

A Composition Estimate for Fourier Series with the Supremum in Time Inside the Sum, and a Question of Chruściel

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October 9, 2026

Abstract

P. T. Chruściel asked (Oberwolfach Reports 18 (2021)) whether the classical composition estimate in the Sobolev space $H^s(\mathbb{T}^2)$, $s > 1$, survives when the functions depend on a parameter t and the supremum over t is taken for each Fourier coefficient separately, inside the weighted sum. With $\mathcal{N}_s(u) = \sum_{\ell \in \mathbb{Z}^2} (1 + |\ell|)^{2s} \sup_t |u_\ell(t)|^2$: given a smooth function G with $G(0) = 0$, is there a constant $C_2 = C_2(G, s)$ such that $\mathcal{N}_s(G \circ \varphi) \leq C_2 \mathcal{N}_s(\varphi)$ whenever $\mathcal{N}_s(\varphi) \leq 1$? This estimate is the hypothesis of a conditional existence theorem for the Einstein–Maxwell equations for Robinson–Trautman metrics stated there. We show that the answer is yes, in every dimension $d \geq 1$ and for every $s > \frac{d}{2}$, with explicit constants, and for every $G \in C^k$ with $G(0) = 0$ and $k > M + \frac{3}{2}$, where M is the exponent in the following bound. The proof rests on a polynomial bound for exponentials: $\mathcal{N}_s(e^{i\lambda\varphi})^{1/2} \leq C(1 + |\lambda|)^M$ uniformly over the real-valued families with $\mathcal{N}_s(\varphi) \leq 1$, with $M = s + \frac{d}{2}$ if $s > 1 + \frac{d}{2}$ and M arbitrarily close to $(s + \frac{d}{2})/(s - \frac{d}{2})$ otherwise. For an explicit family the left side is at least $c\lambda^{s+d/2}$, whereas the quantity with the supremum outside the sum grows like λ^s on this family; so the bound is sharp for $s > 1 + \frac{d}{2}$, and the loss $\lambda^{d/2}$ is the loss of Parseval’s identity under the mode-by-mode supremum. Some smoothness of G is necessary: for every $k < s + \frac{d}{2}$ the estimate fails for some $G \in C^k$ with $G(0) = 0$. The method is known from weighted Fourier algebras (Leblanc) and from modulation spaces (Reich and Sickel), and the necessity argument follows Katznelson and Leblanc; we did not find the estimate for this space in the literature, and the two closest composition theorems that we know, for modulation spaces, do not cover the case which corresponds to it by analogy. The exact growth rate of the exponentials for $\frac{d}{2} < s \leq 1 + \frac{d}{2}$ and the minimal smoothness of G are not determined, and the constants are very large. This is an unrefereed note.

1 Introduction

1.1 The question

In his contribution [Chr21] to the Oberwolfach workshop “Mathematical Aspects of General Relativity” of 2021, Chruściel describes an approach to the Einstein–Maxwell equations for Robinson–Trautman metrics in which the data are prescribed partly on an initial and partly on a final characteristic surface, and he reports that the nonlinear problem with small data can be solved if a certain functional inequality holds [Chr21, Section 3.3]. The inequality is formulated on the flat square torus $S = \mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$. We recall it; the numbers (5)–(12) below are the equation numbers of [Chr21].

The functions $f_\ell(x) = \frac{1}{2\pi} e^{i\ell \cdot x}$ are taken as an orthonormal basis of $L^2(S)$ (5). For a function φ on S and a smooth function $G: \mathbb{R} \rightarrow \mathbb{R}$ with $G(0) = 0$, let φ_ℓ and $(G \circ \varphi)_\ell$ be the coefficients of φ and

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of $G \circ \varphi$ in this basis (6), (7). The classical composition estimate (8) says that for $s > 1$

$$\|G \circ \varphi\|_{H^s(S)} \leq C(\|\varphi\|_{L^\infty(S)}) \|\varphi\|_{H^s(S)},$$

which, with another function C , is equivalent to (9):

$$\sum_{\ell} (1 + |\ell|)^{2s} |(G \circ \varphi)_{\ell}|^2 \leq C(\|\varphi\|_{L^\infty(S)}) \sum_{\ell} (1 + |\ell|)^{2s} |\varphi_{\ell}|^2.$$

Now let $\varphi = \varphi(t)$ depend on a parameter $t \in [t_-, t_+]$, and put

$$\mathcal{N}_s(\varphi) = \sum_{\ell} (1 + |\ell|)^{2s} \sup_{t \in [t_-, t_+]} |\varphi_{\ell}(t)|^2. \quad (1.1)$$

The question of [Chr21] is: given a smooth function G with a zero of first order at the origin and a real number $s > 1$, is there a constant $C_2 = C_2(G, s)$ such that for all $\varphi(t)$ satisfying

$$\mathcal{N}_s(\varphi) \leq 1 \quad (10)$$

we have

$$\mathcal{N}_s(G \circ \varphi) \leq C_2 \mathcal{N}_s(\varphi) ? \quad (11)$$

Here $(G \circ \varphi)_{\ell}(t)$ is the ℓ -th coefficient of $x \mapsto G(\varphi(t, x))$. The source points out that an application of (8) for each t gives a different inequality,

$$\sup_t \sum_{\ell} (1 + |\ell|)^{2s} |(G \circ \varphi)_{\ell}(t)|^2 \leq C \left(\sup_t \|\varphi(t)\|_{L^\infty(S)} \right) \sup_t \sum_{\ell} (1 + |\ell|)^{2s} |\varphi_{\ell}(t)|^2, \quad (12)$$

in which the supremum over t stands outside the sums. Theorem 1 of [Chr21], attributed there to a paper of Chruściel and Tod which is listed as reference [9] without journal or preprint number, asserts under the assumption (11) the existence of a unique smooth solution of the Einstein–Maxwell Robinson–Trautman equations on $[u_-, u_+] \times \mathbb{T}^2$ for sufficiently small spectral data prescribed partly at u_- and partly at u_+ . The section ends with the sentence “The question then arises, whether (11) holds true” [Chr21, p. 2171]. The question is record OWR-8415343-001 of the UnsolvedMath collection [ula26], where it is listed as open.

Four points of reading have to be fixed.

Index set. The source prints $\ell \in \mathbb{N}^2$ in (5)–(12) (in the report and in the arXiv version). The exponentials f_{ℓ} form an orthonormal basis of $L^2(\mathbb{T}^2)$ only for $\ell \in \mathbb{Z}^2$, and the literal reading would make (11) false for trivial reasons: for $\varphi = A \cos(x_1 - 2x_2) + B \cos(2x_1 - x_2)$ no coefficient with index in $\mathbb{N}^2$ is different from zero, while φ^2 contains the term $AB \cos(x_1 + x_2)$; so for $G(u) = u + u^2$ the right side of (11) vanishes and the left side does not. We read $\ell \in \mathbb{Z}^2$. A second consistent reading, the real orthonormal basis of products of cosines and sines indexed by pairs of non-negative integers, gives an equivalent quantity and the same answer (Corollary 6.1(b)).

Moduli. For the exponential basis the coefficients are complex, and the squares in (9)–(12) are squared moduli, as in (1.1).

Supremum. The supremum over t is taken for each ℓ separately. The source does not specify any regularity in t . We take the pointwise supremum over an arbitrary index set; our constants do not depend on this set. Restricting to families which are continuous or smooth in t changes neither the positive nor the negative results (Remark 7.2).

The function G and the constant. We use only $G(0) = 0$; a zero of first order ($G'(0) \neq 0$) is not needed for a positive answer and can be arranged in the negative results. The constant C_2 may depend on G and s only. The condition $G(0) = 0$ is necessary for (11) (take $\varphi = 0$), the restriction to a ball like (10) is necessary for a linear bound (take $G(u) = u^2$), and $s > 1$ is necessary because $H^1(\mathbb{T}^2)$ is not an algebra. The text of the record in [ula26] ends with “ $\leq C_2$ ”: the factor $\mathcal{N}_s(\varphi)$ on the right of (11) is missing there, which is a weaker statement. We prove (11) as it stands in the source.

1.2 Why the position of the supremum matters

Write $c_\ell(u)$ for the Fourier coefficients of a function on the torus and call $m_\ell = \sup_t |c_\ell(\varphi(t))|$ the *envelope* of the family φ (precise definitions follow in Section 1.3). The left side of (10) is the square of the weighted ℓ^2 norm of the envelope; the right side of (12) contains the supremum of the H^s norms of the members. The first quantity dominates the second, but not conversely: for a family of n exponentials $e^{ik_j \cdot x}$ with different frequencies the first is the sum, and the second the maximum, of the n numbers $(1 + |k_j|)^{2s}$.

The usual proofs of (8) work for one function at a time and use the orthogonality of the exponentials at fixed t , in the form of Parseval's identity or of Littlewood–Paley decompositions; they give (12). For the envelope there is no Parseval identity. What survives is the algebra property: the envelope of a product is dominated by the convolution of the envelopes, and the weighted ℓ^2 space with $s > \frac{d}{2}$ is a convolution algebra. Hence (11) holds for polynomials G without constant term, and by summation of the power series for functions G which are analytic in a sufficiently large disc around 0, in particular for entire functions (Proposition 2.5). This elementary argument does not reach functions which are not analytic.

That the algebra property alone does not settle the case of smooth functions is shown by a classical theorem. In the Wiener algebra $A(\mathbb{T})$ of absolutely convergent Fourier series, which is a Banach algebra of the same kind without weight, only analytic functions operate: if F is defined on an open interval and $F \circ f \in A(\mathbb{T})$ for every $f \in A(\mathbb{T})$ with values in this interval, then F is analytic there (Katznelson [Kat59, Théorème II.1], with a corollary for closed intervals; for general groups see Helson, Kahane, Katznelson and Rudin [HKKR59]). So the weight has to enter. It enters through the following fact, which fails without weight: on the unit ball (10) the tails $\sum_{|\ell| > \rho} m_\ell$ of the envelopes tend to zero uniformly as $\rho \rightarrow \infty$.

1.3 Results

Notation. Throughout, $d \geq 1$ is an integer, $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$, and

$$s > \frac{d}{2}, \quad \sigma := s - \frac{d}{2} > 0.$$

For $\ell \in \mathbb{Z}^d$, $|\ell|$ is the Euclidean norm and $\langle \ell \rangle := 1 + |\ell|$ (this is the weight of the source, not $(1 + |\ell|^2)^{1/2}$); also $\langle \rho \rangle := 1 + \rho$ for real $\rho \geq 0$. For $a: \mathbb{Z}^d \rightarrow [0, \infty]$ put

$$\|a\|_1 := \sum_\ell a_\ell, \quad \|a\|_s := \left(\sum_\ell \langle \ell \rangle^{2s} a_\ell^2 \right)^{1/2}, \quad (a * b)_\ell := \sum_k a_{\ell-k} b_k.$$

For $u \in L^1(\mathbb{T}^d)$ let $c_\ell(u) := (2\pi)^{-d} \int_{\mathbb{T}^d} u(x) e^{-i\ell \cdot x} dx$. Let I be a non-empty set. A *family* is a map $t \mapsto u(t)$ from I to $L^1(\mathbb{T}^d)$; its *envelope* and its norm are

$$\text{env}(u)_\ell := \sup_{t \in I} |c_\ell(u(t))| \in [0, \infty], \quad \|u\|_X := \|\text{env}(u)\|_s.$$

Sums, products and compositions of families are taken for each t . The norm depends only on the set $\{u(t) : t \in I\}$ of members. In the normalisation of the source ($d = 2$, $I = [t_-, t_+]$) we have $\varphi_\ell = 2\pi c_\ell(\varphi)$, hence

$$\mathcal{N}_s(u) = (2\pi)^d \|u\|_X^2, \tag{1.2}$$

and (10) is the ball $\|\varphi\|_X \leq \frac{1}{2\pi}$. For $d \neq 2$ we take (1.2) as the definition of $\mathcal{N}_s$. The following constants depend only on d and s :

$$c_s := \max\{1, 2^{s-1}\}, \quad K_\tau := \left(\sum_{\ell \in \mathbb{Z}^d} \langle \ell \rangle^{-2\tau} \right)^{1/2} \quad (\tau > \frac{d}{2}), \quad C_T := \left(1 + \frac{\sqrt{d}}{2}\right)^{s+\sigma} \left(\frac{\omega_d}{2\sigma}\right)^{1/2},$$

where $\omega_d = 2\pi^{d/2}/\Gamma(\frac{d}{2})$ is the surface measure of the unit sphere of $\mathbb{R}^d$ ($\omega_1 = 2$). Finally we fix a number θ with

$$0 \leq \theta \leq 1, \quad \theta < \sigma, \quad (1.3)$$

and put

$$\begin{aligned} \kappa &:= 1 + \frac{1-\theta}{\sigma}, & M &:= (s + \frac{d}{2})\kappa, & N &:= \lceil s + \frac{d}{2} + \frac{1}{2} \rceil, & B &:= \max\{1, c_s C_T\}^{1/\sigma}, \\ B_R &:= 1 + 2(N-1)B + 2\sqrt{d} K_{s-\theta} B, & C_A &:= e 2^{d+s+1} B_R^{s+d/2} + e c_s 2^{2d+2} B_R^d. \end{aligned} \quad (1.4)$$

The choice $\theta = 0$ is always admissible and gives $M = (s + \frac{d}{2})(1 + \frac{1}{\sigma})$. If $\sigma > 1$ one may take $\theta = 1$, which gives $M = s + \frac{d}{2}$. If $\sigma \leq 1$, the exponents M fill the interval $((s + \frac{d}{2})/\sigma, (s + \frac{d}{2})(1 + \frac{1}{\sigma})]$. In all cases the infimum of the admissible exponents is

$$M_* := (s + \frac{d}{2}) \max\{1, \frac{1}{\sigma}\}, \quad (1.5)$$

and it is attained if and only if $\sigma > 1$.

Theorem 1.1 (Exponentials). *Let ψ be a family of real-valued functions on $\mathbb{T}^d$, indexed by an arbitrary set, with $\|\psi\|_X \leq 1$. Then for every $\lambda \in \mathbb{R}$*

$$\|e^{i\lambda\psi}\|_X \leq C_A (1 + |\lambda|)^M,$$

with M and C_A from (1.4); the constant does not depend on the index set.

Theorem 1.2 (Composition; measure form). *Let $r > 0$ and let μ be a complex Borel measure on $\mathbb{R}$ with*

$$J_\mu(r) := \int_{\mathbb{R}} |\lambda| (1 + r|\lambda|)^M d|\mu|(\lambda) < \infty.$$

Let $G(u) := \int_{\mathbb{R}} (e^{i\lambda u} - 1) d\mu(\lambda)$ for real u . Then for every family φ of real-valued functions with $\|\varphi\|_X \leq r$

$$\|G \circ \varphi\|_X \leq C_B \|\varphi\|_X, \quad C_B := 2 c_s K_s C_A J_\mu(r).$$

Corollary 1.3 (C^k functions; the answer to the question). *Let k be an integer with $k > M + \frac{3}{2}$, let $G \in C^k(\mathbb{R}; \mathbb{C})$ with $G(0) = 0$, let $r > 0$, and let $H \in C^k(\mathbb{R}; \mathbb{C})$ be a function with compact support which coincides with G on $[-K_s r, K_s r]$ (for instance G times a cut-off function).*

(a) *For every family φ of real-valued functions on $\mathbb{T}^d$ with $\|\varphi\|_X \leq r$ we have $\|G \circ \varphi\|_X \leq C_B \|\varphi\|_X$ with*

$$C_B := 2 c_s K_s C_A \cdot (2\pi)^{-1/2} (1 + r)^M 2^{k-1} (\|H\|_{L^2} + \|H^{(k)}\|_{L^2}) \left(\frac{2}{2k - 2M - 3} \right)^{1/2}. \quad (1.6)$$

If $k > M + 2$, the same holds with

$$C_B := \frac{2}{\pi} c_s K_s C_A (1 + r)^M \left(\|H\|_{L^1} + \frac{\|H^{(k)}\|_{L^1}}{k - M - 2} \right). \quad (1.7)$$

(b) *Let $d = 2$ and $s > 1$. If $\mathcal{N}_s(\varphi) \leq 1$, then $\mathcal{N}_s(G \circ \varphi) \leq C_2 \mathcal{N}_s(\varphi)$ with $C_2 := C_B^2$, where C_B is the constant of (a) for $r = \frac{1}{2\pi}$. It depends on s , on the choice of θ , and on G through the extension H of its restriction to $[-\frac{K_s}{2\pi}, \frac{K_s}{2\pi}]$; it does not depend on the index set of the families. In particular the answer to the question of Section 1.1 is yes for every smooth G with $G(0) = 0$, and more generally for every $G \in C^k$ with $G(0) = 0$ and $k > M_* + \frac{3}{2}$.*

Since M can be chosen with $M_* \leq M < k - \frac{3}{2}$ as soon as $k > M_* + \frac{3}{2}$, the last sentence follows from the first. For the source ($d = 2$) this means: (11) holds for $G \in C^k$, $G(0) = 0$, if $k > s + \frac{5}{2}$ in the case $s \geq 2$, and if $k > \frac{s+1}{s-1} + \frac{3}{2}$ in the case $1 < s < 2$; for $s = 2$ the smallest such integer is $k = 5$.

The next two statements show how far these results are from optimal. Let

$$\mathcal{E}(\lambda) := \sup\{\|e^{i\lambda\varphi}\|_X : \varphi \text{ a family of real-valued functions on } \mathbb{T}^d, \|\varphi\|_X \leq 1\}$$

and let $\gamma_* := \inf\{\gamma : \mathcal{E}(\lambda) = O(\lambda^\gamma) \text{ as } \lambda \rightarrow \infty\}$ be the growth exponent of the exponentials. By Theorem 1.1, $\gamma_* \leq M_*$. We write $\|u\|_{H^s(\mathbb{T}^d)} := \|(|c_\ell(u)|)_\ell\|_s$ for single functions.

Proposition 1.4 (Lower bound). *There are a family Φ of real-valued C^∞ functions on $\mathbb{T}^d$, indexed by a cube in $\mathbb{R}^d$, with $\|\Phi\|_X \leq 1$, and constants $c_* > 0$, $\lambda_* \geq 1$ and $C < \infty$, depending only on d , s and on a profile function fixed in the construction, such that:*

- (a) $\|e^{i\lambda\Phi}\|_X \geq c_* \lambda^{s+d/2}$ for all $\lambda \geq \lambda_*$; for each such λ a finite subfamily of Φ suffices;
- (b) $\sup_{u \in \Phi} \|e^{i\lambda u}\|_{H^s(\mathbb{T}^d)} \leq C(1 + \lambda)^s$ for all $\lambda \geq 0$, and this supremum is at least $c_* \lambda^s$ for $\lambda \geq \lambda_*$.

Consequently $\gamma_* \geq s + \frac{d}{2}$, and for $s > 1 + \frac{d}{2}$

$$c_* \lambda^{s+d/2} \leq \mathcal{E}(\lambda) \leq C_A(1 + \lambda)^{s+d/2} \quad (\lambda \geq \lambda_*),$$

so that the exponent of Theorem 1.1 with $\theta = 1$ cannot be lowered. On the family Φ , and for $\lambda \geq \lambda_*$, the norm of $e^{i\lambda\Phi}$ with the supremum inside the sum exceeds the norm with the supremum outside by a factor of at least $c \lambda^{d/2}$, with a constant $c > 0$.

The last sentence is the quantitative difference between (11) and (12). (That the two quantities are not comparable is already seen for $G(u) = u$, see Section 1.2.) Its reason is visible in the proof: every member $e^{i\lambda u}$ has $\sum_\ell |c_\ell|^2 = 1$ by Parseval's identity, but the envelope of the family is bounded below on about λ^d modes simultaneously, because every such mode has its own member with the matching linear phase.

Proposition 1.5 (Some smoothness is necessary). *Let $r > 0$ and let $k \geq 0$ be an integer with $k < s + \frac{d}{2}$. There is a real-valued function $G \in C^k(\mathbb{R})$ with $G(0) = 0$, and with $G'(0) = 1$ if $k \geq 1$, such that*

$$\|G \circ (r\Phi)\|_X = \infty \quad \text{although} \quad \|r\Phi\|_X \leq r,$$

where Φ is the family of Proposition 1.4. The functions G with $\|G \circ (r\Phi)\|_X = \infty$ form a residual subset of the Banach space $\{G \in C^k([-K_s r, K_s r]; \mathbb{R}) : G(0) = 0\}$, and the same is true in the Hölder classes $C^{k,\alpha}$ with $k + \alpha < s + \frac{d}{2}$. In particular, for $d = 2$ and every integer k with $1 \leq k < s + 1$ the inequality (11) fails for some $G \in C^k$ with a zero of first order at the origin, even if its right side is replaced by a constant.

For the source ($d = 2$, $s > 1$) the results are summarised in Table 1.

Further results: (11) for analytic G with small constants (Proposition 2.5); variants for other weights, for the real basis and without $G(0) = 0$ (Corollary 6.1); several functions and complex-valued families (Corollary 6.2); a Lipschitz estimate (Corollary 6.3); the exact threshold $\beta = s + \frac{d}{2}$ in the scale of the weighted Wiener algebras $A_\beta(\mathbb{T})$ when $s > 1 + \frac{d}{2}$ (Corollary 8.2); and for $d = 1$, $\frac{1}{2} < s < 1$ the bound $\mathcal{E}(\lambda) \geq c \lambda^{3/2}$, so that $s + \frac{d}{2}$ is not the growth exponent there (Proposition 9.1).

	$1 < s < 2$	$s = 2$	$s > 2$
growth exponent γ_* of the exponentials	$s + 1 \leq \gamma_* \leq \frac{s+1}{s-1}$	$\gamma_* = 3$	$\gamma_* = s + 1$
(11) holds for all $G \in C^k$, $G(0) = 0$, if	$k > \frac{s+1}{s-1} + \frac{3}{2}$	$k \geq 5$	$k > s + \frac{5}{2}$
(11) fails for some $G \in C^k$, $G(0) = 0$, if	$k < s + 1$	$k \leq 2$	$k < s + 1$

Table 1: The case of the source, $d = 2$. First row: Theorem 1.1 and Proposition 1.4; second row: Corollary 1.3; third row: Proposition 1.5.

1.4 The method, and what is known

The proof of Theorem 1.1 follows a known scheme. Leblanc [Leb69] studied the weighted algebras $A_\alpha(\mathbb{T})$ of the functions $f = \sum_n a_n e^{inx}$ with $\|f\|_\alpha = \sum_n (1 + |n|)^\alpha |a_n| < \infty$ and asked for which β all functions of A_β operate on the real-valued functions of A_α . Katznelson had shown that $\beta \geq \alpha + \frac{1}{2}$ is necessary [Kat59, Corollaire, p. 102]. This is also sufficient as soon as $\|e^{inf}\|_\alpha \leq C_f |n|^{\alpha+1/2}$, which Leblanc notes for $\alpha \geq 1$. For $\alpha < 1$ he proves that $\beta \geq 1 + \frac{1}{2\alpha}$ is necessary [Leb69, Théorème 1] and that $\beta > 1 + \frac{1}{2\alpha}$, with strict inequality, is sufficient [Leb69, Théorème 2]. In the proof of the second theorem he splits f at the frequency $\|f\|_\alpha^{1/\alpha}$, treats the exponentials of the high-frequency parts by the exponential series, shows that the coefficients of the exponential of the low part are negligible beyond a certain frequency, and estimates the remaining ones by the Cauchy–Schwarz inequality together with Parseval’s identity. For the modulation spaces $M_{p,q}^s(\mathbb{R}^n)$ the scheme was carried out by Reich and Sickel [RS16], with Taylor polynomials for the low part: for $1 < p < \infty$, $1 \leq q \leq \infty$ and $s > n/q'$ they prove $\|e^{iu} - 1\| \leq C\|u\|(1 + \|u\|)^{(s+n/q)(1+1/(s-n/q'))}$ for real-valued u [RS16, Proposition 4.10], and deduce a composition theorem for functions which are inverse Fourier transforms of measures with a finite moment [RS16, Theorem 4.13]. According to [RS16] the technique goes back to the thesis of Bourdaud and was used by Bourdaud, Reissig and Sickel [BRS03]; we have not read these earlier works. For $q = 2$ the exponent of Reich and Sickel is our M with $\theta = 0$, and Theorem 1.2 has the form of their Theorem 4.13.

What is different in the space of the question is the treatment of the low part. Under the mode-by-mode supremum Parseval’s identity is lost, and only the bound $|c_\ell| \leq 1$ for the coefficients of a unimodular function remains. This is why the exponent is governed by $s + \frac{d}{2}$ here, by $\alpha + \frac{1}{2}$ in A_α , and by s in H^s ; Proposition 1.4 shows that the loss is real. The refinement $\theta > 0$, which gives the sharp exponent for $s > 1 + \frac{d}{2}$, rests on a recursion for the Fourier coefficients (Lemma 3.1) and corresponds to Leblanc’s remark for $\alpha \geq 1$; we did not see a counterpart in [RS16]. The scheme of Proposition 1.5 (closed graph or Baire category, exponentials as test functions, a lower bound for the norms of exponentials) is that of Katznelson [Kat59, Théorème 5.7 and Corollaire] and of Leblanc [Leb69, Théorème 1]; our contribution to it is the lower bound of Proposition 1.4.

Is the estimate (11) itself known? The quantity $\mathcal{N}_s$ has the structure of a modulation norm of type $M_{\infty,2}^s$, and the quantity in (12) that of an amalgam norm (Remark 9.3; this is an analogy, not a theorem). The two closest composition theorems that we know are the one of Reich and Sickel just quoted and the theorem of Kato, Sugimoto and Tomita [KST20, Theorem 1.1]: if $1 \leq p < \infty$, $\frac{4}{3} \leq q < \infty$ (or $p = q = \infty$) and $s > n/q'$, then $F(f) \in M_{s,q}^{p,q}(\mathbb{R}^n)$ for every real-valued $f \in M_{s,q}^{p,q}(\mathbb{R}^n)$ and every $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$. Both exclude $p = \infty$ with $q < \infty$, which is the pattern of the question: Reich and Sickel write that they do not know whether their Proposition 4.10 extends to $p = 1$ or $p = \infty$ [RS16, Remark 4.11], and Kato, Sugimoto and Tomita point out in their proof that the cases $p = \infty$, $q < \infty$ are excluded. Bhimani and Solanki [BS25] study weighted Fourier algebras $A_\omega^q(\Gamma, \mathcal{X})$ of functions with values in a Banach algebra $\mathcal{X}$; the space of the question is of this type, with $\Gamma = \mathbb{T}^d$, $\omega(\ell) = \langle \ell \rangle^s$, $q = 2$ and $\mathcal{X}$ the bounded functions of t . Their theorems concern

real-analytic nonlinearities and a converse for $1 \leq q < 2$. For $q = 2$ they remark that this converse fails when there is no weight [BS25, Remark 1.3]; they prove no composition theorem for functions which are not analytic. The survey of Bourdaud and Sickel [BS11] on composition operators in Besov and Lizorkin–Triebel spaces does not treat norms with a supremum over a parameter inside. Two papers of Kobayashi and Sato [KS22a, KS22b], which by their titles concern operating functions on modulation spaces and on weighted spaces of Fourier coefficients on the circle, were not accessible.

We found the estimate (11) neither in print nor on arXiv; the searches are described in the paragraph “Scope and priority” at the end. A search that finds nothing is not a proof of novelty, and we make no priority claim. The method is not new.

1.5 What stays open

The growth exponent. For $\frac{d}{2} < s < 1 + \frac{d}{2}$ we only know $s + \frac{d}{2} \leq \gamma_* \leq (s + \frac{d}{2})/(s - \frac{d}{2})$, and for $s = 1 + \frac{d}{2}$ we know $\gamma_* = s + \frac{d}{2}$ but not whether $\mathcal{E}(\lambda) = O(\lambda^{s+d/2})$. For $d = 1$ and $\frac{1}{2} < s < 1$ the lower end can be raised to $\frac{3}{2}$ (Proposition 9.1), so the lower bound of Proposition 1.4 is not the truth there; for $d \geq 2$, hence for the source with $1 < s \leq 2$, we have no such improvement.

The minimal smoothness of G . (11) holds for all $G \in C^k$ with $G(0) = 0$ if $k > M_* + \frac{3}{2}$ and fails for some such G if $k < s + \frac{d}{2}$. Even for $s > 1 + \frac{d}{2}$ a gap of $\frac{3}{2}$ derivatives remains; in the scale $A_\beta(\mathbb{T})$ there is no gap (Corollary 8.2).

The constants are explicit and very large (Section 9.1); we did not try to optimise them.

The application. The paper of Chruściel and Tod to which [Chr21] refers was not found (see “Scope and priority”). We therefore could not check in which form (11) is used there, for which functions G , and with which regularity in t .

2 Notation and elementary tools

We keep the notation of Section 1.3. If $\|\text{env}(u)\|_1 < \infty$, then for every t the Fourier series of $u(t)$ converges absolutely and uniformly, and $u(t)$ is almost everywhere equal to its sum, by the uniqueness theorem for Fourier coefficients (see, for example, [Kat04]). We then take the continuous function $\sum_\ell c_\ell(u(t))e^{i\ell \cdot x}$ as $u(t)$; it satisfies $|u(t, x)| \leq \|\text{env}(u)\|_1$. By Lemma 2.1(a) this applies whenever $\|u\|_X < \infty$. The proofs below use the supremum over t only pointwise: every inequality is proved for each t , and then the supremum is taken. No measurability or continuity in t is used, and no integral of functions with values in a Banach space is formed.

Lemma 2.1 (Cauchy–Schwarz and lattice sums). *Let $a: \mathbb{Z}^d \rightarrow [0, \infty]$ and $h_d := \frac{\sqrt{d}}{2}$.*

- (a) $\|a\|_1 \leq K_s \|a\|_s$, and for $0 \leq \theta < \sigma$: $\sum_\ell \langle \ell \rangle^\theta a_\ell \leq K_{s-\theta} \|a\|_s$.
- (b) For $\tau > \frac{d}{2}$: $K_\tau^2 \leq (1 + h_d)^{2\tau} \omega_d / (2\tau - d)$.
- (c) For $\rho \geq 0$: $\sum_{|\ell| > \rho} a_\ell \leq T_s(\rho) \|a\|_s$ with $T_s(\rho) := (\sum_{|\ell| > \rho} \langle \ell \rangle^{-2s})^{1/2} \leq C_T \langle \rho \rangle^{-\sigma}$.

Proof. The three inequalities between sums follow from the Cauchy–Schwarz inequality after writing $a_\ell = \langle \ell \rangle^{-s} \cdot \langle \ell \rangle^s a_\ell$ and $\langle \ell \rangle^\theta a_\ell = \langle \ell \rangle^{\theta-s} \cdot \langle \ell \rangle^s a_\ell$. For the lattice sums let $Q_\ell := \ell + [-\frac{1}{2}, \frac{1}{2}]^d$; these cubes tile $\mathbb{R}^d$ and have volume 1. For $x \in Q_\ell$ we have $|x - \ell| \leq h_d$, hence $1 + |x| \leq 1 + |\ell| + h_d \leq (1 + h_d) \langle \ell \rangle$, that is, $\langle \ell \rangle^{-2\tau} \leq (1 + h_d)^{2\tau} (1 + |x|)^{-2\tau}$; and $|\ell| > \rho$ implies $|x| > \rho - h_d$. With $r_0 := \max\{\rho - h_d, 0\}$, polar coordinates and $t^{d-1} \leq (1+t)^{d-1}$,

$$\sum_{|\ell| > \rho} \langle \ell \rangle^{-2\tau} \leq (1 + h_d)^{2\tau} \int_{|x| \geq r_0} \frac{dx}{(1 + |x|)^{2\tau}} \leq (1 + h_d)^{2\tau} \omega_d \int_{r_0}^\infty (1+t)^{d-1-2\tau} dt = \frac{(1 + h_d)^{2\tau} \omega_d}{(2\tau - d)(1 + r_0)^{2\tau-d}}.$$

The same computation over all ℓ (with $r_0 = 0$) gives (b). For $\tau = s$ we have $2\tau - d = 2\sigma$, and $1 + r_0 \geq (1 + \rho)/(1 + h_d)$: if $\rho \geq h_d$ this is $(1 + \rho - h_d)(1 + h_d) = 1 + \rho + h_d(\rho - h_d) \geq 1 + \rho$, and if $\rho < h_d$ it is $1 + h_d \geq 1 + \rho$. Hence $T_s(\rho)^2 \leq (1 + h_d)^{2s+2\sigma} \omega_d(\rho)^{-2\sigma} / (2\sigma) = C_T^2 \langle \rho \rangle^{-2\sigma}$. $\square$

Lemma 2.2 (Weighted Young inequality). *For $a, b: \mathbb{Z}^d \rightarrow [0, \infty]$: $\|a * b\|_1 = \|a\|_1 \|b\|_1$ and*

$$\|a * b\|_s \leq c_s (\|a\|_s \|b\|_1 + \|a\|_1 \|b\|_s) \leq 2c_s K_s \|a\|_s \|b\|_s.$$

Proof. We have $\langle \ell \rangle \leq \langle \ell - k \rangle + \langle k \rangle$, and $(x + y)^s \leq c_s(x^s + y^s)$ for $x, y \geq 0$ (convexity of $x \mapsto x^s$ if $s \geq 1$, subadditivity if $s \leq 1$). Hence $\langle \ell \rangle^s (a * b)_\ell \leq c_s ((\langle \cdot \rangle^s a) * b + a * (\langle \cdot \rangle^s b))_\ell$, and Young's inequality $\|f * g\|_{\ell^2} \leq \|f\|_{\ell^2} \|g\|_{\ell^1}$ gives the first bound. The second follows from Lemma 2.1(a). $\square$

Lemma 2.3 (Convolution powers). *For $a: \mathbb{Z}^d \rightarrow [0, \infty)$ with $\|a\|_1 < \infty$ and every integer $k \geq 1$: $\|a^{*k}\|_1 = \|a\|_1^k$ and $\|a^{*k}\|_s \leq k c_s^{k-1} \|a\|_1^{k-1} \|a\|_s$.*

Proof. For $k = 1$ this is an equality. If it holds for k , then by Lemma 2.2 $\|a^{*k} * a\|_s \leq c_s (k c_s^{k-1} \|a\|_1^k \|a\|_s + \|a\|_1^k \|a\|_s) \leq (k + 1) c_s^k \|a\|_1^k \|a\|_s$, because $c_s \geq 1$. $\square$

Lemma 2.4 (Envelopes of sums and products). *Let u, w be families on the same index set with $\text{env}(u), \text{env}(w) \in \ell^1(\mathbb{Z}^d)$. Then, pointwise on $\mathbb{Z}^d$,*

$$\text{env}(u + w) \leq \text{env}(u) + \text{env}(w), \quad \text{env}(uw) \leq \text{env}(u) * \text{env}(w),$$

and consequently $\|uw\|_X \leq 2c_s K_s \|u\|_X \|w\|_X$.

Proof. For each t the functions $u(t)$ and $w(t)$ have absolutely convergent Fourier series, so $c_\ell(u(t)w(t)) = \sum_k c_{\ell-k}(u(t))c_k(w(t))$ and $|c_\ell(u(t)w(t))| \leq \sum_k \text{env}(u)_{\ell-k} \text{env}(w)_k$. Take the supremum over t , and apply Lemma 2.2. $\square$

The algebra property already gives (11) for analytic functions. We record this with its constant, for comparison with Corollary 1.3.

Proposition 2.5 (Analytic nonlinearities). *Let $G(u) = \sum_{k \geq 1} g_k u^k$ be a power series with radius of convergence $\rho_G > 0$, and $\tilde{G}(x) := \sum_{k \geq 1} |g_k| x^k$. For every family φ of complex-valued functions with $c_s K_s \|\varphi\|_X < \rho_G$,*

$$\|G \circ \varphi\|_X \leq \tilde{G}'(c_s K_s \|\varphi\|_X) \|\varphi\|_X.$$

In particular, for $d = 2$: if $\rho_G > \frac{c_s K_s}{2\pi}$, then (11) holds on the set (10) with $C_2 = \tilde{G}'(\frac{c_s K_s}{2\pi})^2$.

Proof. Let $m := \text{env}(\varphi)$. Then $|\varphi(t, x)| \leq \|m\|_1 \leq K_s \|m\|_s < \rho_G$, so $G(\varphi(t)) = \sum_k g_k \varphi(t)^k$ converges uniformly and $c_\ell(G \circ \varphi(t)) = \sum_k g_k c_\ell(\varphi(t)^k)$. By Lemma 2.4 and induction $|c_\ell(\varphi(t)^k)| \leq (m^{*k})_\ell$, hence $\text{env}(G \circ \varphi) \leq \sum_k |g_k| m^{*k}$. By Lemmas 2.1 and 2.3, $\|m^{*k}\|_s \leq k (c_s K_s \|m\|_s)^{k-1} \|m\|_s$, and the sum over k is $\tilde{G}'(c_s K_s \|m\|_s) \|m\|_s$. $\square$

3 The low part: unimodular families

The exponential of a real-valued function is unimodular, so all its Fourier coefficients are at most 1 in modulus. For a band-limited function the coefficients are moreover negligible beyond a radius which is controlled by the size of the gradient. Both facts are uniform in a family.

Lemma 3.1 (Unimodular low-frequency families). *Let $\rho \geq 0$, $\lambda \in \mathbb{R}$, and let ψ be a family of real-valued trigonometric polynomials $\psi(t) = \sum_{|k| \leq \rho} c_k(t) e^{ik \cdot x}$ with finite envelope m . Put*

$$\nu := |\lambda| \sum_{|k| \leq \rho} |k| m_k, \quad e := \text{env}(e^{i\lambda\psi}).$$

Then

(i) $e_\ell \leq 1$ for all $\ell \in \mathbb{Z}^d$;

(ii) for every integer $N \geq 1$ and every $\ell \neq 0$ with $|\ell| \geq 2(N-1)\rho$: $e_\ell \leq (2\sqrt{d}\nu/|\ell|)^N$.

Consequently $e \leq p$ pointwise, where, with $R := \max\{2(N-1)\rho, 2\sqrt{d}\nu\}$,

$$p_\ell := \min\{1, (R/|\ell|)^N\} \quad (\ell \neq 0), \quad p_0 := 1. \quad (3.1)$$

Proof. (i) Since $\psi(t)$ is real-valued, $|e^{i\lambda\psi(t,x)}| = 1$ and $|c_\ell(e^{i\lambda\psi(t)})| \leq (2\pi)^{-d} \int_{\mathbb{T}^d} 1 dx = 1$.

(ii) Fix t and put $U := e^{i\lambda\psi(t)} \in C^\infty(\mathbb{T}^d)$. For $j = 1, \dots, d$ we have $\partial_j U = i\lambda(\partial_j \psi(t))U$. The ℓ -th Fourier coefficient of the left side is $i\ell_j c_\ell(U)$ (integration by parts on the torus). On the right side, $\partial_j \psi(t) = \sum_{|k| \leq \rho} ik_j c_k(t) e^{ik \cdot x}$ is a trigonometric polynomial and $c_\ell(e^{ik \cdot x} U) = c_{\ell-k}(U)$. Hence $\ell_j c_\ell(U) = i\lambda \sum_{|k| \leq \rho} k_j c_k(t) c_{\ell-k}(U)$, and

$$|\ell_j| |c_\ell(U)| \leq |\lambda| \sum_{|k| \leq \rho} |k_j| m_k e_{\ell-k} \leq \nu \max_{|k| \leq \rho} e_{\ell-k}.$$

The right side does not depend on t . Taking the supremum over t and choosing j with $|\ell_j| = \max_i |\ell_i| \geq |\ell|/\sqrt{d}$ we get

$$e_\ell \leq \frac{\sqrt{d}\nu}{|\ell|} \max_{|k| \leq \rho} e_{\ell-k} \quad (\ell \neq 0). \quad (3.2)$$

For real x put $\eta(x) := \sup\{e_\ell : |\ell| \geq x\} \leq 1$; this function is non-increasing. If $x > 0$, $|\ell| \geq x$ and $|k| \leq \rho$, then $|\ell - k| \geq x - \rho$, so (3.2) gives $e_\ell \leq (\sqrt{d}\nu/x) \eta(x - \rho)$; hence $\eta(x) \leq (\sqrt{d}\nu/x) \eta(x - \rho)$ for all $x > 0$. Now let $N \geq 1$ and $x > 0$ with $x \geq 2(N-1)\rho$. For $0 \leq i \leq N-1$ we have $x - i\rho \geq x - (N-1)\rho \geq x/2 > 0$, so N applications give

$$\eta(x) \leq \prod_{i=0}^{N-1} \frac{\sqrt{d}\nu}{x - i\rho} \cdot \eta(x - N\rho) \leq \left(\frac{2\sqrt{d}\nu}{x}\right)^N.$$

With $x = |\ell|$ this is (ii).

For the last statement let $\ell \neq 0$. If $|\ell| \geq R$, then $|\ell| \geq 2(N-1)\rho$ and $2\sqrt{d}\nu \leq R$, so $e_\ell \leq (R/|\ell|)^N$ by (ii), and $e_\ell \leq 1$ by (i). If $|\ell| < R$, then $p_\ell = 1$. (If $R = 0$, then $\nu = 0$, every $e^{i\lambda\psi(t)}$ is constant in x , and $e_\ell = 0 = p_\ell$ for $\ell \neq 0$.) $\square$

Lemma 3.2 (Norms of the profile). *Let $R \geq 0$, let N be an integer with $N \geq s + \frac{d}{2} + \frac{1}{2}$, and let p be defined by (3.1). Then*

$$\|p\|_s \leq 2^{d+s+1} \langle R \rangle^{s+d/2}, \quad \|p\|_1 \leq 2^{2d+2} \langle R \rangle^d.$$

Proof. For $x \geq 0$ the number of $\ell \in \mathbb{Z}^d$ with $|\ell| \leq x$ is at most $(2x+1)^d \leq 2^d \langle x \rangle^d$. If $R = 0$, then p is 1 at the origin and 0 elsewhere, and both norms are 1. Let $R > 0$. The ball $|\ell| \leq R$ contributes at most $2^d \langle R \rangle^d \cdot \langle R \rangle^{2s}$ to $\|p\|_s^2$ and at most $2^d \langle R \rangle^d$ to $\|p\|_1$. On the shell $2^j R < |\ell| \leq 2^{j+1} R$ ($j \geq 0$)

we have $p_\ell \leq 2^{-jN}$ and $\langle \ell \rangle \leq 1 + 2^{j+1}R \leq 2^{j+1}\langle R \rangle$, and the number of its points is at most $(2^{j+2}R + 1)^d \leq 2^d(2^{j+1}\langle R \rangle)^d$. Hence the shells contribute at most

$$\sum_{j \geq 0} 2^d 2^{(j+1)d} \langle R \rangle^d \cdot 2^{2s(j+1)} \langle R \rangle^{2s} \cdot 2^{-2jN} = 2^{2d+2s} \langle R \rangle^{2s+d} \sum_{j \geq 0} 2^{-j(2N-2s-d)} \leq 2^{2d+2s+1} \langle R \rangle^{2s+d}$$

to $\|p\|_s^2$, because $2N - 2s - d \geq 1$, and at most $\sum_{j \geq 0} 2^d 2^{(j+1)d} \langle R \rangle^d 2^{-jN} = 2^{2d} \langle R \rangle^d \sum_{j \geq 0} 2^{-j(N-d)} \leq 2^{2d+1} \langle R \rangle^d$ to $\|p\|_1$, because $N - d$ is an integer greater than σ , hence at least 1. Therefore $\|p\|_s^2 \leq (2^d + 2^{2d+2s+1}) \langle R \rangle^{2s+d} \leq 2^{2d+2s+2} \langle R \rangle^{2s+d}$ and $\|p\|_1 \leq (2^d + 2^{2d+1}) \langle R \rangle^d \leq 2^{2d+2} \langle R \rangle^d$. $\square$

4 The high part: a majorant

If the envelope of a family is small in ℓ^1 , the exponential series can be summed term by term. Real-valuedness is not needed here.

Lemma 4.1 (Majorant for the exponential). *Let ψ be a family of complex-valued functions with $\text{env}(\psi) \leq a$, where $a: \mathbb{Z}^d \rightarrow [0, \infty)$ and $\|a\|_1 < \infty$, and let $\lambda \in \mathbb{R}$. Then*

$$\text{env}(e^{i\lambda\psi} - 1) \leq v := \sum_{k \geq 1} \frac{|\lambda|^k}{k!} a^{*k}, \quad \|v\|_1 = e^{|\lambda|\|a\|_1} - 1, \quad \|v\|_s \leq |\lambda| \|a\|_s e^{c_s |\lambda| \|a\|_1}.$$

Proof. We have $|\psi(t, x)| \leq \|a\|_1$, so $e^{i\lambda\psi(t)} - 1 = \sum_{k \geq 1} (i\lambda)^k \psi(t)^k / k!$ converges uniformly in x and $c_\ell(e^{i\lambda\psi(t)} - 1) = \sum_{k \geq 1} (i\lambda)^k c_\ell(\psi(t)^k) / k!$. By Lemma 2.4 and induction, $|c_\ell(\psi(t)^k)| \leq (a^{*k})_\ell$. This gives the pointwise bound. The formula for $\|v\|_1$ follows from $\|a^{*k}\|_1 = \|a\|_1^k$, and by Lemma 2.3

$$\|v\|_s \leq \sum_{k \geq 1} \frac{|\lambda|^k}{k!} k c_s^{k-1} \|a\|_1^{k-1} \|a\|_s = |\lambda| \|a\|_s \sum_{k \geq 1} \frac{(c_s |\lambda| \|a\|_1)^{k-1}}{(k-1)!}. \quad \square$$

5 Proof of Theorem 1.1

Let ψ be a family of real-valued functions with envelope m , $\|m\|_s \leq 1$, and let $\lambda \in \mathbb{R}$. For $\lambda = 0$ the claim is $\|1\|_X = 1 \leq C_A$. Let $\lambda \neq 0$.

Step 0: the cut-off frequency. Put $\langle \rho \rangle := \max\{1, (c_s C_T |\lambda|)^{1/\sigma}\}$ and $\rho := \langle \rho \rangle - 1 \geq 0$. Then $|\lambda| C_T \langle \rho \rangle^{-\sigma} \leq 1/c_s$ (if the maximum is the second term this is an equality; otherwise $c_s C_T |\lambda| \leq 1$). By Lemma 2.1(c),

$$|\lambda| T_s(\rho) \leq \frac{1}{c_s}. \quad (5.1)$$

The number ρ depends on λ , d and s only: the tail bound of Lemma 2.1(c) is uniform on the unit ball. This is the step which uses the weight.

Step 1: splitting. For each t let $\psi_L(t) := \sum_{|k| \leq \rho} c_k(\psi(t)) e^{ik \cdot x}$ and $\psi_H(t) := \psi(t) - \psi_L(t)$. Since $\psi(t)$ is real-valued, $c_{-k}(\psi(t)) = \overline{c_k(\psi(t))}$; the set $\{|k| \leq \rho\}$ is symmetric, so $\psi_L(t)$ is a real-valued trigonometric polynomial. The envelopes are $\text{env}(\psi_L) = m \mathbf{1}_{|k| \leq \rho}$ and $\text{env}(\psi_H) = m_H := m \mathbf{1}_{|k| > \rho}$. By Lemma 2.1(c) and (5.1),

$$|\lambda| \|m_H\|_1 \leq |\lambda| T_s(\rho) \|m\|_s \leq \frac{1}{c_s}.$$

Put $U(t) := e^{i\lambda\psi_L(t)}$ and $V(t) := e^{i\lambda\psi_H(t)} - 1$. Then $e^{i\lambda\psi(t)} = U(t) + U(t)V(t)$.

Step 2: the high part. By Lemma 4.1 with $a = m_H$ we have $\text{env}(V) \leq v$ with $\|v\|_1 \leq e^{1/c_s} - 1$ and $\|v\|_s \leq |\lambda| \|m_H\|_s e \leq e|\lambda|$. Moreover $c_s \|v\|_1 \leq c_s(e^{1/c_s} - 1) \leq e - 1$, because $y \mapsto (e^y - 1)/y$ is

increasing and $1/c_s \leq 1$. Note that the high part need not be small in the norm of X ; only $|\lambda| \|m_H\|_1$ is small.

Step 3: the low part. By Lemma 3.1, applied to the family ψ_L with $N = \lceil s + \frac{d}{2} + \frac{1}{2} \rceil$, we have $\text{env}(U) \leq p$, where p is the profile (3.1) with $R := \max\{2(N-1)\rho, 2\sqrt{d}\nu\}$ and $\nu := |\lambda| \sum_{|k| \leq \rho} |k| m_k$. For $|k| \leq \rho$ we have $|k| \leq \langle k \rangle^{1-\theta} \langle k \rangle^\theta \leq \langle \rho \rangle^{1-\theta} \langle k \rangle^\theta$, because $0 \leq \theta \leq 1$. So by Lemma 2.1(a), which needs $\theta < \sigma$,

$$\nu \leq |\lambda| \langle \rho \rangle^{1-\theta} \sum_k \langle k \rangle^\theta m_k \leq |\lambda| K_{s-\theta} \langle \rho \rangle^{1-\theta}.$$

Step 4: assembly. Both p and v are in ℓ^1 (Lemmas 3.2 and 4.1), so Lemma 2.4 applies: $\text{env}(e^{i\lambda\psi}) \leq \text{env}(U) + \text{env}(UV) \leq p + p * v$. By Lemma 2.2, Step 2 and Lemma 3.2,

$$\begin{aligned} \|e^{i\lambda\psi}\|_X &\leq \|p\|_s + c_s (\|p\|_s \|v\|_1 + \|p\|_1 \|v\|_s) \leq e \|p\|_s + e c_s |\lambda| \|p\|_1 \\ &\leq e 2^{d+s+1} \langle R \rangle^{s+d/2} + e c_s 2^{2d+2} |\lambda| \langle R \rangle^d. \end{aligned} \quad (5.2)$$

Step 5: the size of R . We have $\langle \rho \rangle \leq B(1 + |\lambda|)^{1/\sigma}$: either $\langle \rho \rangle = 1 \leq B$, or $\langle \rho \rangle = (c_s C_T |\lambda|)^{1/\sigma} \leq B |\lambda|^{1/\sigma}$. Since $\kappa - \frac{1}{\sigma} = 1 - \frac{\theta}{\sigma} > 0$ and $B \geq 1$, this gives $\langle \rho \rangle \leq B(1 + |\lambda|)^\kappa$ and $|\lambda| \langle \rho \rangle^{1-\theta} \leq (1 + |\lambda|) B^{1-\theta} (1 + |\lambda|)^{(1-\theta)/\sigma} \leq B(1 + |\lambda|)^\kappa$. Therefore, by Step 3,

$$\langle R \rangle \leq 1 + 2(N-1)\rho + 2\sqrt{d}\nu \leq 1 + 2(N-1)\langle \rho \rangle + 2\sqrt{d} K_{s-\theta} |\lambda| \langle \rho \rangle^{1-\theta} \leq B_R (1 + |\lambda|)^\kappa.$$

Step 6: conclusion. Insert this into (5.2). The first term is at most $e 2^{d+s+1} B_R^{s+d/2} (1 + |\lambda|)^{(s+d/2)\kappa}$. The second is at most $e c_s 2^{2d+2} B_R^d (1 + |\lambda|)^{1+d\kappa}$, and $1 + d\kappa \leq (s + \frac{d}{2})\kappa$, because $\sigma\kappa = \sigma + 1 - \theta \geq 1$. This is the claim with $M = (s + \frac{d}{2})\kappa$ and the constant C_A of (1.4). $\square$

Remark 5.1 (Where the exponent comes from). The low part is a unimodular function whose coefficients are negligible beyond $|\ell| \approx R$. On the ball $|\ell| \leq R$ only the bound 1 is used, which costs $\|1_{|\ell| \leq R}\|_s \approx R^{s+d/2}$; Proposition 1.4 shows that for $R \approx \lambda$ this cost is real. The radius is $R \approx \max\{\rho, \nu\}$ with $\rho \approx \lambda^{1/\sigma}$. With $\theta = 0$ one only uses $\nu \leq |\lambda| \rho K_s$, so $R \approx \lambda^{1+1/\sigma}$; this is the radius which a Taylor polynomial of degree about λ of the exponential of the low part would give, and it leads to the exponent $(s + \frac{d}{2})(1 + \frac{1}{\sigma})$ of [RS16, Proposition 4.10] with $q = 2$. With $\theta > 0$ the weight is also used to bound the gradient of the low part, and for $\sigma > 1$ one gets $\nu \lesssim \lambda$ and $R \approx \lambda$.

Remark 5.2 (The Wiener algebra). In $A(\mathbb{T})$ (no weight, ℓ^1 norm) the proof breaks down at Step 0: the unit ball of ℓ^1 has no uniform tail decay. And indeed no bound of the form $\sup\{\|e^{i\lambda u}\|_A : u \text{ real-valued}, \|u\|_A \leq 1\} \leq C(1 + |\lambda|)^M$ can hold there: by the Fourier inversion argument of Section 6 it would make every compactly supported C^k function with $k > M + 2$ operate on the real-valued functions of $A(\mathbb{T})$, in contradiction to the theorem of Katznelson quoted in Section 1.2.

6 Proof of Theorem 1.2 and corollaries

Proof of Theorem 1.2. Let $m := \text{env}(\varphi)$. Then $\|m\|_1 \leq K_s \|m\|_s \leq K_s r$, so $|\varphi(t, x)| \leq K_s r$. The function G is continuous, because $|e^{i\lambda u} - 1| \leq |\lambda| |u|$ and $\int |\lambda| d\mu \leq J_\mu(r) < \infty$.

(i) *One exponential.* Fix $\lambda \in \mathbb{R}$, and let b^λ be the envelope of the family $(\tau, t) \mapsto e^{i\tau\lambda\varphi(t)}$ indexed by $[0, 1] \times I$. The family $(\tau, t) \mapsto \tau\varphi(t)/r$ is real-valued with envelope m/r , of norm at most 1, and $e^{i\tau\lambda\varphi(t)} = e^{i(\lambda r)\tau\varphi(t)/r}$. So Theorem 1.1, which holds for arbitrary index sets, gives $\|b^\lambda\|_s \leq C_A(1 + r|\lambda|)^M$; in particular $b^\lambda \in \ell^1$. For real u we have $e^{i\lambda u} - 1 = i\lambda u \int_0^1 e^{i\tau\lambda u} d\tau$. By Fubini's theorem (the integrand is continuous on $[0, 1] \times \mathbb{T}^d$) and the product formula for absolutely convergent Fourier series,

$$|c_\ell(e^{i\lambda\varphi(t)} - 1)| = |\lambda| \left| \int_0^1 c_\ell(\varphi(t) e^{i\tau\lambda\varphi(t)}) d\tau \right| \leq |\lambda| (m * b^\lambda)_\ell \quad \text{for all } t \in I, \ell \in \mathbb{Z}^d. \quad (6.1)$$

This is the place where $G(0) = 0$ enters: the representation of G contains only the differences $e^{i\lambda u} - 1$, and each of them carries one factor u , hence one factor m .

(ii) Put $D_\ell(\lambda) := \sup_t |c_\ell(e^{i\lambda\varphi(t)} - 1)|$. By (6.1) and Lemma 2.2,

$$\|D(\lambda)\|_s \leq 2c_s K_s |\lambda| \|m\|_s \|b^\lambda\|_s \leq 2c_s K_s C_A |\lambda| (1 + r|\lambda|)^M \|\varphi\|_X.$$

(iii) For fixed t we have $G(\varphi(t, x)) = \int_{\mathbb{R}} (e^{i\lambda\varphi(t, x)} - 1) d\mu(\lambda)$, and the integrand is bounded by $|\lambda| K_s r$, which is $|\mu|$ -integrable. By Fubini's theorem $c_\ell(G \circ \varphi(t)) = \int_{\mathbb{R}} c_\ell(e^{i\lambda\varphi(t)} - 1) d\mu(\lambda)$, hence $|c_\ell(G \circ \varphi(t))| \leq \int_{\mathbb{R}} D_\ell(\lambda) d|\mu|(\lambda)$ for every t . Here D_ℓ is a supremum of continuous functions of λ , hence lower semicontinuous, hence a Borel function, whatever the index set is. Taking the supremum over t on the left we get $\text{env}(G \circ \varphi)_\ell \leq \int_{\mathbb{R}} D_\ell(\lambda) d|\mu|(\lambda)$.

(iv) Minkowski's integral inequality in the weighted space $\ell^2(\mathbb{Z}^d; \langle \ell \rangle^{2s})$ (see, for example, [Fol99]) and (ii) give

$$\|G \circ \varphi\|_X \leq \int_{\mathbb{R}} \|D(\lambda)\|_s d|\mu|(\lambda) \leq 2c_s K_s C_A \int_{\mathbb{R}} |\lambda| (1 + r|\lambda|)^M d|\mu|(\lambda) \cdot \|\varphi\|_X.$$

Note the order of the operations: Fubini's theorem for scalar functions at fixed (ℓ, t) ; then the supremum over t at fixed ℓ , under the integral; then Minkowski's inequality in a scalar sequence space. The supremum over t stays inside the sum over ℓ throughout. $\square$

Proof of Corollary 1.3. (a) Let $\widehat{H}(\lambda) := \int_{\mathbb{R}} H(u) e^{-i\lambda u} du$. Then $|\widehat{H}(\lambda)| \leq \|H\|_{L^1}$, and k integrations by parts give $|\widehat{H}(\lambda)| \leq \|H^{(k)}\|_{L^1} |\lambda|^{-k}$. As $k \geq 2$, $\widehat{H}$ is integrable, and $H(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{H}(\lambda) e^{i\lambda u} d\lambda$ for all u . Let $d\mu := \frac{1}{2\pi} \widehat{H}(\lambda) d\lambda$. Then $\int_{\mathbb{R}} (e^{i\lambda u} - 1) d\mu(\lambda) = H(u) - H(0) = H(u)$ for all real u , since $H(0) = G(0) = 0$. A family φ with $\|\varphi\|_X \leq r$ takes its values in $[-K_s r, K_s r]$, where $H = G$; so $G \circ \varphi = H \circ \varphi$, and Theorem 1.2 applies with $J_\mu(r) = \frac{1}{2\pi} \int_{\mathbb{R}} |\lambda| (1 + r|\lambda|)^M |\widehat{H}(\lambda)| d\lambda$.

For (1.7) we use $|\lambda| (1 + r|\lambda|)^M \leq (1 + r)^M$ for $|\lambda| \leq 1$ and $\leq (1 + r) |\lambda|^{M+1}$ for $|\lambda| \geq 1$:

$$J_\mu(r) \leq \frac{(1 + r)^M}{2\pi} \left(\int_{|\lambda| \leq 1} \|H\|_{L^1} d\lambda + \int_{|\lambda| > 1} \|H^{(k)}\|_{L^1} |\lambda|^{M+1-k} d\lambda \right) = \frac{(1 + r)^M}{\pi} \left(\|H\|_{L^1} + \frac{\|H^{(k)}\|_{L^1}}{k - M - 2} \right),$$

which needs $k > M + 2$. For (1.6) we use $|\lambda| (1 + r|\lambda|)^M \leq (1 + r)^M (1 + |\lambda|)^{M+1}$, the Cauchy-Schwarz inequality, $(1 + |\lambda|)^k \leq 2^{k-1} (1 + |\lambda|^k)$ and Plancherel's theorem ($\|\widehat{H}\|_{L^2} = (2\pi)^{1/2} \|H\|_{L^2}$, $\|\lambda^k \widehat{H}\|_{L^2} = (2\pi)^{1/2} \|H^{(k)}\|_{L^2}$):

$$\begin{aligned} J_\mu(r) &\leq \frac{(1 + r)^M}{2\pi} \|(1 + |\lambda|)^k \widehat{H}\|_{L^2} \|(1 + |\lambda|)^{M+1-k}\|_{L^2} \\ &\leq \frac{(1 + r)^M}{(2\pi)^{1/2}} 2^{k-1} (\|H\|_{L^2} + \|H^{(k)}\|_{L^2}) \left(\frac{2}{2k - 2M - 3} \right)^{1/2}, \end{aligned}$$

where $\int_{\mathbb{R}} (1 + |\lambda|)^{-2(k-M-1)} d\lambda = \frac{2}{2k-2M-3}$ needs $k > M + \frac{3}{2}$. (Then $k \geq 3$, because $M \geq s + \frac{d}{2} > 1$.)

(b) For $d = 2$, (1.2) gives $\mathcal{N}_s(u) = 4\pi^2 \|u\|_X^2$ for every family. If $\mathcal{N}_s(\varphi) \leq 1$, then $\|\varphi\|_X \leq \frac{1}{2\pi} = r$ and $\mathcal{N}_s(G \circ \varphi) = 4\pi^2 \|G \circ \varphi\|_X^2 \leq C_B^2 \cdot 4\pi^2 \|\varphi\|_X^2 = C_B^2 \mathcal{N}_s(\varphi)$. A smooth function is in C^k for every k . $\square$

In the following corollaries $C_B(G; r)$ denotes a constant with which the conclusion of Corollary 1.3(a) holds for the function G and the radius r .

Corollary 6.1 (Variants of the answer). *Let $d = 2$, $s > 1$ and $G \in C^k(\mathbb{R}; \mathbb{C})$ with $k > M + \frac{3}{2}$.*

(a) *Without the assumption $G(0) = 0$: if $\mathcal{N}_s(\varphi) \leq 1$, then $\mathcal{N}_s(G \circ \varphi)^{1/2} \leq 2\pi |G(0)| + C_B (G - G(0); \frac{1}{2\pi}) \mathcal{N}_s(\varphi)^{1/2}$.*

- (b) Let $G(0) = 0$, and let $\mathcal{N}'$ be defined like $\mathcal{N}_s$, but with a weight w_ℓ such that $a^{-1}\langle \ell \rangle^{2s} \leq w_\ell \leq a\langle \ell \rangle^{2s}$ for some $a \geq 1$, or with the real orthonormal basis of the products of $\frac{1}{\sqrt{2\pi}}, \frac{\cos nx_j}{\sqrt{\pi}}, \frac{\sin nx_j}{\sqrt{\pi}}$ in x_1 and x_2 and the weight $(1 + (n_1^2 + n_2^2)^{1/2})^{2s}$ (then $a = 4$). If $\mathcal{N}'(\varphi) \leq 1$, then $\mathcal{N}'(G \circ \varphi) \leq a^2 C_B(G; \frac{\sqrt{a}}{2\pi})^2 \mathcal{N}'(\varphi)$.

Proof. (a) $G \circ \varphi = G(0) + (G - G(0)) \circ \varphi$, and the constant function $G(0)$ has $\|G(0)\|_X = |G(0)|$.

(b) In both cases $a^{-1}\mathcal{N}_s \leq \mathcal{N}' \leq a\mathcal{N}_s$. For a weight this is clear. For the real basis fix (n_1, n_2) with $n_1, n_2 \geq 0$: the at most four basis functions with these frequencies span the same space as the at most four exponentials $f_{(\pm n_1, \pm n_2)}$, the two orthonormal systems are related by a unitary matrix, and all these indices carry the same weight. So for every t a squared coefficient in one system is at most the sum of the squared moduli of the coefficients in the other; hence the supremum over t of a squared coefficient in one system is at most the sum of the suprema in the other, and summing over the at most four members gives $a = 4$. Now if $\mathcal{N}'(\varphi) \leq 1$, then $\mathcal{N}_s(\varphi) \leq a$, $\|\varphi\|_X \leq \frac{\sqrt{a}}{2\pi}$, and $\mathcal{N}'(G \circ \varphi) \leq a\mathcal{N}_s(G \circ \varphi) \leq aC_B^2\mathcal{N}_s(\varphi) \leq a^2C_B^2\mathcal{N}'(\varphi)$. $\square$

With the supremum inside, the quantity $\mathcal{N}_s$ is not invariant under unitary changes of basis inside the eigenspaces of the Laplacian, whose dimensions are unbounded; the equivalence in (b) is special to the basis named there.

Corollary 6.2 (Several functions; complex-valued families). *Let $n \geq 1$, let $k > M + n + 1$ be an integer, $G \in C^k(\mathbb{R}^n; \mathbb{C})$ with $G(0) = 0$, and $r > 0$. There is a constant $C < \infty$ such that for all families $\varphi^1, \dots, \varphi^n$ of real-valued functions on the same index set with $\|\varphi\| := (\sum_j \|\varphi^j\|_X^2)^{1/2} \leq r$ we have $\|G(\varphi^1, \dots, \varphi^n)\|_X \leq C\|\varphi\|$. For a family φ of complex-valued functions this applies to $(\operatorname{Re} \varphi, \operatorname{Im} \varphi)$, for which $\|(\operatorname{Re} \varphi, \operatorname{Im} \varphi)\| \leq \sqrt{2}\|\varphi\|_X$.*

Proof. Choose $H \in C^k(\mathbb{R}^n; \mathbb{C})$ with compact support and $H = G$ on the cube $[-K_s r, K_s r]^n$, and let $\widehat{H}(\lambda) = \int H(u) e^{-i\lambda \cdot u} du$. Integrating by parts k times in the coordinate j with $|\lambda_j| \geq |\lambda|/\sqrt{n}$ gives $|\widehat{H}(\lambda)| \leq \min\{\|H\|_{L^1}, n^{k/2} \max_j \|\partial_j^k H\|_{L^1} |\lambda|^{-k}\}$. As $k > n$, $\widehat{H} \in L^1(\mathbb{R}^n)$ and $G(u) = \int_{\mathbb{R}^n} (e^{i\lambda \cdot u} - 1) d\mu(\lambda)$ on the cube, with $d\mu := (2\pi)^{-n} \widehat{H} d\lambda$. For $\lambda \neq 0$ write $\lambda = |\lambda|\omega$. The family $\psi_\omega := \sum_j \omega_j \varphi^j$ is real-valued with $\operatorname{env}(\psi_\omega) \leq \sum_j |\omega_j| \operatorname{env}(\varphi^j)$, so $\|\psi_\omega\|_X \leq \sum_j |\omega_j| \|\varphi^j\|_X \leq \|\varphi\| \leq r$, and $e^{i\lambda \cdot \varphi} = e^{i|\lambda|\psi_\omega}$. Steps (i) and (ii) of the proof of Theorem 1.2, for ψ_ω and the parameter $|\lambda|$, give $\|\operatorname{env}(e^{i\lambda \cdot \varphi} - 1)\|_s \leq 2c_s K_s C_A |\lambda| (1 + r|\lambda|)^M \|\varphi\|$, and steps (iii) and (iv), with $\mathbb{R}^n$ in place of $\mathbb{R}$, give the claim with $C := 2c_s K_s C_A \int_{\mathbb{R}^n} |\lambda| (1 + r|\lambda|)^M d\mu(\lambda)$, which is finite because $k > M + 1 + n$. For complex φ : $c_\ell(\operatorname{Re} \varphi) = \frac{1}{2}(c_\ell(\varphi) + \overline{c_{-\ell}(\varphi)})$, so $\operatorname{env}(\operatorname{Re} \varphi)_\ell \leq \frac{1}{2}(\operatorname{env}(\varphi)_\ell + \operatorname{env}(\varphi)_{-\ell})$ and $\|\operatorname{Re} \varphi\|_X \leq \|\varphi\|_X$; the same holds for $\operatorname{Im} \varphi$. $\square$

Corollary 6.3 (Differences). *Let $k > M + \frac{3}{2}$ be an integer, $G \in C^{k+1}(\mathbb{R}; \mathbb{C})$, $r > 0$ and $F := G' - G'(0)$. For families φ, ψ of real-valued functions on the same index set with $\|\varphi\|_X \leq r$ and $\|\psi\|_X \leq r$,*

$$\|G \circ \varphi - G \circ \psi\|_X \leq \left(|G'(0)| + 2\sqrt{2} c_s K_s r C_B(F; \sqrt{2}r) \right) \|\varphi - \psi\|_X.$$

Proof. For real a, b : $G(a) - G(b) = (a - b)(G'(0) + \int_0^1 F(b + \tau(a - b)) d\tau)$. Let χ be the real-valued family $(\tau, t) \mapsto (1 - \tau)\psi(t) + \tau\varphi(t)$ on $[0, 1] \times I$. Its envelope is at most $\max\{\operatorname{env}(\varphi), \operatorname{env}(\psi)\}$ pointwise, so $\|\chi\|_X \leq (\|\varphi\|_X^2 + \|\psi\|_X^2)^{1/2} \leq \sqrt{2}r$, and Corollary 1.3(a), for $F \in C^k$ with $F(0) = 0$ and the radius $\sqrt{2}r$, gives $\|F \circ \chi\|_X \leq C_B(F; \sqrt{2}r)\sqrt{2}r$. Put $w(t) := \int_0^1 F(\chi(\tau, t)) d\tau$; by Fubini's theorem $\operatorname{env}(w) \leq \operatorname{env}(F \circ \chi)$. Since $G \circ \varphi(t) - G \circ \psi(t) = (\varphi(t) - \psi(t))(G'(0) + w(t))$, Lemma 2.4 gives $\|G \circ \varphi - G \circ \psi\|_X \leq |G'(0)|\|\varphi - \psi\|_X + 2c_s K_s \|F \circ \chi\|_X \|\varphi - \psi\|_X$. $\square$

Remark 6.4 (On vector-valued integrals). The proof of Theorem 1.2 forms no integral of functions with values in the space of the families with finite norm. This space is not separable when the index set is infinite, but no difficulty of measurability is hidden here: for a family φ with $\|\varphi\|_X < \infty$ and envelope m the map $\lambda \mapsto e^{i\lambda\varphi} - 1$ is continuous in the norm $\|\cdot\|_X$. Indeed, Lemma 4.1 with $a = m$ gives $\|e^{i\lambda\varphi} - 1\|_X \leq |h| \|m\|_s e^{c_s|h|\|m\|_1}$, and by Lemma 2.4

$$\|e^{i\lambda\varphi} - e^{i\lambda'\varphi}\|_X \leq 2c_s K_s \|e^{i\lambda'\varphi}\|_X \|e^{i(\lambda-\lambda')\varphi} - 1\|_X \longrightarrow 0 \quad (\lambda \rightarrow \lambda').$$

A continuous map of $\mathbb{R}$ into a normed space has separable range and is therefore strongly measurable. The space of the families with finite norm is complete (it consists of the arrays $(c_\ell(t))$ with $\sum_\ell \langle \ell \rangle^{2s} \sup_t |c_\ell(t)|^2 < \infty$), and under the hypotheses of Theorem 1.2 the norm of $e^{i\lambda\varphi} - 1$ is $|\mu|$ -integrable by step (ii) of the proof. So $G \circ \varphi = \int (e^{i\lambda\varphi} - 1) d\mu(\lambda)$ can also be read as a Bochner integral in this space, as in the proof of [RS16, Theorem 4.13]; we do not use this.

7 The lower bound

The profile. Fix $\beta \in C^\infty(\mathbb{R})$ with $\beta \geq 0$, $\beta > 0$ exactly on $(1, 2\pi - 1)$, $\int \beta = 1$ and $\beta(2\pi - x) = \beta(x)$, and let β_{per} be its 2π -periodisation; it is C^∞ , even, and vanishes on $[-1, 1]$. The function $1 - 2\pi\beta_{\text{per}}$ is C^∞ , 2π -periodic, even, and has integral 0 over a period. Hence

$$\psi_0(x) := \int_0^x (1 - 2\pi\beta_{\text{per}}(y)) dy$$

is a real-valued, odd, 2π -periodic C^∞ function with $c_0(\psi_0) = 0$ and $\psi_0(x) = x$ for $|x| \leq 1$. Put $h(x) := \psi_0(x) - x$ on $[-1, 2\pi - 1]$; then $h = 0$ on $[-1, 1]$, $h' = -2\pi\beta < 0$ on $(1, 2\pi - 1)$ and $h(2\pi - 1) = -2\pi$. For $n \in \mathbb{Z}$ let

$$b(n) := c_n(e^{in\psi_0}) = \frac{1}{2\pi} \int_{-1}^{2\pi-1} e^{inh(x)} dx = \frac{1}{2\pi} \left(2 + \int_1^{2\pi-1} e^{inh(x)} dx \right).$$

Lemma 7.1. $b(n) \rightarrow \frac{1}{\pi}$ as $|n| \rightarrow \infty$. In particular there is $n_0 \geq 1$ with $|b(n)| \geq \frac{1}{2\pi}$ for all $|n| \geq n_0$.

Proof. The map $x \mapsto -h(x)$ is a C^∞ diffeomorphism of $(1, 2\pi - 1)$ onto $(0, 2\pi)$ with derivative $2\pi\beta > 0$. The substitution $u = -h(x)$ gives $\int_1^{2\pi-1} e^{inh(x)} dx = \int_0^{2\pi} e^{-inu} g(u) du$ with $g(u) := 1/(2\pi\beta(x(u))) > 0$ and $\int_0^{2\pi} g = 2\pi - 2$. So g is integrable, and the integral tends to 0 by the Riemann–Lebesgue lemma. $\square$

The family. Let $\|\psi_0\|_{H^s(\mathbb{T})} = (\sum_n (1 + |n|)^{2s} |c_n(\psi_0)|^2)^{1/2}$, $c := (\sqrt{d} \|\psi_0\|_{H^s(\mathbb{T})})^{-1}$, and for $a \in [-c, c]^d$

$$\psi^a(x) := \sum_{j=1}^d a_j \psi_0(x_j), \quad \Phi := (\psi^a)_{a \in [-c, c]^d}.$$

These are real-valued C^∞ functions on $\mathbb{T}^d$. The Fourier coefficients of ψ^a vanish except at the points ne_j ($n \neq 0$, $1 \leq j \leq d$) of the coordinate axes, where they are $a_j c_n(\psi_0)$. Hence $\text{env}(\Phi)_{ne_j} = c |c_n(\psi_0)|$ and

$$\|\Phi\|_X^2 = \sum_{j=1}^d \sum_{n \neq 0} (1 + |n|)^{2s} c^2 |c_n(\psi_0)|^2 = d c^2 \|\psi_0\|_{H^s(\mathbb{T})}^2 = 1.$$

Proof of Proposition 1.4. (a) Let $\lambda > 0$ and $\ell \in \mathbb{Z}^d$ with $|\ell_j| \leq c\lambda$ for all j . Then $a := \ell/\lambda \in [-c, c]^d$, $\lambda\psi^a(x) = \sum_j \ell_j \psi_0(x_j)$ and $e^{i\lambda\psi^a(x)} e^{-i\ell \cdot x} = \prod_j e^{i\ell_j(\psi_0(x_j) - x_j)}$, so

$$c_\ell(e^{i\lambda\psi^{\ell/\lambda}}) = \prod_{j=1}^d b(\ell_j). \quad (7.1)$$

By Lemma 7.1, $\text{env}(e^{i\lambda\Phi})_\ell \geq (2\pi)^{-d}$ for every ℓ with $n_0 \leq |\ell_j| \leq c\lambda$ for all j . Let $c\lambda \geq \max\{4, 2n_0\}$. The interval $[\frac{c\lambda}{2}, c\lambda]$ contains at least $\frac{c\lambda}{2} - 1$ integers, so the number of integers n with $\frac{c\lambda}{2} \leq |n| \leq c\lambda$ is at least $c\lambda - 2 \geq \frac{c\lambda}{2}$, and for ℓ with all coordinates in this range $\langle \ell \rangle \geq \frac{c\lambda}{2}$. Therefore

$$\|e^{i\lambda\Phi}\|_X^2 \geq \left(\frac{c\lambda}{2}\right)^d (2\pi)^{-2d} \left(\frac{c\lambda}{2}\right)^{2s},$$

which is (a) with $c_* := (2\pi)^{-d} \min\{1, \frac{c}{2}\}^{s+d/2}$ and $\lambda_* := \max\{1, \max\{4, 2n_0\}/c\}$. Only the finitely many members $\psi^{\ell/\lambda}$, $|\ell_j| \leq c\lambda$, were used.

(b) For such a member with $\frac{c\lambda}{2} \leq |\ell_j| \leq c\lambda$ for all j , (7.1) gives $\|e^{i\lambda\psi^{\ell/\lambda}}\|_{H^s} \geq \langle \ell \rangle^s \prod_j |b(\ell_j)| \geq (\frac{c\lambda}{2})^s (2\pi)^{-d} \geq c_* \lambda^s$. For the upper bound let $\nu \in \mathbb{R}$. For every integer $j \geq 0$ the j -th derivative of $e^{i\nu\psi_0}$ is $e^{i\nu\psi_0}$ times a polynomial in ν of degree at most j whose coefficients are bounded functions (products of derivatives of ψ_0). By Parseval's identity $\sum_k |k|^{2j} |c_k(e^{i\nu\psi_0})|^2 \leq A_j^2 (1+|\nu|)^{2j}$, and $\sum_k |c_k(e^{i\nu\psi_0})|^2 = 1$; hence $\|e^{i\nu\psi_0}\|_{H^j(\mathbb{T})} \leq A'_j (1+|\nu|)^j$. For $j \leq s \leq j+1$, Hölder's inequality for the sums, $\|u\|_{H^s} \leq \|u\|_{H^j}^{j+1-s} \|u\|_{H^{j+1}}^{s-j}$, gives $\|e^{i\nu\psi_0}\|_{H^s(\mathbb{T})} \leq A_*(1+|\nu|)^s$ for all real ν . Now let $u(x) = \prod_j u_j(x_j)$ with $u_j := e^{i\nu_j\psi_0}$ and $|\nu_j| \leq c\lambda$; every $e^{i\lambda\psi^a}$, $a \in [-c, c]^d$, is of this form. Then $c_k(u) = \prod_j c_{k_j}(u_j)$, $\langle k \rangle \leq \sum_j (1+|k_j|)$ and $(\sum_{j=1}^d y_j)^{2s} \leq d^{2s-1} \sum_j y_j^{2s}$, so

$$\|u\|_{H^s(\mathbb{T}^d)}^2 \leq d^{2s-1} \sum_{j=1}^d \|u_j\|_{H^s(\mathbb{T})}^2 \prod_{i \neq j} \sum_k |c_k(u_i)|^2 = d^{2s-1} \sum_{j=1}^d \|u_j\|_{H^s(\mathbb{T})}^2 \leq d^{2s} A_*^2 (1+c\lambda)^{2s},$$

which is the upper bound in (b), with $C := d^s A_* \max\{1, c\}^s$.

The consequences stated in Proposition 1.4 follow: (a) gives $\mathcal{E}(\lambda) \geq c_* \lambda^{s+d/2}$, Theorem 1.1 with $\theta = 1$ gives the upper bound when $\sigma > 1$, and the quotient of (a) and (b) is at least $(c_*/C) \lambda^{s+d/2} (1+\lambda)^{-s} \geq c \lambda^{d/2}$. $\square$

Remark 7.2 (Regularity in t). The positive results hold for all families, in particular for those which are continuous or smooth in t . The negative results do not need irregular families. The family Φ is indexed by a cube; composing with a continuous map of $[t_-, t_+]$ onto $[-c, c]^d$ (a Peano curve) gives a family indexed by an interval, continuous in (t, x) , with the same members. With a smooth curve one gets for each λ a family which is C^∞ in (t, x) and for which Proposition 1.4(a) and (b) hold: take a polygonal path in the cube through the finitely many points ℓ/λ used in the proof and parametrise it smoothly; its members belong to Φ . Proposition 1.5 then holds in the following form for smooth families: the proof of Proposition 8.1 applies verbatim to $P(G) := \sup_\varphi \|G \circ \varphi\|_X$, the supremum over the subfamilies φ of $r\Phi$ which are C^∞ in (t, x) , because a supremum of lower semicontinuous seminorms is a lower semicontinuous seminorm; so for a residual set of G the estimate (11) fails on smooth families.

8 Necessity of smoothness

For $\lambda \geq 1$ let $S_\lambda(u) := \sin(\lambda u)$ and $C_\lambda(u) := \cos(\lambda u) - 1$. Proposition 1.5 is a special case of the following statement, whose proof follows the classical scheme of [Kat59, Théorème 5.7 and Corollaire]

and [Leb69, Théorème 1], with Baire's theorem in place of the closed graph theorem; what is specific here is the lower bound of Proposition 1.4.

Proposition 8.1. *Let $r > 0$ and $b := K_s r$. Let $\mathfrak{B}$ be a real Banach space whose elements are continuous real-valued functions on an interval containing $[-b, b]$, such that $\sup_{[-b, b]} |G| \leq C_{\mathfrak{B}} \|G\|_{\mathfrak{B}}$ for all $G \in \mathfrak{B}$. Assume that $S_{\lambda}, C_{\lambda} \in \mathfrak{B}$ for all λ in an unbounded set $\Lambda_0 \subset [1, \infty)$ and that*

$$\liminf_{\lambda \rightarrow \infty, \lambda \in \Lambda_0} \lambda^{-(s+d/2)} (\|S_{\lambda}\|_{\mathfrak{B}} + \|C_{\lambda}\|_{\mathfrak{B}}) = 0. \quad (8.1)$$

Then the set of all $G \in \mathfrak{B}$ with $\|G \circ (r\Phi)\|_X < \infty$ is meagre in $\mathfrak{B}$. In particular $\|G \circ (r\Phi)\|_X = \infty$ for some $G \in \mathfrak{B}$.

Proof. The members of $r\Phi$ take their values in $[-b, b]$, because $\|r\Phi\|_X \leq r$. Put $p(G) := \|G \circ (r\Phi)\|_X \in [0, \infty]$.

p is a seminorm with values in $[0, \infty]$: $\text{env}((G_1 + G_2) \circ r\Phi) \leq \text{env}(G_1 \circ r\Phi) + \text{env}(G_2 \circ r\Phi)$, and $\|\cdot\|_s$ satisfies the triangle inequality.

p is lower semicontinuous. If $G_n \rightarrow G$ in $\mathfrak{B}$, then $G_n \rightarrow G$ uniformly on $[-b, b]$, so for every member u of $r\Phi$ and every ℓ we have $|c_{\ell}(G \circ u)| = \lim_n |c_{\ell}(G_n \circ u)| \leq \liminf_n \text{env}(G_n \circ r\Phi)_{\ell}$. Hence $\text{env}(G \circ r\Phi)_{\ell} \leq \liminf_n \text{env}(G_n \circ r\Phi)_{\ell}$, and by Fatou's lemma for sums $p(G) \leq \liminf_n p(G_n)$.

No set $\{p \leq j\}$, $j \in \mathbb{N}$, has an interior point. These sets are closed, convex and symmetric. If one of them contained a ball around some G_0 , it would contain the ball of the same radius ε around $-G_0$ and hence, by convexity, around 0; by homogeneity we would get $p(G) \leq A\|G\|_{\mathfrak{B}}$ for all $G \in \mathfrak{B}$, with $A := 2j/\varepsilon$. Since $e^{i\lambda u} = 1 + C_{\lambda}(u) + iS_{\lambda}(u)$, this would give for $\lambda \in \Lambda_0$

$$\|e^{i\lambda r\Phi}\|_X \leq 1 + p(C_{\lambda}) + p(S_{\lambda}) \leq 1 + A(\|C_{\lambda}\|_{\mathfrak{B}} + \|S_{\lambda}\|_{\mathfrak{B}}).$$

But $\|e^{i\lambda r\Phi}\|_X \geq c_*(\lambda r)^{s+d/2}$ for $\lambda r \geq \lambda_*$ by Proposition 1.4(a). Dividing by $\lambda^{s+d/2}$ and using (8.1) we get $c_* r^{s+d/2} \leq 0$, a contradiction.

So $\{p < \infty\} = \bigcup_j \{p \leq j\}$ is a countable union of closed sets without interior points, that is, a meagre set; by Baire's theorem its complement is not empty. $\square$

Proof of Proposition 1.5. Let $\mathfrak{B} := \{G \in C^k([-b, b]; \mathbb{R}) : G(0) = 0\}$ with the norm $\max_{j \leq k} \sup |G^{(j)}|$. For $\lambda \geq 1$ we have $\|S_{\lambda}\|_{\mathfrak{B}} + \|C_{\lambda}\|_{\mathfrak{B}} \leq 3\lambda^k$, so (8.1) holds because $k < s + \frac{d}{2}$. In the Hölder class $\{G \in C^{k, \alpha}([-b, b]; \mathbb{R}) : G(0) = 0\}$, $0 < \alpha \leq 1$, with the norm $\|G\|_{C^k + \sup_{x \neq y} |G^{(k)}(x) - G^{(k)}(y)|/|x - y|^{\alpha}}$, the inequality $|\sin x - \sin y| \leq \min\{2, |x - y|\} \leq 2^{1-\alpha} |x - y|^{\alpha}$ and its analogue for the cosine give $\|S_{\lambda}\| + \|C_{\lambda}\| \leq 3\lambda^k + 4\lambda^{k+\alpha}$, and (8.1) holds if $k + \alpha < s + \frac{d}{2}$. By Proposition 8.1 the functions G with $p(G) = \infty$ form the complement of a meagre set. Take one of them and extend it to a C^k function on $\mathbb{R}$, for instance by its Taylor polynomials of degree k at $\pm b$; the values outside $[-b, b]$ do not matter. If $k \geq 1$, put $G_1 := G + (1 - G'(0))\text{id}$. Then $G_1(0) = 0$, $G'_1(0) = 1$ and $p(G_1) \geq p(G) - |1 - G'(0)|p(\text{id}) = \infty$, since $p(\text{id}) = \|r\Phi\|_X \leq r$. For $d = 2$ and $r = \frac{1}{2\pi}$ we have $\mathcal{N}_s(r\Phi) \leq 1$ and $\mathcal{N}_s(G_1 \circ r\Phi) = \infty$. $\square$

Corollary 8.2 (The scale $A_{\beta}(\mathbb{T})$). *Let $s > 1 + \frac{d}{2}$ and $\beta \geq 0$, and let $A_{\beta}(\mathbb{T})$ be the space of the functions $G(u) = \sum_{n \in \mathbb{Z}} g_n e^{inu}$ with $\|G\|_{A_{\beta}} := \sum_n |g_n| (1 + |n|)^{\beta} < \infty$. The following are equivalent:*
(i) $\|G \circ \varphi\|_X < \infty$ for every real-valued $G \in A_{\beta}(\mathbb{T})$ and every family φ of real-valued functions with $\|\varphi\|_X < \infty$; (ii) $\beta \geq s + \frac{d}{2}$. *In this case $\|G \circ \varphi\|_X \leq C_A (1 + \|\varphi\|_X)^{s+d/2} \|G\|_{A_{s+d/2}}$, with C_A for $\theta = 1$.*

Proof. If $\beta \geq s + \frac{d}{2}$: $G \circ \varphi(t) = \sum_n g_n e^{in\varphi(t)}$ converges uniformly, so $\text{env}(G \circ \varphi) \leq \sum_n |g_n| \text{env}(e^{in\varphi})$. Theorem 1.1 with $\theta = 1$, applied to the family $\varphi/\|\varphi\|_X$ and the parameter $n\|\varphi\|_X$, gives $\|e^{in\varphi}\|_X \leq C_A (1 + |n|\|\varphi\|_X)^{s+d/2} \leq C_A (1 + \|\varphi\|_X)^{s+d/2} (1 + |n|)^{s+d/2}$ (if $\|\varphi\|_X = 0$, then $e^{in\varphi} = 1$ and the bound is trivial). If $\beta < s + \frac{d}{2}$: apply Proposition 8.1, with any $r > 0$, to the space of the real-valued functions of $A_{\beta}(\mathbb{T})$, with $\Lambda_0 = \{1, 2, 3, \dots\}$; here $\|S_n\|_{A_{\beta}} + \|C_n\|_{A_{\beta}} = 2(1 + n)^{\beta} + 1$. $\square$

This threshold $s + \frac{d}{2}$ is the analogue of the threshold $\alpha + \frac{1}{2}$ for A_α quoted in Section 1.4.

9 Remarks

9.1 The constants

All constants of Theorems 1.1 and 1.2 are explicit. Table 2 gives values for $d = 2$, computed in floating point, with upper bounds for K_s and $K_{s-\theta}$ (summation over a square, an integral bound for the rest); all constants are non-decreasing in these numbers.

s	θ	M	least $k > M + \frac{3}{2}$	$K_s \leq$	C_T	$B_R \leq$	$\log_{10} C_A \leq$
1.5	0	7.5	10	1.916	7.305	1006.2	9.35
1.5	0.4	5.5	8	1.916	7.305	1989.0	10.08
2	0	6	8	1.299	8.818	171.6	8.66
2	0.9	3.3	5	1.299	8.818	364.8	9.64
3	1	4	6	1.050	18.170	100.6	10.25
4	1	5	7	1.011	43.235	92.1	12.37

Table 2: The constants of Theorem 1.1 for $d = 2$ (values rounded upwards, except M and C_T).

A worked example. Let $d\mu(\lambda) := \frac{1}{2\pi i} \operatorname{sgn}(\lambda) e^{-\sqrt{|\lambda|}} d\lambda$, so that

$$G(u) = \int_{\mathbb{R}} (e^{i\lambda u} - 1) d\mu(\lambda) = \frac{1}{\pi} \int_0^\infty e^{-\sqrt{\lambda}} \sin(\lambda u) d\lambda.$$

This function is real-valued and C^∞ , $G(0) = 0$, $G'(0) = \frac{12}{\pi}$, and $G^{(2j+1)}(0) = (-1)^j \frac{2}{\pi} \Gamma(4j+4)$; so its Taylor series at 0 has radius of convergence zero, and Proposition 2.5 does not apply. Here $J_\mu(r) = \frac{1}{\pi} \int_0^\infty \lambda(1+r\lambda)^M e^{-\sqrt{\lambda}} d\lambda \leq \frac{2^M}{\pi} (6 + \Gamma(2M+4))$ for $r \leq 1$, by $(1+\lambda)^M \leq 2^{M-1}(1+\lambda^M)$ and $\int_0^\infty \lambda^a e^{-\sqrt{\lambda}} d\lambda = 2\Gamma(2a+2)$. In the normalisation of the source ($d = 2$, $r = \frac{1}{2\pi}$), Theorem 1.2 gives $C_2 = (2c_s K_s C_A J_\mu(r))^2$ with

$$\log_{10} C_2 \leq 45.6, \quad 34.0, \quad 39.0 \quad \text{for} \quad (s, \theta) = (2, 0), (2, 0.9), (3, 1);$$

if $J_\mu(r)$ is evaluated numerically instead of by the closed bound, these numbers become 33.2, 27.6 and 31.0. So the constants are explicit but very large, and we did not try to optimise them. For comparison, for the entire function $G(u) = e^{2u} - 1$ Proposition 2.5 gives $C_B \leq 2e^{2c_s K_s r}$, that is

$$C_2 \leq 20.9 \quad (d = 2, s = 2),$$

and $C_2 \leq 22.5$ for $s = \frac{3}{2}$, $C_2 \leq 58.0$ for $s = 3$. We mention this function because in the equations (3), (4) of [Chr21] the unknown $\log P$ enters through the factor $P^2 = e^{2\log P}$ which the metric $2P^{-2} d\zeta d\bar{\zeta}$ gives to its Laplace operator and to the norm of covectors. Whether the proof of Theorem 1 of [Chr21] needs (11) for functions which are not entire, we cannot say (Section 1.5).

9.2 Dimension one

For $d = 1$ and $s < 1$ the exponent $s + \frac{d}{2}$ of Proposition 1.4 is not the growth exponent.

Proposition 9.1. *Let $d = 1$ and $\frac{1}{2} < s < 1$. There are $c_E > 0$ and λ_E such that for every $\lambda \geq \lambda_E$ there is a finite family ψ of real-valued C^∞ functions on $\mathbb{T}$ with $\|\psi\|_X \leq 1$ and $\|e^{i\lambda\psi}\|_X \geq c_E \lambda^{3/2}$. Hence $\gamma_* \geq \frac{3}{2} > s + \frac{1}{2}$.*

Proof. Let ψ_0 , $b(n)$ and n_0 be as in Section 7, $c' := (2\|\psi_0\|_{H^s(\mathbb{T})})^{-1}$ and $C_3 := \|\psi_0''\|_\infty + \|\psi_0'\|_\infty^2$. For $\lambda \geq 2$ let $L := \lfloor c'\lambda \rfloor$, and let K be the largest integer with $(1+K)^s \leq \sqrt{2}\lambda$; then $K \geq 1$, because $2^s < 2 < \sqrt{2}\lambda$. Because $s < 1$, the number $K > (\sqrt{2}\lambda)^{1/s} - 2$ grows faster than $L \leq c'\lambda$. So there is $\lambda_1 \geq 2$ such that

$$K > L \quad \text{and} \quad \frac{2C_3L^2}{(K-L)^2} \leq \frac{1}{32\pi} \quad \text{for all } \lambda \geq \lambda_1. \quad (9.1)$$

Let $\lambda_E := \max\{\lambda_1, 2n_0/c'\}$ and $\lambda \geq \lambda_E$; then $L \geq 2n_0$. Put

$$\psi_\ell(x) := \frac{\ell}{\lambda} \psi_0(x) + \frac{1}{2\lambda} \cos(Kx), \quad n_0 \leq \ell \leq L.$$

As $K \geq 1$, the envelope of this family is at most $\frac{L}{\lambda}|c_k(\psi_0)| + \frac{1}{4\lambda}\mathbf{1}_{|k|=K}$, so

$$\|\psi\|_X \leq c'\|\psi_0\|_{H^s(\mathbb{T})} + \frac{\sqrt{2}(1+K)^s}{4\lambda} \leq \frac{1}{2} + \frac{1}{2}.$$

Let $w(y) := e^{\frac{i}{2}\cos y} = \sum_{m \in \mathbb{Z}} a_m e^{imy}$. By Lemma 4.1, $\sum_m |a_m| \leq e^{1/2} < 2$. Moreover $w = 1 + \frac{i}{2}\cos y + \varrho$ with $|\varrho| \leq \frac{1}{8}$ (as $|e^{i\vartheta} - 1 - i\vartheta| \leq \frac{1}{2}\vartheta^2$), so $a_1 = \frac{i}{4} + c_1(\varrho)$ and $|a_1| \geq \frac{1}{8}$. Since $e^{i\lambda\psi_\ell(x)} = e^{i\ell\psi_0(x)} w(Kx)$,

$$c_{\ell+K}(e^{i\lambda\psi_\ell}) = \sum_m a_m c_{\ell+(1-m)K}(e^{i\ell\psi_0}) = a_1 b(\ell) + \sum_{m \neq 1} a_m c_{\ell+(1-m)K}(e^{i\ell\psi_0}).$$

Two integrations by parts give, for $j \neq 0$ and $\ell \geq 1$, $|c_j(e^{i\ell\psi_0})| \leq \|(e^{i\ell\psi_0})''\|_\infty / j^2 \leq C_3 \ell^2 / j^2$. For $m \neq 1$ and $1 \leq \ell \leq L$ we have $|\ell + (1-m)K| \geq K - L > 0$. Hence, by Lemma 7.1 and (9.1), for $n_0 \leq \ell \leq L$

$$|c_{\ell+K}(e^{i\lambda\psi_\ell})| \geq \frac{1}{8} \cdot \frac{1}{2\pi} - \frac{2C_3L^2}{(K-L)^2} \geq \frac{1}{32\pi},$$

that is, $\text{env}(e^{i\lambda\psi})_{\ell+K} \geq \frac{1}{32\pi}$. These are $L - n_0 + 1$ different frequencies with $\langle \ell + K \rangle \geq 1 + K$. We have $L - n_0 + 1 \geq c'\lambda - n_0 \geq \frac{c'\lambda}{2}$, and by the maximality of K and $s \leq 1$, $(1+K)^{2s} \geq 2^{-2s}(2+K)^{2s} > 2^{-2s}(\sqrt{2}\lambda)^2 \geq \frac{\lambda^2}{2}$. Hence

$$\|e^{i\lambda\psi}\|_X^2 \geq \frac{c'\lambda}{2} \cdot \frac{\lambda^2}{2} \cdot \frac{1}{(32\pi)^2},$$

which is the claim with $c_E := \sqrt{c'}/(64\pi)$. $\square$

The mechanism is again a loss of orthogonality: one mode of high frequency K in its linear regime costs a fixed part of the norm, whatever K is, and transfers the roughly λ coefficients of size about $\frac{1}{\pi}$ of the low part to the frequencies near K . For $d \geq 2$ we have $s > 1$, and a bound $\lambda^{1+d/2}$ of this kind would be weaker than Proposition 1.4; we did not examine it. With the supremum over the finite families of smooth functions in the ball of radius r in place of the single family $r\Phi$, the proof of Proposition 8.1 and Proposition 9.1 show that for $d = 1$ and $\frac{1}{2} < s < 1$ the conclusion of Proposition 1.5, in the form that $\|G \circ \varphi\|_X$ is unbounded on this ball, extends to the classes $C^{1,\alpha}$ with $\alpha < \frac{1}{2}$.

9.3 A second route, and an analogy

Remark 9.2 (A second route for $s \geq 1$). For a family of trigonometric polynomials the computation which leads to (3.2) gives, without any restriction on the spectrum, $e_\ell \leq \frac{\sqrt{d}|\lambda|}{|\ell|} \sum_k |k| m_k e_{\ell-k}$ for $\ell \neq 0$, where m and e are the envelopes of ψ and of $e^{i\lambda\psi}$. From this inequality alone, together with

$e_\ell \leq 1$ and the tail bound of Lemma 2.1(c), the second verification run (see “Verification”) derived for $s \geq 1$ a bound $\|e^{i\lambda\psi}\|_X \leq C'(1 + |\lambda|)^M$ with the same exponent M and another explicit constant: one multiplies by $\langle \ell \rangle^s$, splits the convolution according to the sizes of $|k|$ and $|\ell - k|$, and absorbs the weighted tail of e beyond a radius $R \approx \max\{\nu, |\lambda|^{1/\sigma}\}$. No factorisation, no majorant series and no iteration of the recursion are needed; the assumption $s \geq 1$ is automatic for $d \geq 2$. We do not reproduce this proof, and no statement of this note depends on it.

Remark 9.3 (An analogy with modulation spaces). A modulation norm of a function f on $\mathbb{R}^n$ is formed from a time–frequency decomposition of f by taking the L^p norm in the position variable first and a weighted ℓ^q norm in the frequency variable afterwards; by [RS16, Proposition 2.4] an equivalent norm of $M_{p,q}^s$ is $(\sum_k \langle k \rangle^{sq} \|\square_k f\|_{L^p}^q)^{1/q}$ with frequency-uniform pieces $\square_k f$. The quantity $\mathcal{N}_s$ has this pattern with $p = \infty$ and $q = 2$, if the parameter t is read as the position variable and $c_\ell(u(t))$ as the piece of frequency ℓ . The quantity in (12) is formed in the other order, the weighted ℓ^2 norm in the frequency first and the supremum afterwards, which is the pattern of an amalgam norm. We use this only as an orientation. It explains why (12) is classical, whereas (11) corresponds to the case $p = \infty$, $q < \infty$, which the theorems of [RS16] and [KST20] quoted in Section 1.4 exclude. We do not claim that Theorems 1.1 and 1.2 imply, or follow from, statements about the spaces $M_{\infty,2}^s(\mathbb{R}^d)$: such a transference was only sketched during the verification and has not been written out.

10 Computations and reproducibility

All theorems, propositions, lemmas and corollaries of this note are proved without computation. The computations test the inequalities of the proofs on sampled and on optimised families, search for families with fast growth, evaluate the constants, and illustrate the lower bounds. The reproducibility package has five parts.

Programs written with this note (`writing_stage/`). The program `constants_table.py` computes Table 2 and the numbers of Section 9.1. The program `check_prop_E.py` tests the proof of Proposition 9.1 as printed, for $s = 0.6, 0.75, 0.9$ and $\lambda = 16, \dots, 1024$: the norm of the family is at most 0.71, the coefficients $c_{\ell+K}(e^{i\lambda\psi_\ell})$ agree with $a_1 b(\ell)$ within the error term of the proof, the smallest of their moduli lies for each s and λ between 0.082 and 0.094 (the proof needs $\frac{1}{32\pi} \approx 0.0099$), and the resulting lower bound for $\|e^{i\lambda\psi}\|_X$ divided by $\lambda^{3/2}$ stays between 0.072 and 0.084. The program `make_figure.py` produced Figure 1 from saved outputs.

The programs with which the results were first obtained (`original/`). For $d = 1$ ($s = 0.75, 1, 1.5, 2.5$, $\lambda \leq 30$) and for $d = 2$ ($s = 1.25, 1.5, 2, 3$, $\lambda \leq 8$), the program `check_lemmas.py` draws an envelope with $\|m\|_s = 1$ and random real-valued functions below it, and checks on a grid Lemmas 2.2, 3.1, 3.2 and 4.1 and the pointwise bound $p + p * v$ of Step 4 (36 configurations, all passed). A second program does the same for Lemma 3.1(ii) with coefficients maximised numerically (500 cases, no violation). The program `growth_experiment.py` maximises single coefficients of $e^{i\lambda\psi}$ over the functions below a flat envelope on $\{1, \dots, K\}$ or on $\{3^j\}$, for $d = 1$, $s = 1$ and $\lambda = 8, \dots, 1024$; the resulting lower bounds for the norm grow polynomially, with local slopes between 0.82 and 2.10. The program `lower_bound_illustration.py` computes $b(n)$ for $n \leq 20000$ and the profile $\beta \propto \exp(-1/((x-1)(2\pi-1-x)))$: the values decrease from 0.431 to 0.3306 (the limit is $\frac{1}{\pi} = 0.3183$), so that $n_0 = 1$ on this range. It also computes the lower bound $Q_C(\lambda)$ for $\|e^{i\lambda\Phi}\|_X$ which (7.1) gives in $d = 1$; its local slopes are 1.49, 2.48–2.49 and 3.42–3.49 for $s = 1, 2, 3$, against 1.00, 2.00 and 2.96–3.00 for the largest H^s norm of a single member.

Three independent verification runs, each with its own programs and their recorded outputs. The first two are in `verification_run_A/` and `verification_run_B/`. The programs of the third, in `independent_run_2/`, were written from the text of this note: enclosures of the lattice sums K_τ with all numbers of Table 2 and of Section 9.1; tests of Lemmas 2.1 and 3.2 in $d = 1, 2, 3$, of Lemmas 2.2

and 3.1 and of the six steps of Section 5 in $d = 1, 2$, and of Lemma 2.3 and (6.1) in $d = 1$; the proof of Proposition 9.1 for two profiles and all members of the family, every coefficient being computed by sampling the function; the numbers $b(n)$, (7.1), the lower bounds of Proposition 1.4 and their local slopes; Corollary 6.1(b) and the bounds (1.6), (1.7). See the next paragraph.

All programs except the 26 optimisation runs of the first verification run were run again on 9 October 2026, the last time from the archive of the package; the outputs are identical to the recorded ones up to the fields which record running times.

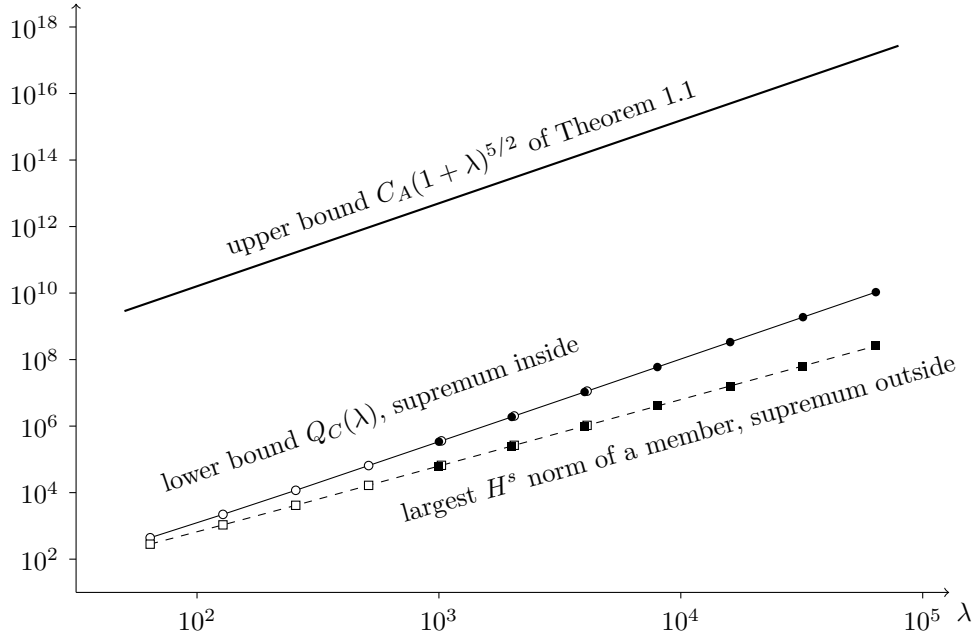


Figure 1: Dimension $d = 1$, $s = 2$ (so that $s > 1 + \frac{d}{2}$); both axes are logarithmic. Thick line: the upper bound of Theorem 1.1 with $\theta = 1$, $C_A \approx 1.54 \cdot 10^5$. Circles: the computed lower bound $Q_C(\lambda) = (\sum_{|\ell| \leq c\lambda} \langle \ell \rangle^{2s} b(\ell)^2)^{1/2}$ for $\|e^{i\lambda\Phi}\|_X$; local slopes 2.48–2.49 for $\lambda \geq 10^3$. Squares: the largest H^s norm of a single member $e^{i\lambda u}$; local slopes 2.00. Filled marks are from the original program, open marks from the program of the second verification run. The two upper curves are parallel and differ by a factor of about $1.5 \cdot 10^7$: the exponent of Theorem 1.1 is attained, the constant is not approached.

Verification. The results were first obtained with complete proofs and with the programs in `original/`. A first independent reading of that version, AI-assisted and without knowledge of the reasoning behind it, found no error in the proofs; it led to the restriction of Corollary 6.1(b) to the basis named there, to several clarifications, and to the correction of a round-off error in one program. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined that first version, and neither read the report of the other; the third examined the final text of this note.

The first run examined the statement, Theorems 1.1 and 1.2 and Corollaries 1.3 and 6.1 to 6.3. It compared the question with the source in the publisher’s file and in the arXiv version (index set, position of the supremum, the factor $\mathcal{N}_s(\varphi)$ in (11), the text of the corpus record). It re-derived every lemma of Sections 2 to 4, the six steps of Section 5 with the exponent M and the constant C_A , the chain of Fubini’s theorem, semicontinuity and Minkowski’s inequality in the proof of Theorem 1.2, and the count of derivatives, including the condition $k > M + \frac{3}{2}$. It found no statement that is false or unproved and required no correction. Its programs tested the lemmas ($1.16 \cdot 10^7$ comparisons for

Lemma 3.1(ii) on optimised members in $d = 1, 2$; the pointwise bound of Step 4 in 192 configurations) and recomputed the constants; two rows of the first table of constants were valid but not the sharpest values which the formulas give, and Table 2 contains the sharper ones. It also searched for fast growth: 26 optimisation runs in $d = 1$ and $d = 2$, with $4.9 \cdot 10^6$ function evaluations, for lacunary, dissociate, dense, random and sawtooth envelopes, found polynomial growth in all cases and no value above $4.5 \cdot 10^{-6}$ times the bound $C_A(1 + \lambda)^M$. This run also found the phenomenon of Proposition 9.1, with a proof.

The second run examined Propositions 1.4 and 1.5 and the literature. It re-derived both proofs and confirmed them, and it proved the stronger forms which are printed here as Proposition 8.1 and Corollary 8.2 (one family, a residual set of functions, non-integer smoothness). It proved independently, by another construction (one chirp $e^{i\lambda a \cos x}$ and small high-frequency parts with matched phases; van der Corput's lemma), that $\mathcal{E}(\lambda) \geq c\lambda^{3/2}$ for $d = 1$ and every $s > \frac{1}{2}$. It obtained the bound of Theorem 1.1 for $s \geq 1$ by the route of Remark 9.2, and it re-derived the lemmas and the proofs of Theorem 1.1, of Theorem 1.2 and of Corollary 1.3(a) line by line. It read the sources listed below and required five corrections of the statements about the literature, which concern [BRS03], the hypotheses of [KST20], the argument of [Leb69], the credit to [Kat59, Leb69] for the scheme of Proposition 1.5, and the reference [BS25]; all are applied in Section 1.4. Its programs recomputed the numbers $b(n)$ and the lower bounds for two profiles, checked (7.1) in $d = 2$ to 10^{-8} , and tested the inequalities of the second route (2520 tests, no violation).

When this note was written, the following was changed after these two runs. Corollary 1.3 states the condition $k > M + \frac{3}{2}$ with its constant, which the first run had re-derived. Proposition 1.4 is stated for one family which does not depend on λ . Proposition 1.5 is derived from Proposition 8.1; this proposition and Corollary 8.2 are taken from the report of the second run and were rewritten for real-valued functions. The proof of Proposition 9.1 follows the construction of the first run, with the profile of Section 7 and with constants chosen so that only elementary bounds are needed. Proposition 2.5, which the first run proved for $G(u) = e^{2u} - 1$, is stated for power series.

The third run examined the final text, and these changed parts first. It re-derived line by line Proposition 8.1 and Corollary 8.2, the proof of Proposition 1.5 with the Hölder classes, Proposition 9.1, and the statement and the proof of Proposition 1.4; then every lemma and every constant of Sections 2 to 6 (the cut-off frequency and its uniformity, the bound 1 for the coefficients of the low part, the recursion (3.2), the majorant, the weighted Young inequality, the exponent M for $\sigma > 1$ and for $\sigma \leq 1$, the two conditions on k in Corollary 1.3 with their constants), Proposition 2.5 and the remarks. It found no wrong theorem, proposition, lemma or corollary and no gap in a proof. It settled one point. The first run had remarked that the map $\lambda \mapsto e^{i\lambda\varphi} - 1$ need not be strongly measurable, and this remark was not taken over when the note was written. The map is continuous (Remark 6.4, which was added); so the remark was not correct, and the proof of Theorem 1.2 forms no vector-valued integral in any case. The third run compared Section 1.1 with the source, in the publisher's file and in the arXiv version, and the statements of Section 1.4 with the texts of [Leb69, Kat59, RS16, KST20, BS25], and it repeated the searches. It required twelve corrections of the text and of the package, which were made. Those which concern the mathematics or the literature are the following. The proof of Proposition 9.1 now names the values of λ for which each of its steps holds. The statement of Proposition 1.4 on the two norms is made for $\lambda \geq \lambda_*$. Two sentences which said that the theorems of [RS16, KST20] exclude exactly the case of the question were made exact: the hypotheses of these theorems do not admit the case which corresponds to the question by analogy. The theorem of Katznelson quoted in Section 1.2 is stated for an open interval. The description of [BS25] was made exact. Small precisions were added in Section 1.1 (the relation between (8) and (9)), in the proof of Proposition 1.5 (the Hölder space) and in the proof of Corollary 8.2. And in the program `check_prop_E.py` the number K is now that of the proof also in the three cases in which $(1 + K)^s = \sqrt{2}\lambda$ holds exactly; the program had taken

K one smaller there, and no tested inequality was affected.

The programs of the third run recomputed Table 2 and the numbers of Section 9.1 with two-sided enclosures of the lattice sums: all printed values are confirmed, those given as upper bounds being upper bounds, and $5.1570 < K_{1,1} < 5.1572$ for $d = 2$, against the bound 5.172 used in two rows of the table. They tested the lemmas and the six steps of Section 5 in $3.6 \cdot 10^6$ comparisons without a violation. They tested the proof of Proposition 9.1 in 42 cases (two profiles, three values of s , $\lambda = 16, \dots, 1024$, all members of the family): all inequalities hold, the moduli of the coefficients are at least 0.082, and the lower bound divided by $\lambda^{3/2}$ lies between 0.073 and 0.098. They reproduced the numbers $b(n)$, the local slopes of Figure 1 (between 2.47 and 2.50 for the lower bound, 2.00 for the largest H^s norm of a member), the factor $1.5 \cdot 10^7$, and (7.1) in $d = 2$ to 10^{-14} . The third run also ran the programs of the package again from the archive, and three of the 26 optimisation runs of the first run; all recorded outputs were reproduced. No proof depends on a computation.

Scope and priority. The note answers the question of [Chr21, Section 3.3] as posed: (11) holds for every smooth G with $G(0) = 0$ (Corollary 1.3), in every dimension, with explicit constants. It does not determine the exact growth rate of the exponentials for $\frac{d}{2} < s \leq 1 + \frac{d}{2}$ or the minimal smoothness of G , and its constants are very large (Sections 1.5 and 9.1). It says nothing about the application: the paper of Chruściel and Tod cited in [Chr21] was not found on arXiv, in INSPIRE, zbMATH or Crossref, or on the publication list of its first author, so the way in which (11) is used there could not be checked. INSPIRE lists four works citing [Chr21]; by their titles, and for the three which are on arXiv by a search of their texts, none of them treats (11).

The question is due to Chruściel [Chr21]. The method of Theorems 1.1 and 1.2 is that of Leblanc [Leb69] and of Reich and Sickel [RS16] (and, according to [RS16], of earlier work of Bourdaud and of [BRS03]), as explained in Section 1.4; the scheme of Proposition 1.5 is that of Katznelson [Kat59] and Leblanc. The tools are classical [Fol99, Kat04]. What we did not find in the literature is the estimate (11) for the space of the question, with the lower bounds of Propositions 1.4 and 9.1 and the statements derived from them. The two closest composition theorems that we know, [RS16, Theorem 4.13] and [KST20, Theorem 1.1], are theorems on modulation spaces; their hypotheses do not admit the case $p = \infty$, $q = 2$, which corresponds to the question only by the analogy of Remark 9.3. We make no priority claim.

What was read. Of [Chr21]: Section 3.3, in the publisher's file of the report and in the arXiv version v1 (the only one); the two texts of this section agree up to notation ($\ln P$ in the report, $\log P$ on arXiv), including the index set. [Leb69]: all five pages, in the journal's scan. [Kat59]: in the Numdam scan, pages 101–103 (Théorème 5.7, its Corollaire and the Remarques) and the statement of Théorème II.1; the rest was not read. [RS16]: in the arXiv version v1 (the only one), the introduction, Proposition 2.4 and Section 4.3 from Proposition 4.7 to the end; the published version was not compared. [KST20]: in the arXiv version v1 (the only one), Sections 1–4 and the beginning of Section 5; the published version was not compared. [BS25]: version 2, the abstract and Section 1; the proofs were not read. [BS11]: in the file on the site of the RIMS, the introduction and Sections 4.2.2 and 6. Not read: [BRS03], [KS22a] and [KS22b], of which no accessible copy was found (what we say about them is taken from [RS16] and from their titles), and [HKKR59] (the case of the circle was read in [Kat59]). The textbooks [Fol99, Kat04] are cited for standard facts; no particular statement in them was looked up. The bibliographic data were checked with Crossref, zbMATH and arXiv. The third verification run read again, in files which it fetched anew: Section 3.3 of [Chr21] in both versions, with the entry [9] of its list of references; [Leb69] completely; pages 101–103, 110 and 111 of [Kat59]; and in [RS16, KST20, BS25] the statements quoted in Section 1.4 with their surroundings, in [RS16] also Lemma 4.12 and the proof of Theorem 4.13.

The searches and the bibliographic checks were made on 9 October 2026, by the first investigation,

by the three verification runs and at the writing. Together they comprise 69 queries to the arXiv API (for composition, superposition and operating functions in modulation, Fourier–Lebesgue and amalgam spaces and in weighted Fourier algebras, for vector-valued variants, for the paper of Chruściel and Tod, and for later work of its authors on Robinson–Trautman metrics), 29 to zbMATH, 33 to Crossref, further queries to OpenAlex, INSPIRE and Semantic Scholar (among them the lists of works citing [Chr21], [RS16] and [KST20]), and six web searches. No statement of the estimate was found. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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