2$ can generate a quantum group whose classical version O_4 contains reflections.

Remark 4.11 (relation to known generation theorems). (1) Brannan, Collins and Vergnioux proved $O_4^+ = \langle S_4, O_2^+ \hat{*} O_2^+ \rangle$, where $S_4 \subset O_4$ is the group of permutation matrices [BCV17, Theorem 4.2]. In Theorem 1.2 the second generator $\widehat{\mathbb{F}}_2$ is a quantum subgroup of $O_2^+ \hat{*} O_2^+$, and the first one is connected.

(2) *The structure of Theorem 1.2.* Let v be the fundamental matrix of the free unitary quantum group U_2^+ . In the basis $f_a, f_b, f_{a-1}, f_{b-1}$ of $\mathbb{C}^4$ the matrix $v \oplus \bar{v}$ defines a quantum subgroup $U_2^+ \leq O_4^+$. Its diagonal torus is the copy of $\widehat{\mathbb{F}}_2$ given by (1), and its classical version U_2 lies in SO_4 . For a sign vector s , the space $\Theta_m(e_s \otimes V^{\otimes m})$ carries the representation $v^{s_1} \otimes \dots \otimes v^{s_m}$ of U_2^+ , where $v^+ = v$ and $v^- = \bar{v}$. Its vectors fixed by U_2 are those of $\Theta_m(e_s \otimes I'_m)$ if $\sum_i s_i = 0$, and only the zero vector otherwise; its vectors fixed by $\widehat{\mathbb{F}}_2$ are spanned by the $\Theta_m(e_s \otimes e'_t)$ with $W(s, t) = e$; and $\Theta_m(e_s \otimes R'_s)$ consists of vectors fixed by U_2^+ , being spanned by the vectors of the noncrossing pairings which match

the colours. Hence Lemma 4.9 is the fixed-vector form of the statement $U_2^+ = \langle U_2, \widehat{\mathbb{F}}_2 \rangle$, which is the case $N = 2$ of [Chi20, Theorem 3.1]. Given this statement, Theorem 1.2 amounts to $O_4^+ = \langle SO_4, U_2^+ \rangle$, and this is what Lemma 4.5 and Section 4.8 prove.

(3) *The vanishing lemma.* Let $\mathbb{T}$ be the circle group and $\mathcal{O}(\mathbb{T} \hat{*} G) = \mathcal{O}(\mathbb{T}) * \mathcal{O}(G)$. By Wang's description of the irreducible representations of such dual free products [Wan95], those of $\mathbb{T} \hat{*} SU_2$ are the products $z^{n_0} \otimes V_{d_1} \otimes z^{n_1} \otimes \dots \otimes V_{d_k} \otimes z^{n_k}$ with $d_j \geq 1$ and $n_j \neq 0$ for $0 < j < k$, where z generates $\mathcal{O}(\mathbb{T})$. The vectors of such a representation fixed by SU_2 are the invariant vectors of $V_{d_1} \otimes \dots \otimes V_{d_k}$, and those fixed by $\mathbb{T} \hat{*} T$, for the maximal torus T of SU_2 with generating character c , are spanned by the $|\tau\rangle$ with $z^{n_0} c^{\tau_1} z^{n_1} \dots c^{\tau_k} z^{n_k} = e$. After conjugation by z^{n_0} one sees that Lemma 4.8 is exactly the fixed-vector form of the case $G = SU_2$ of [Chi20, Theorem 3.4]: $\mathbb{T} \hat{*} G = \langle \mathbb{T} \hat{*} T, G \rangle$ for a compact connected Lie group G with maximal torus T . In [Chi20], Theorem 3.1 for $N = 2$ is deduced from Theorem 3.4 and from Banica's embedding of $\mathcal{O}(U_2^+)$ into $\mathcal{O}(\mathbb{T}) * \mathcal{O}(SU_2)$ [Ban97]; and in [Chi19, Section 3] Theorem 3.4 is obtained as a consequence of a general theorem on free products, [Chi19, Theorem 2.2]. (Numbers refer to arXiv:1904.03974v1 and arXiv:1904.13190v1, the only arXiv versions of [Chi20] and [Chi19].)

The statements of Lemmas 4.8 and 4.9 are thus due to Chirvasitu. We give complete proofs, because the arguments for these statements in the two arXiv texts use an identification of a subalgebra with a free product which does not hold in general; see Remark 4.12. The results of the literature which depend on the theorem $U_N^+ = \langle U_N, \widehat{\mathbb{F}}_N \rangle$, namely [Chi20, Theorem 3.3], [FFS23, Proposition 4.6] and the remark of [FFS23] on the quantum automorphism group of $(M_N(\mathbb{C}), \text{tr})$, are not used in the proofs of this note; they appear only in the account of what was known in Section 1.3 and in the discussion of consistency in Section 7.1. The base case $N = 2$ of that theorem is, in its fixed-vector form, Lemma 4.9.

Remark 4.12 (on two proofs in the arXiv versions). In arXiv:1904.03974v1 the proof of [Chi20, Theorem 3.4] chooses maximal tori $T_1, \dots, T_k$ of G for which the product map from $T_1 \times \dots \times T_k$ to G is onto. It considers the iterated coproduct followed by the restrictions to the tori,

$$\mathcal{O}(\mathbb{T}) * \mathcal{O}(G) \longrightarrow (\mathcal{O}(\mathbb{T}) * \mathcal{O}(T_1)) \otimes \dots \otimes (\mathcal{O}(\mathbb{T}) * \mathcal{O}(T_k)). \quad (10)$$

It identifies the algebra generated in the target by the diagonal copy of $\mathcal{O}(\mathbb{T})$ and by $\mathcal{O}(T_1) \otimes \dots \otimes \mathcal{O}(T_k)$ with the free product of these two algebras, and concludes that (10) is injective.

(i) For $k \geq 2$ and $G \neq \{e\}$ the canonical map from the free product $\mathcal{O}(\mathbb{T}) * (\mathcal{O}(T_1) \otimes \dots \otimes \mathcal{O}(T_k))$ onto that algebra is not injective. Let z be the generator of $\mathcal{O}(\mathbb{T})$ and l_1, l_2 non-trivial characters of T_1, T_2 . The elements $(z \otimes z)(l_1 \otimes 1)(z \otimes z)^{-1} = z l_1 z^{-1} \otimes 1$ and $1 \otimes l_2$ of the target commute (we write two tensor factors only), whereas $z l_1 z^{-1} \cdot l_2$ and $l_2 \cdot z l_1 z^{-1}$ are different reduced words of a free product of groups, hence different elements of the free product of the algebras.

(ii) Let $G = SU_2$, $k = 3$, $T_1 = T_3 = T_z$ and $T_2 = T_x$, the maximal tori which cover the rotations about the z -axis and the x -axis. Their product map is onto (Euler angles), but the map (10) is not injective. To see this, let ψ_+, ψ_- be the two diagonal coefficients of V_1 and ψ_0 the diagonal coefficient of V_2 at its vector of weight zero, in bases of weight vectors for T_z , and let c be the generating character of T_z . The iterated coproduct of the diagonal coefficient ψ at a vector of weight τ , restricted to T_z in the first and in the third factor, is $c^\tau \otimes \psi \otimes c^\tau$; and ψ_+ and ψ_- have the same restriction ψ' to T_x , since the elements of T_x have equal diagonal entries. Hence (10) maps both $\psi_+ z \psi_0 z^{-1} \psi_-$ and $\psi_- z \psi_0 z^{-1} \psi_+$ to $1 \otimes \psi' z \psi'_0 z^{-1} \psi' \otimes 1$, where ψ'_0 is the restriction of ψ_0 to T_x . These two elements are different coefficients of the irreducible representation $\alpha = V_1 \otimes z \otimes V_2 \otimes z^{-1} \otimes V_1$, of dimension 12; so their difference is a non-zero element of the kernel.

More precisely, on the coefficient space of an irreducible representation α of $\mathbb{T} \hat{*} SU_2$ the map (10) is injective if and only if the products $P_{\gamma_1}^1 \dots P_{\gamma_k}^k$ span the algebra of all endomorphisms of the space of α , where the P_γ^i are the projections onto the isotypic components of the restriction

of α to $\mathbb{T} \hat{*} T_i$. For the representation α above and the tori (T_z, T_x, T_z) , exact computations in rational arithmetic give a span of dimension 142, not 144 (programs `vb_chirvasitu_eq8_exact.py` and `r2_tori.py` of the reproducibility package, written independently of each other; a third program, `r2_tori_direct.py`, computes the map from its definition and finds a kernel of dimension two on this coefficient space). With the four tori (T_z, T_x, T_z, T_x) the span is everything for this α . With four, five and six tori alternating between T_z and T_x it is everything on each of the 20 representations tested, of dimension up to 125 (ranks modulo a prime), and with four to ten random tori no deficit was found on the representations tested in floating-point arithmetic. The same identification is used in the proof of [Chi19, Proposition 2.3], with the group of order two in place of $\mathbb{T}$; for the representation $V_1 \otimes t \otimes V_2 \otimes t \otimes V_1$, where t generates this group, and the tori (T_z, T_x, T_z) the same computations give 142 of 144 (programs `vb_chirvasitu_z2_exact.py`, `r2_tori.py`).

(iii) This concerns the arXiv versions arXiv:1904.03974v1 and arXiv:1904.13190v1, which are the texts we read; the published version of [Chi20] was not available to us. Nothing indicates that the statements themselves fail, and Lemmas 4.8 and 4.9 prove the cases needed here.

5 The case $N \geq 5$ and the answer to the final problem

For $2 \leq j \leq N$ let $O_j^+ \leq O_N^+$ be the quantum subgroup given by $u \mapsto u_j \oplus 1_{N-j}$, where u_j is the fundamental matrix of O_j^+ , placed in the first j coordinates; similarly $SO_j \subset O_j \subset O_N$. In the notation of [BCV17], $O_{N-1}^+ \leq O_N^+$ is the stabiliser $O_{N-1}^{+, \xi}$ of the unit vector $\xi = e_N$. We shall use the following remark.

Restriction. Let $H \leq O_j^+ \leq O_N^+$. The space $(\mathbb{C}^N)^{\otimes m}$ is the direct sum of the subspaces $M_{L,c}$, where $L \subset \{1, \dots, m\}$ and $c: \{1, \dots, m\} \setminus L \rightarrow \{j+1, \dots, N\}$: the tensors with a vector of $\mathbb{C}^j$ in each leg of L and the basis vector $e_{c(i)}$ in each leg $i \notin L$. The restriction of $u^{\otimes m}$ to H preserves each $M_{L,c} \cong (\mathbb{C}^j)^{\otimes |L|}$ and acts on it as $u_j^{\otimes |L|}$. Hence a vector x lies in $\text{Fix}_H(u^{\otimes m})$ if and only if each of its components lies in $\text{Fix}_H(u_j^{\otimes |L|})$.

Theorem 5.1 (Brannan, Collins and Vergnioux [BCV17, Theorem 4.1(1)]). *Let $N \geq 4$. Then $O_N^+ = \langle O_N, O_{N-1}^+ \rangle$; more precisely, for every $k \geq 0$,*

$$\text{Fix}_{O_N}(u^{\otimes k}) \cap \text{Fix}_{O_{N-1}^+}(u^{\otimes k}) = \text{Fix}_{O_N^+}(u^{\otimes k}).$$

In [BCV17] the theorem is stated for the stabiliser of every real unit vector, and it is proved in the form of the displayed equality (the number is that of the arXiv version; the published version was not consulted). It is the one external theorem on which the case $N \geq 5$ of Theorem 1.3 rests. For the convenience of the reader, Appendix A recalls its proof for $N \geq 5$.

Proof of Theorem 1.3. The quantum subgroups SO_4 and $\widehat{\mathbb{F}}_2$ of O_N^+ , the latter given by π_N , are contained in $O_4^+ \leq O_N^+$, where they are the quantum subgroups of Theorem 1.2. Let $x \in \text{Fix}_{SO_N}(u^{\otimes m}) \cap \text{Fix}_{\widehat{\mathbb{F}}_2}(u^{\otimes m})$.

Step 1. The vector x is fixed by $SO_4 \subset SO_N$ and by $\widehat{\mathbb{F}}_2$. By Theorem 1.2, applied to each component of x as in the remark on restriction, $x \in \text{Fix}_{O_4^+}(u^{\otimes m})$.

Step 2. In particular x is fixed by $O_4 \leq O_4^+$, hence by the reflection $\text{diag}(-1, 1, \dots, 1)$. Being fixed by SO_N as well, x is fixed by O_N .

Step 3. Suppose that $x \in \text{Fix}_{O_j^+}(u^{\otimes m})$ for some j with $4 \leq j < N$. Then x is fixed by $O_{j+1} \subset O_N$ and by $O_j^+ \leq O_{j+1}^+$. By Theorem 5.1 for O_{j+1}^+ , where $j+1 \geq 5$, applied to each component of x , we get $x \in \text{Fix}_{O_{j+1}^+}(u^{\otimes m})$. By induction on j , $x \in \text{Fix}_{O_N^+}(u^{\otimes m})$.

This proves the equality of fixed vectors, and Proposition 2.2 gives $O_N^+ = \langle SO_N, \widehat{\mathbb{F}}_2 \rangle$. For $2 \leq k \leq N/2$, the quantum subgroup $\widehat{\mathbb{F}}_2$ is contained in the dual of $\mathbb{F}_k$ embedded by k rotation blocks (send the other free generators to e), so the fixed vectors of the latter are fixed by $\widehat{\mathbb{F}}_2$, and the same conclusions hold. $\square$

Lemma 5.2 (Magnus). *Let $\mathfrak{M}$ be the algebra of formal power series in two non-commuting variables X, Y over $\mathbb{C}$, let $\mathfrak{m}$ be its ideal of series without constant term, and let $\mu: \mathbb{C}[\mathbb{F}_2] \rightarrow \mathfrak{M}$ be the homomorphism with $\mu(a) = 1 + X$ and $\mu(b) = 1 + Y$. Let I be the augmentation ideal of $\mathbb{C}[\mathbb{F}_2]$.*

(a) μ is injective.

(b) $\mu(I^n) \subset \mathfrak{m}^n$ for all n ; hence $\bigcap_n I^n = \{0\}$.

(c) $\mathbb{F}_2$ is residually torsion-free nilpotent.

Part (c) is the theorem of Magnus [Mag35] for $\mathbb{F}_2$, and (a), (b) are classical as well; with the theorem of Passi quoted in [FFS23, Proposition 4.5], (b) and (c) are equivalent. We include the short proofs.

Proof. For an integer ℓ we have $(1 + X)^\ell = \sum_{k \geq 0} \binom{\ell}{k} X^k$, and for $k \geq 1$ the binomial coefficient $\binom{\ell}{k}$ is a polynomial of degree k in ℓ which vanishes at $\ell = 0$. Every $w \neq e$ in $\mathbb{F}_2$ has a unique reduced form $w = x_1^{\ell_1} \cdots x_r^{\ell_r}$ with $x_i \in \{a, b\}$, $x_i \neq x_{i+1}$ and $\ell_i \neq 0$; we call $(x_1, \dots, x_r)$ its pattern, $(\ell_1, \dots, \ell_r)$ its exponent vector, and put $r(w) = r$, $r(e) = 0$. Let $X_i \in \{X, Y\}$ be the variable of x_i , and let $M = X_1^{k_1} \cdots X_r^{k_r}$ with all $k_i \geq 1$. The monomial M has r maximal runs of equal letters, and a product of r' powers of single variables has at most r' such runs. Hence the coefficient of M in $\mu(w')$ is $\prod_i \binom{\ell'_i}{k_i}$ if w' has the pattern $(x_1, \dots, x_r)$ and the exponent vector ℓ' , and it is zero for all other w' with $r(w') \leq r$.

(a) Let $f = \sum_w c_w w \neq 0$ with $\mu(f) = 0$, and let r be the largest value of $r(w)$ with $c_w \neq 0$. If $r = 0$, then $\mu(f)$ is a non-zero constant. Let $r \geq 1$, fix a pattern occurring among the w with $r(w) = r$ and $c_w \neq 0$, and let $\mathcal{E} \subset (\mathbb{Z} \setminus \{0\})^r$ be the finite set of the exponent vectors of these w ; write c_ℓ for the coefficient of the word with exponent vector $\ell \in \mathcal{E}$. The coefficient of M in $\mu(f)$ gives $\sum_{\ell \in \mathcal{E}} c_\ell \prod_i \binom{\ell_i}{k_i} = 0$ for all $k_1, \dots, k_r \geq 1$. Fix $\ell^0 \in \mathcal{E}$, and let $\mathcal{E}_i$ be the set of i -th entries of the elements of $\mathcal{E}$. The polynomials $\binom{\ell}{k}$, $1 \leq k \leq |\mathcal{E}_i|$, span the polynomials of degree at most $|\mathcal{E}_i|$ in ℓ vanishing at 0; so a linear combination p_i of them satisfies $p_i(\ell_i^0) = 1$ and $p_i(\ell) = 0$ for the other $\ell \in \mathcal{E}_i$. The corresponding combination of the equations is $c_{\ell^0} = \sum_{\ell \in \mathcal{E}} c_\ell \prod_i p_i(\ell_i) = 0$, a contradiction.

(b) For $g \in \mathbb{F}_2$ the series $\mu(g)$ has constant term 1, so $\mu(g - 1) \in \mathfrak{m}$; the elements $g - 1$ span I . Hence $\mu(I^n) \subset \mathfrak{m}^n$ and $\mu(\bigcap_n I^n) \subset \bigcap_n \mathfrak{m}^n = \{0\}$. Apply (a).

(c) The series with constant term 1 form a group $1 + \mathfrak{m}$, and $D_n = \{w \in \mathbb{F}_2 : \mu(w) \in 1 + \mathfrak{m}^n\}$ is a normal subgroup of $\mathbb{F}_2$ with $\bigcap_n D_n = \{e\}$, by (a). The quotient $\mathbb{F}_2/D_n$ embeds into the group $(1 + \mathfrak{m})/(1 + \mathfrak{m}^n)$. This group is nilpotent, because the commutator of elements of $1 + \mathfrak{m}^i$ and $1 + \mathfrak{m}^j$ lies in $1 + \mathfrak{m}^{i+j}$. It is torsion-free: if the non-zero homogeneous part of lowest degree of $v \in \mathfrak{m} \setminus \mathfrak{m}^n$ is v_0 , then $(1 + v)^r$ is $1 + rv_0$ plus terms of higher degree, for every integer r . $\square$

Proof of Corollary 1.4. A residually torsion-free nilpotent group is torsion-free: a non-trivial element of finite order would have a non-trivial image of finite order in a torsion-free quotient. So for $N = 2, 3$ the answer is negative by Theorem 1.1. For $N \geq 4$ the group $\mathbb{F}_2$ has the required properties by Theorem 1.3 and Lemma 5.2(c). $\square$

6 The nilpotent residual

Let A be a bialgebra over $\mathbb{C}$, with counit ε ; any field of characteristic zero would do. As in [FFS23] we write $K = \ker \varepsilon$, $K_0 = A$, $K_n = K^n$ for the span of the products of n elements of K , and $K_\infty = \bigcap_n K_n$; thus $K_\infty = I^\infty$. Let $\text{gr}_n = K_n/K_{n+1}$, and let $[K, K]$ be the span of the commutators $[a, b]$ with $a, b \in K$.

Lemma 6.1. *If $[K, K] \subset K_3$, then $[K, K] \subset K_n$ for all n . Hence $[A, A] \subset K_\infty$, and A/K_∞ is commutative.*

Proof. Step 1: the graded algebra. For $a \in K$ let $\delta(a) = \Delta(a) - a \otimes 1 - 1 \otimes a$. It is annihilated by $\varepsilon \otimes \text{id}$ and by $\text{id} \otimes \varepsilon$. Since $A = \mathbb{C}1 \oplus K$, the intersection of the kernels $K \otimes A$ and $A \otimes K$ of these maps is $K \otimes K$; so $\delta(a) \in K \otimes K$. Let $F_n = \sum_{i+j=n} K_i \otimes K_j \subset A \otimes A$. Then $\Delta(K) \subset F_1$, the product of n elements of $F_1 = K \otimes A + A \otimes K$ lies in F_n because K is an ideal, and therefore $\Delta(K_n) \subset F_n$. Moreover F_n/F_{n+1} is canonically isomorphic to $\bigoplus_{i+j=n} \text{gr}_i \otimes \text{gr}_j$: choose complements C_i of K_{i+1} in K_i for $0 \leq i \leq n$; modulo $K_{n+1} \otimes A + A \otimes K_{n+1} \subset F_{n+1}$ the space F_n becomes $\bigoplus_{i+j \geq n} C_i \otimes C_j$ and F_{n+1} becomes $\bigoplus_{i+j \geq n+1} C_i \otimes C_j$, the sums being over $i, j \leq n$. So Δ induces a homomorphism of graded algebras $\bar{\Delta}: \text{gr } A \rightarrow \text{gr } A \otimes \text{gr } A$, where $\text{gr } A = \bigoplus_n \text{gr}_n$. The algebra $\text{gr } A$ is generated by gr_1 , and the elements of gr_1 are *primitive*: $\bar{\Delta}(\bar{a}) = \bar{a} \otimes 1 + 1 \otimes \bar{a}$, because $\delta(a) \in K \otimes K \subset F_2$.

Step 2: primitive elements. Assume $[K, K] \subset K_3$. Then the elements of gr_1 commute in $\text{gr } A$, so $\text{gr } A$ is commutative and there is a surjective homomorphism of graded algebras $\theta: \text{Sym}(\text{gr}_1) \rightarrow \text{gr } A$ which is the identity on gr_1 . Let Δ_S be the coproduct of the symmetric algebra $\text{Sym} = \text{Sym}(\text{gr}_1)$ for which gr_1 is primitive. Then $(\theta \otimes \theta)\Delta_S = \bar{\Delta}\theta$, since both sides are algebra homomorphisms which agree on gr_1 . For a homogeneous element f of degree n of Sym let $f_{(1, n-1)}$ be the component of bidegree $(1, n-1)$ of $\Delta_S(f)$; multiplying its two legs gives nf , as one sees for a product $f = v_1 \cdots v_n$ of elements of gr_1 .

The map θ is injective. Otherwise let $f \neq 0$ be a homogeneous element of minimal degree n of $J = \ker \theta$. Then $n \geq 2$, and $\Delta_S(f)$ lies in $\ker(\theta \otimes \theta) = J \otimes \text{Sym} + \text{Sym} \otimes J$, whose part of bidegree $(1, n-1)$ is zero, because J vanishes in the degrees 1 and $n-1$. So $f_{(1, n-1)} = 0$ and $nf = 0$, a contradiction.

A primitive element $\bar{c}$ of $\text{gr } A$ of degree $n \geq 2$ is zero. Let $f = \theta^{-1}(\bar{c})$. Then $(\theta \otimes \theta)(\Delta_S(f) - f \otimes 1 - 1 \otimes f) = 0$, so $\Delta_S(f) = f \otimes 1 + 1 \otimes f$, since $\theta \otimes \theta$ is injective. Hence $f_{(1, n-1)} = 0$, $nf = 0$ and $\bar{c} = 0$.

Step 3: induction. Assume $[K, K] \subset K_n$ for some $n \geq 3$. Let $a, b \in K$, $c = [a, b] \in K_n$, and write $\delta(a) = \sum a' \otimes a''$, $\delta(b) = \sum b' \otimes b''$ with $a', a'', b', b'' \in K$. Since $\Delta(c) = [\Delta(a), \Delta(b)]$,

$$\Delta(c) - c \otimes 1 - 1 \otimes c = [a \otimes 1 + 1 \otimes a, \delta(b)] + [\delta(a), b \otimes 1 + 1 \otimes b] + [\delta(a), \delta(b)].$$

Here $[a \otimes 1, b' \otimes b''] = [a, b'] \otimes b'' \in K_n \otimes K$ and $[1 \otimes a, b' \otimes b''] = b' \otimes [a, b''] \in K \otimes K_n$, similarly for the second bracket, and $[a' \otimes a'', b' \otimes b''] = [a', b'] \otimes a''b'' + b'a' \otimes [a'', b''] \in K_n \otimes K_2 + K_2 \otimes K_n$. All these terms lie in F_{n+1} . So the class of c in gr_n is primitive, hence zero by Step 2: $c \in K_{n+1}$. Thus $[K, K] \subset K_{n+1}$, and by induction $[K, K] \subset K_n$ for all n . Finally $[A, A] = [\mathbb{C}1 + K, \mathbb{C}1 + K] = [K, K]$. $\square$

The proof uses neither an antipode nor any finiteness. The lemma is an elementary instance of the classical principle that a connected graded bialgebra generated by primitive elements is governed, in characteristic zero, by the Lie algebra of its primitive elements (Milnor and Moore [MM65]); for group algebras compare Quillen's theorem on the associated graded ring of a group ring [Qui68]. We do not use these results.

Proof of Corollary 1.5. Let $A = \mathcal{O}(O_N^+)$. The ideal K_∞ is a Hopf ideal [FFS23, Proposition 4.1], and it is invariant under the involution, because $K^* = K$; so it is a Hopf $*$ -ideal. A quantum subgroup

$\pi: A \rightarrow \mathcal{O}(H)$ maps $K_n(A)$ into $K_n(\mathcal{O}(H))$, hence $K_\infty(A)$ into $K_\infty(\mathcal{O}(H))$. By [FFS23, Lemma 4.2], $K_\infty(\mathcal{O}(G)) = \{0\}$ for a compact Lie group G if and only if G is connected. (If G is not connected, the indicator function p of the identity component is a representative function with $1 - p = (1 - p)^n \in K_n$ for all n .)

(a) Let $N \geq 4$. The Hopf $*$ -ideal $K_\infty(A)$ is mapped into $K_\infty(\mathcal{O}(SO_N)) = \{0\}$ by π_{SO_N} , since SO_N is connected, and into $K_\infty(\mathbb{C}[\mathbb{F}_2]) = \{0\}$ by π_N , by Lemma 5.2(b). So it is contained in both kernels. By Theorem 1.3 and Proposition 2.2, in the direction “equality of fixed vectors implies that no non-zero Hopf $*$ -ideal lies in both kernels”, we get $K_\infty(A) = \{0\}$. This is the argument of [FFS23, Proposition 4.5(3)], whose notion of topological generation is condition (C) of Section 2.3.

(b) Let $N \in \{2, 3\}$ and $x_{ij} = u_{ij} - \delta_{ij}$. The x_{ij} generate the ideal K , so every element of K is congruent modulo K_2 to a linear combination of them. The relation $uu^t = 1$ reads

$$x_{ij} + x_{ji} + \sum_k x_{ik}x_{jk} = 0 \quad (1 \leq i, j \leq N).$$

Hence $x_{ij} + x_{ji} \in K_2$ and $x_{ii} \in K_2$, so K/K_2 is spanned by the classes of the x_{ij} with $i < j$; and subtracting the relation for (j, i) from that for (i, j) gives

$$\sum_k [x_{ik}, x_{jk}] = 0 \quad (1 \leq i, j \leq N).$$

Modulo K_3 a commutator of elements of K depends only on their classes modulo K_2 . For $N = 2$ the space K/K_2 is spanned by one class, so $[K, K] \subset K_3$. For $N = 3$ the identities for $(i, j) = (1, 2), (1, 3), (2, 3)$, together with $x_{ii} \in K_2$ and $x_{ji} \equiv -x_{ij}$ modulo K_2 , give $[x_{13}, x_{23}] \in K_3$, $[x_{12}, x_{23}] \in K_3$ and $[x_{12}, x_{13}] \in K_3$; so the three classes spanning K/K_2 commute modulo K_3 , and again $[K, K] \subset K_3$. By Lemma 6.1, A/K_∞ is commutative. It is a CQG algebra generated by the entries of an orthogonal matrix, so $A/K_\infty = \mathcal{O}(H)$ for a closed subgroup $H \subset O_N$. Since $K_\infty(\mathcal{O}(H)) = K_\infty(A)/K_\infty(A) = \{0\}$, the group H is connected, so $H \subset SO_N$, that is, $\ker(A \rightarrow \mathcal{O}(SO_N)) \subset K_\infty(A)$. Conversely $K_\infty(A)$ is mapped into $K_\infty(\mathcal{O}(SO_N)) = \{0\}$. Hence $K_\infty(A) = \ker(A \rightarrow \mathcal{O}(SO_N))$. $\square$

Remark 6.2 (why this argument stops at $N = 3$). For $N \geq 4$ the identities $\sum_k [x_{ik}, x_{jk}] = 0$ only say that sums of at least two commutators of generators lie in K_3 , and $[x_{12}, x_{34}] \notin K_3$. Indeed $\pi_N(x_{12}) = S(a)$ and $\pi_N(x_{34}) = S(b)$, and with the map μ of Lemma 5.2 we get $\mu(S(a)) \in -iX + \mathfrak{m}^2$ and $\mu(S(b)) \in -iY + \mathfrak{m}^2$, hence $\mu([S(a), S(b)]) \in -[X, Y] + \mathfrak{m}^3$, which is not contained in $\mathfrak{m}^3$, whereas $\mu(\pi_N(K_3)) \subset \mathfrak{m}^3$. So the hypothesis of Lemma 6.1 fails for $N \geq 4$, as it must by Corollary 1.5(a).

For the *Gaussian part* $\text{Gauss}(G)$ of a compact quantum group, the smallest quantum subgroup through which all Gaussian generating functionals factor, we refer to [FFS23, Section 3].

Corollary 6.3. $\text{Gauss}(O_N^+) = SO_N$ for $N = 2$ and $N = 3$.

Proof. By [FFS23, Theorem A] the Gaussian part is contained in the strongly connected component of the identity, which is SO_N by Corollary 1.5(b). By [FFS23, Corollary 3.8] the Gaussian part of a quantum subgroup is contained in the Gaussian part of the ambient quantum group, and $\text{Gauss}(SO_N) = SO_N$ because SO_N is connected [FFS23, Corollary 5.3]. Hence $SO_N = \text{Gauss}(SO_N) \subset \text{Gauss}(O_N^+) \subset SO_N$. $\square$

For $N = 2$ this follows from [FFS23] alone, with their Lemma 4.7. For $N = 3$ it agrees with the classification of the Gaussian generating functionals of O_N^+ in [FFS25, Theorem 4.1] (number of the arXiv version) by antisymmetric matrices $L_1, \dots, L_d$ with $\sum_r \overline{L_r} L_r$ real: if L_r is the matrix of the vector product with $a_r \in \mathbb{C}^3$, the condition says that $\sum_r a_r a_r^*$ is real, and then all scalar products $\sum_r \overline{(L_r)_{ij}} (L_r)_{kl}$ are real, so that the functional factors through O_3 by [FFS23, Lemma 6.2].

7 Remarks and open questions

7.1 Consistency with known results

The statements used in this subsection for comparison are not used in the proofs of this note.

Torsion. By [Chi20, Theorem 3.3], O_N^+ is topologically generated by O_N and the dual of $\mathbb{Z}_2^{*N}$ for all $N \geq 2$. The classical version of that dual contains a reflection, so O_N^+ is also generated by SO_N and the dual of $\mathbb{Z}_2^{*N}$. This does not contradict Theorem 1.1, since $\mathbb{Z}_2^{*N}$ has elements of order two; for $N = 2, 3$ the torsion cannot be removed, and for $N \geq 4$ Theorem 1.3 removes it.

Classical versions and quantum subgroups. The classical version O_N of O_N^+ is not connected, while O_N^+ is strongly connected for $N \geq 4$. There is no contradiction: strong connectedness passes neither to quantum subgroups nor to the classical version. For quantum subgroups, the dual of $\mathbb{Z}_2$ is a quantum subgroup of $\widehat{\mathbb{F}}_2$. For classical versions, let Γ be the group of triples $(i, j, k) \in \mathbb{Z}^3$ with the product $(i, j, k)(i', j', k') = (i + i', j + j', k + k' + 2ji')$. It is torsion-free and nilpotent of class two, so $\widehat{\Gamma}$ is strongly connected [FFS23, Proposition 4.5]. Its commutator subgroup consists of the elements $(0, 0, 2n)$, so its abelianisation is $\mathbb{Z}^2 \times \mathbb{Z}_2$, and the classical version of $\widehat{\Gamma}$, the dual of the abelianisation, is the group $\mathbb{T}^2 \times \mathbb{Z}_2$, which is not connected. Example 4.10 shows the mechanism for O_4^+ .

Hopf subalgebras. The Hopf $*$ -subalgebra $\mathcal{O}(PO_N^+) \subset \mathcal{O}(O_N^+)$ generated by the products $u_{ij}u_{kl}$ is isomorphic to the subalgebra of $\mathcal{O}(U_N^+)$ generated by the products $u_{ij}^*u_{kl}$ of its generators (see the proof of [Chi20, Theorem 3.3]). Since U_N^+ is strongly connected [FFS23, Proposition 4.6] and $K_n(A_0) \subset K_n(A)$ for a Hopf $*$ -subalgebra $A_0 \subset A$, the quantum group PO_N^+ is strongly connected for all $N \geq 2$; compare the remark on the quantum automorphism group of $(M_N(\mathbb{C}), \text{tr})$ in [FFS23, Section 4.2]. This does not contradict Corollary 1.5(b) for $N = 2, 3$, because the inclusion $K_n(A_0) \subset K_n(A) \cap A_0$ can be strict: $\mathbb{C}[\mathbb{Z}]$, with $\mathbb{Z}$ generated by the product of the two generators of $\mathbb{Z}_2 * \mathbb{Z}_2$, has $K_\infty = \{0\}$, while $\mathbb{C}[\mathbb{Z}_2 * \mathbb{Z}_2]$ is generated by projections, so that $K_\infty = K_1$ there [FFS23, Proposition 4.10].

7.2 A question on the Heisenberg group

Let H_3 be the discrete Heisenberg group, the quotient of $\mathbb{F}_2$ by the third term of its lower central series. Its dual is a quantum subgroup of $\widehat{\mathbb{F}}_2 \leq O_4^+$, and it is Gaussian, as is SO_N [FFS23, Theorem 5.5, Corollary 5.3]; the dual of H_3 inside O_4^+ is used in [FFS23, proof of Proposition 6.5]. At the end of [FFS23, Section 6], Franz, Freslon and Skalski ask whether U_2^+ is topologically generated by U_2 and the dual of H_3 , and more generally for which quotients Γ of $\mathbb{F}_2$ it is generated by U_2 and $\widehat{\Gamma}$. The analogue for O_4^+ is:

Question 7.1. Is O_4^+ topologically generated by SO_4 and the dual of H_3 ? For which quotients Γ of $\mathbb{F}_2$ is $O_4^+ = \langle SO_4, \widehat{\Gamma} \rangle$?

Such a group Γ cannot be abelian. For H_3 the statement has been tested only: for every $m \leq 12$ the vectors of $(\mathbb{C}^4)^{\otimes m}$ fixed by SO_4 and by the dual of H_3 are those fixed by O_4^+ (program `vb_fix_heis.py` of the package; the computation is exact for these m). We make no claim beyond this. If the equality held for all m , the proof of Theorem 1.3, in which $\widehat{\mathbb{F}}_2$ enters only through Theorem 1.2, would give $O_N^+ = \langle SO_N, \widehat{H}_3 \rangle$ for all $N \geq 4$, and by [FFS23, Lemma 6.6] the quantum group O_N^+ would be Gaussian for $N \geq 4$. Whether O_N^+ is its own Gaussian part is stated as unknown in the introduction of [FFS23], and the corresponding problem for U_N^+ is described as open in the introduction of [FFS25].

7.3 What stays open

1. The general question of [Fre21]: for which Hopf algebras the nilpotent residual vanishes. This is a programme; the present note contributes only the case of O_N^+ .

2. The Gaussian part of O_N^+ for $N \geq 4$. By [FFS23, Theorem B] it is neither classical nor a group dual, and by Corollary 1.5(a) the inclusion $\text{Gauss}(O_N^+) \subset (O_N^+)^{00}$ of [FFS23, Theorem A] gives no restriction; see Question 7.1.
3. For $N \geq 4$: which torsion-free quotients of $\mathbb{F}_k$, embedded by rotation blocks, generate O_N^+ together with SO_N .

8 Computations and reproducibility

All theorems, propositions, lemmas and corollaries of this note are proved without computation. The programs of the reproducibility package only test the statements for small tensor powers and small parameters. Ranks are computed over $\mathbb{Q}$ or modulo primes below 2^{31} ; most reported equalities have the form “an upper bound from a rank modulo a prime meets a lower bound from explicit vectors” and are then exact for the given parameters, and a few programs use floating-point arithmetic. The file `README.md` of the package says for each program which of its results are exact and which are only tests. The package has five parts.

The programs with which the results were first obtained (original/). In the model of Section 4.2: $\dim X_m = |\text{NC}_2(m)|$ for $m \leq 10$ (Proposition 4.3); Lemma 4.9 for all 2^m sign vectors and Lemma 4.5 for all noncrossing pairings, $m \leq 12$; Lemma 4.6 for all pairs (s, t) , $m \leq 9$; Lemma 4.7 on 637,420 words and Lemma 4.8 on 42,884 configurations, with controls in which a hypothesis is violated. Directly in $\mathbb{C}^N$, with the Brauer and determinant tensors of SO_N : the intersection in Theorem 1.3 has dimension 1, 0, 2, 0, 5, 0, 14 for $m = 2, \dots, 8$ and $N = 4, 5$, and 1, 2, 5, 14 for even $m \leq 8$ and $N = 6$.

Two independent verification runs (verification_run_A/, verification_run_B/), each with its own programs. Run A: the equality of Theorem 1.2 directly in $\mathbb{C}^4$ for $m \leq 10$; Lemma 4.9 for all sign vectors, $m \leq 12$; Proposition 4.3 for $m \leq 10$ with all 43,640 equations; Lemmas 4.6 to 4.8 on 87,380 pairs, 3,368,420 words and 299,202 configurations; the equalities of Theorem 1.3 for $N = 5, 6, 7$, $m \leq 8$, and of Theorem 5.1 for $N = 4, 5, 6$, $k \leq 8$. Run B: the equality of Theorem 1.3 for (N, m) up to $(4, 12)$, $(5, 10)$, $(6, 8)$, $(7, 7)$; the fixed-vector form of $U_2^+ = \langle U_2, \widehat{\mathbb{F}}_2 \rangle$ for all colour words of length at most 12; $\mathcal{O}(O_3^+)/K_n = \mathcal{O}(SO_3)/K_n$ for $n \leq 10$, in accordance with Corollary 1.5(b); and the computations quoted in Remark 4.12 and after Question 7.1.

A program written with this note (writing_stage/) tests Lemma 5.2(a): the images under μ of the 485 elements of $\mathbb{F}_2$ of word length at most 5 are linearly independent.

A third independent verification run (independent_run_2/, the second round of verification), on the final text, with programs which were written before the other parts of the package were opened. The equality of Theorem 1.2 directly in $\mathbb{C}^4$ for $m \leq 8$ and that of Theorem 1.3 for $N = 5$ ($m \leq 7$), $N = 6$ ($m \leq 6$) and $N = 7$ ($m \leq 5$), from the equations of the Lie algebra of SO_N on the basis tensors with trivial product, all exact; Lemma 4.9 and Lemma 4.6(c) for all 8,191 sign vectors of length at most 12, and Lemma 4.5 for all noncrossing pairings of at most 12 points; Lemma 4.7 on 3,802 trivial words found by enumeration and on 300,000 random ones; Lemma 4.8 on 373,539 configurations, of which 8,016 admit a weight with trivial word, with controls; Lemma A.1 and the statements of Proposition A.2 for $N = 4, 5, 6$ and $k \leq 8, 7, 6$; Lemma 5.2(a) for the same 485 elements; the properties of the group Γ of Section 7.1; the test after Question 7.1 for $m \leq 8$; and the computations of Remark 4.12, as described there.

All programs except a few slow ones were run again when this note was written, one process at a time, and once more from the archive by the third run; the outputs are identical to the recorded ones up to fields that record running times. `README.md` also lists the commands, the running times and the correspondence between the numbering used in the programs and that of this note.

Verification. The results were first obtained with complete proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the first written version of the results, and neither of them used the report of the other; the third examined the final text of this note.

The first run compared the statement with the source and examined Proposition 3.1, Theorems 1.1 to 1.3 and the cited results. It re-derived every lemma of Section 4 and the three steps of the proof of Theorem 1.3, proved Propositions 2.2 and 2.3 and the statement of Remark 2.4, made the observation of Remark 2.5 (up to the standard fact quoted there), re-derived the proof of [BCV17, Theorem 4.1(1)] reproduced in Appendix A, and observed that Theorem 1.2 does not need Banica’s theorem. It found no gap and required no correction; the six clarifications which it recommended were made. The second run examined Lemma 6.1, Corollary 1.5, the consistency with known results, the computations and the relation to the literature. It re-proved Lemma 6.1 and Corollary 1.5, identified Lemma 4.8 with a case of a theorem of Chirvasitu (for Lemma 4.9 this was known from the start), found the preprint [Chi19], which had been overlooked, and made Remark 4.12 precise, adding the exact computation. It found no gap in the proofs and required corrections of credits, of the account of the literature and of wording, all of which were made.

When this note was written, the following was added after these two runs: the proof of Lemma 5.2(a) for the group algebra, which replaces a reference; the presentation of the proof of [BCV17] in Appendix A as an induction starting from Corollary 4.4; the description of the group Γ in Section 7.1 by triples of integers; and Figure 1.

The third run examined the final text, and these additions first. It re-derived the proof of Lemma 5.2 line by line. It compared Lemma A.1, Proposition A.2 and Corollary A.3 with [BCV17, Section 4.1]: the linear independence of the vectors of the noncrossing pairings is used once, in dimension $N - 2 \geq 2$, in Step 3 of the proof of Proposition A.2; and the argument is not circular, since Corollary 4.4 comes from Section 4 alone and Theorem 5.1 is used in the proof of Theorem 1.3 only for $N \geq 5$. It found that the fact quoted in Remark 2.5 is the correct statement for every $N \geq 2$, and verified the properties of the group Γ and Figure 1. It then re-derived all other proofs of the final text, read the two pages of the source again in the publisher’s file, and compared every numbered citation of [FFS23, FFS25, Chi20, Chi19, BCV17, BP17] with the arXiv sources. For Remark 4.12 it derived the criterion again, repeated the computations with two programs of its own, and found the element of the kernel given there, which makes the statement on the three tori independent of the programs. It found no wrong theorem, proposition, lemma or corollary and no gap in a proof. It required twelve corrections, which were made; those which concern the text are corrections of wording and of credit in the introduction and after Proposition 3.1, in Remarks 2.5, 4.11 and 4.12, in Section 7.1 and in the first paragraph of Appendix A, and the present account of the verification and of the package. Its programs (Section 8) agree with all numbers of the other parts of the package wherever the same quantity is computed. No proof depends on a computation.

Scope and priority. The note answers the final problem of [Fre21] for every $N \geq 2$ and determines the nilpotent residual of $\mathcal{O}(O_N^+)$ for every $N \geq 2$. The general question of [Fre21] is a programme and stays open, and the Gaussian part of O_N^+ is not determined for $N \geq 4$. The case $N \geq 5$ relies on the theorem of Brannan, Collins and Vergnioux [BCV17]; the negative answer and the residual for $N = 2$ were known [FFS23]. The problem is due to Freslon, Franz and Skalski [Fre21, FFS23]; the notion of topological generation to Brannan, Collins and Vergnioux [BCV17] and Chirvasitu [Chi20]; the free quantum groups to Wang and to Van Daele and Wang [Wan95, VDW96], in the framework of Woronowicz [Wor87, Wor98]; the description of the fixed vectors of O_N^+ and U_N^+ to Banica [Ban96, Ban97]; the statements of Lemmas 4.8 and 4.9 to Chirvasitu [Chi20, Chi19]; the facts behind Proposition 3.1 to Woronowicz, to Banica, Bhowmick and De Commer and to Banica and

Patri [Wor87, BBD12, BP17]; the embedding of $\widehat{\mathbb{F}}_2$ in O_4^+ and the questions on the Heisenberg group to [FFS23]; and the residual nilpotence of free groups and of group rings to Magnus [Mag35] and Passi [Pas79], the latter as quoted in [FFS23].

What was read, always in the arXiv version, whose numbers we use: the two pages of [Fre21] (in the publisher's file); of [FFS23] (version 3, which carries the final title) the introduction and Sections 3 to 6; of [FFS25] (version 2) the introduction and Section 4; [Chi20] and [Chi19] (version 1 of each) completely; of [BCV17] (version 2) Sections 2.3, 3.3 and 4.1; of [BP17] (version 3) Sections 1 and 6. The published versions of [BCV17] (the site of the publisher refused automated access), [Chi20], [FFS23] and [FFS25] were not consulted. We did not read [Ban96, Ban97, BBD12, BB10, Wan95, VDW96, Wor87, Wor98, DK94, Mag35, Pas79, MM65, Qui68, Bou82]; what we say about them is standard or taken from the papers read. The bibliographic data of all literature references were checked with Crossref, zbMATH or the arXiv listing.

The searches were made on 10 October 2026. When the results were first obtained: eight queries to the arXiv listing interface (strongly connected quantum groups, topological generation, free orthogonal quantum groups, Gaussian parts, augmentation ideals, the authors of the problem), the lists of works citing [FFS23] and [Chi20] in OpenAlex, three queries to zbMATH and two web searches. In the second verification run the arXiv listing interface was not available; OpenAlex (keyword queries, citing works, recent works on free orthogonal quantum groups and by the authors concerned), zbMATH (five queries), Crossref and three web searches were used instead. The first run made two web searches. The third run made twenty-one queries to the arXiv listing interface (the same topics, generation by tori and inner faithfulness, and the recent papers of the authors of the problem and of the generation theorems), used OpenAlex (the works citing [FFS23, FFS25, Chi20, BCV17] and keyword queries), zbMATH (six queries) and Crossref, and made one web search. No answer to the final problem, no determination of the nilpotent residual of $\mathcal{O}(O_N^+)$ for $N \geq 3$, and no statement that O_N^+ is topologically generated by SO_N and the dual of a torsion-free group was found. Preprints not yet indexed by these services may have been missed, and the published versions of the journal articles were not read. A search that finds nothing is not a proof of novelty; no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A The theorem of Brannan, Collins and Vergnioux

We recall the proof of Theorem 5.1 for $N \geq 5$, following [BCV17, Section 4.1] (Lemma 4.5, Proposition 4.6 and the proof of Theorem 4.1 there). The argument is theirs. The changes are that the description of the fixed vectors of O_{N-1}^+ , for which [BCV17] quotes Banica's theorem, is obtained by induction from Corollary 4.4, and that the linear independence of the vectors of the noncrossing pairings, which is quoted there as well [BCV17, Theorem 2.1], is proved in Lemma A.1.

Lemma A.1. *For $n \geq 2$ the vectors $T_q^{(n)}$, $q \in \text{NC}_2(m)$, are linearly independent.*

Proof. Let $g_{\pm} = \frac{1}{\sqrt{2}}(e_1 \pm ie_2)$. Then $\sum_{i \leq n} e_i \otimes e_i = g_+ \otimes g_- + g_- \otimes g_+ + \sum_{3 \leq i \leq n} e_i \otimes e_i$. So, in the basis $g_+, g_-, e_3, \dots, e_n$ of $\mathbb{C}^n$, the coefficient of $T_q^{(n)}$ at $g_{t_1} \otimes \dots \otimes g_{t_m}$ is 1 if $q \sim t$ and 0 otherwise, for $t \in \{\pm 1\}^m$. By Lemma 4.1(a) the matrix of these coefficients at the vectors t^q is triangular with ones on the diagonal. $\square$

A partition p of $\{1, \dots, k\}$ into singletons and pairs is called noncrossing if its pairs do not cross each other; let $\text{NC}_{2,1}(k)$ be the set of these partitions and $s(p)$ the number of singletons of p . For

$\nu \in \{0, 1, 2\}$ and vectors $\zeta_1, \dots, \zeta_{s(p)} \in \mathbb{C}^N$ let $T_p^\nu(\zeta_1, \dots, \zeta_{s(p)}) \in (\mathbb{C}^N)^{\otimes k}$ be the tensor with ζ_l in the leg of the l -th singleton of p and a copy of $\sum_{i \leq N-\nu} e_i \otimes e_i$ in the legs of each pair of p .

Proposition A.2 ([BCV17]). *Let $N \geq 4$, and suppose that $\text{Fix}_{O_{N-1}^+}(u_{N-1}^{\otimes l}) = \text{TL}_{N-1}(l)$ for all $l \geq 0$. Then, for all $k \geq 0$,*

$$\text{Fix}_{O_N}(u^{\otimes k}) \cap \text{Fix}_{O_{N-1}^+}(u^{\otimes k}) = \text{TL}_N(k) = \text{Fix}_{O_N^+}(u^{\otimes k}).$$

Proof. Step 1. By the remark on restriction in Section 5 and by the hypothesis, $\text{Fix}_{O_{N-1}^+}(u^{\otimes k})$ is spanned by the vectors which have e_N in the legs outside a set L and a vector $T_q^{(N-1)}$, $q \in \text{NC}_2(L)$, in the legs of L ; these are the vectors $T_p^1(e_N, \dots, e_N)$, $p \in \text{NC}_{2,1}(k)$. Writing $\sum_{i \leq N} e_i \otimes e_i = \sum_{i \leq N-1} e_i \otimes e_i + e_N \otimes e_N$ at each pair shows that $T_p^0(e_N, \dots, e_N) - T_p^1(e_N, \dots, e_N)$ is a sum of vectors $T_{p'}^1(e_N, \dots, e_N)$, where p' arises from p by splitting some pairs into singletons, so that $s(p') > s(p)$. By descending induction on $s(p)$, the vectors $A_p(e_N)$, $p \in \text{NC}_{2,1}(k)$, span the same space, where $A_p(\zeta) = T_p^0(\zeta, \dots, \zeta)$.

Step 2. Let x be in the intersection on the left, and write $x = \sum_p \lambda_p A_p(e_N)$. Since $\sum_{i \leq N} e_i \otimes e_i$ is fixed by $g \otimes g$ for $g \in O_N$, we have $g^{\otimes k} A_p(\zeta) = A_p(g\zeta)$, and $x = g^{\otimes k} x = \sum_p \lambda_p A_p(g e_N)$ for all $g \in O_N$. So the polynomial map $\zeta \mapsto \sum_p \lambda_p A_p(\zeta)$ is constant on the unit sphere of $\mathbb{R}^N$. Differentiating it along $\cos \theta e_N + \sin \theta e_{N-1}$ at $\theta = 0$ gives

$$\sum_{p: s(p) \geq 1} \lambda_p x_p = 0, \quad x_p = \sum_{l=1}^{s(p)} T_p^0(e_N, \dots, e_N, e_{N-1}, e_N, \dots, e_N),$$

with e_{N-1} in the l -th place.

Step 3. The vectors x_p , $s(p) \geq 1$, are linearly independent. For $p \in \text{NC}_{2,1}(k)$ and $j \in \{N-1, N\}^{s(p)}$ let $y_{p,j} = T_p^2(e_{j_1}, \dots, e_{j_{s(p)}})$. This vector is a combination of basis tensors which have an index in $\{N-1, N\}$ exactly in the legs of the singletons of p , the indices there being given by j . So vectors $y_{p,j}$ with different sets of singletons or different j have disjoint supports, and for a fixed set of singletons and a fixed j they are, in the remaining legs, the vectors $T_q^{(N-2)}$ of the noncrossing pairings q of these legs, which are linearly independent by Lemma A.1, since $N-2 \geq 2$. Hence all $y_{p,j}$ are linearly independent. Writing $\sum_{i \leq N} e_i \otimes e_i = \sum_{i \leq N-2} e_i \otimes e_i + e_{N-1} \otimes e_{N-1} + e_N \otimes e_N$ at each pair shows

$$x_p = \sum_{l=1}^{s(p)} y_{p,j^{(l)}} + (\text{a combination of vectors } y_{p',j'} \text{ with } s(p') > s(p)),$$

where $j^{(l)}$ has the entry $N-1$ in the l -th place and N elsewhere. If $\sum_p c_p x_p = 0$ with some $c_p \neq 0$, let p_0 have the smallest number of singletons among the p with $c_p \neq 0$; then the coefficient of $y_{p_0,j^{(1)}}$ in $\sum_p c_p x_p$ is $c_{p_0} \neq 0$, a contradiction.

Step 4. By Steps 2 and 3, $\lambda_p = 0$ whenever $s(p) \geq 1$, so $x = \sum_{p \in \text{NC}_2(k)} \lambda_p T_p^{(N)} \in \text{TL}_N(k)$. With (7), $\text{TL}_N(k) \subset \text{Fix}_{O_N^+}(u^{\otimes k}) \subset \text{Fix}_{O_N}(u^{\otimes k}) \cap \text{Fix}_{O_{N-1}^+}(u^{\otimes k}) \subset \text{TL}_N(k)$. $\square$

Corollary A.3. *For all $N \geq 4$ and $k \geq 0$, $\text{Fix}_{O_N^+}(u^{\otimes k}) = \text{TL}_N(k)$, and Theorem 5.1 holds for all $N \geq 5$.*

Proof. Induction on N : the case $N = 4$ of the first statement is Corollary 4.4, and if it holds for $N-1 \geq 4$, then Proposition A.2 gives both statements for N . $\square$

The first statement of Corollary A.3 is Banica's theorem [Ban96] for $N \geq 4$; it is not new.

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