

A graphical group whose number of irreducible characters of degree p^3 is not a polynomial in p

Alper Ferudun*

October 27, 2026

Abstract

For a finite simple graph Γ and an odd prime p , let $G_\Gamma(\mathbb{F}_p)$ be the largest group of class at most two and exponent dividing p generated by elements x_v , one for each vertex v , such that x_u and x_v commute whenever u and v are not adjacent. Rossmann asked (Oberwolfach Report 38/2021; Question 1.10 in Bull. London Math. Soc. 2022) how the number of irreducible characters of $G_\Gamma(\mathbb{F}_p)$ of degree p^i depends on p , and suspected that these functions might be wild. We show that they are not always polynomials. Let Γ_{MK} be the bipartite graph on the points and lines of the Möbius–Kantor configuration 8_3 in which a point is joined to the five lines not containing it. For every odd prime p , $G_{\Gamma_{\text{MK}}}(\mathbb{F}_p)$ has exactly $p^{10}(h(p) + (1 + (\frac{-3}{p}))(p-1)^{15})$ irreducible characters of degree p^3 , where h is an explicit integer polynomial of degree 15 obtained by an exact computation; the count is a polynomial on each residue class modulo 3, but on no set of primes containing infinitely many primes $\equiv 1$ and infinitely many $\equiv 2 \pmod{3}$, and this needs no computer. We use the description of character degrees by ranks due to O’Brien and Voll and the fact, classical over the complex numbers, that 8_3 can be realised exactly over the fields containing a root of $t^2 - t + 1$. The counts are polynomials for $i = 1$, and for $i \leq 2$ if Γ is bipartite. Conditionally on the Mnëv–Sturmfels theorem as stated by Belkale and Brosnan, who sketch its proof, some count is not even a polynomial on residue classes (existence only). The general behaviour remains open. This is an unrefereed note.

1 Introduction

1.1 The question

Let Γ be a finite simple graph with distinct vertices $v_1, \dots, v_n$, and let p be an odd prime. In the problem session of the Oberwolfach workshop on computational group theory in 2021, Rossmann [EHNO21, pp. 2081–2082] considered the group

$$G_\Gamma(\mathbb{F}_p) = \langle x_1, \dots, x_n \mid [x_i, x_j] = 1 \text{ whenever } v_i \not\sim v_j, \text{ class} \leq 2, \text{ exponent dividing } p \rangle$$

and asked what its ordinary representation theory looks like; in particular, given Γ and $i \geq 1$, how $c_{\Gamma,i}(p) := \#\{\chi \in \text{Irr}(G_\Gamma(\mathbb{F}_p)) : \chi(1) = p^i\}$ behaves as a function of p . Rossmann remarks that this is equivalent to counting matrices of given rank in $M_\Gamma(\mathbb{F}_p) = \{A \in M_n(\mathbb{F}_p) : A = -A^\top, a_{ij} = 0 \text{ whenever } v_i \not\sim v_j\}$ (see Theorem 2.1), and that the number of conjugacy classes of size p^i is a polynomial in p (see [Ros22, Theorem A]). For the graphical groups $G_\Gamma(\mathbb{F}_q)$ over all finite fields this is Question 1.10 of [Ros22]; the counts are known to be polynomials for edgeless graphs, for paths (Marjoram [Mar99]) and, for odd q , for complete graphs (O’Brien and Voll [OV15, Proposition 2.4]),

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

p	$c(p)$	$N(p)$	$c_{\Gamma_{\text{MK}},3}(p) = p^{10}N(p)$
3	1	12900138432	761740274271168
5	0	37623332218368	367415353695000000000
7	2	6220918600020288	1757255530549462257851712
11	0	5479557249873720000	142125603015462529271385720000
13	2	66433274503723123200	9158391031673893699820022796800

Table 1: The number of irreducible characters of degree p^3 of $G_{\Gamma_{\text{MK}}}(\mathbb{F}_p)$, from (1). For $p \leq 7$ the values of $N(p)$ agree with direct counts (Section 8).

and Rossmann writes that the work of Belkale and Brosnan [BB03] on symmetric matrices leads Rossmann to suspect that these functions “might be rather wild as Γ and i vary” [Ros22, §1.6]. The problem is record OWR-8415341-012 of the UnsolvedMath collection [ula26], listed as open. In view of the remark on conjugacy classes we read its precise core as the question whether $c_{\Gamma,i}(p)$ is always a polynomial in p . We show that it is not, and we determine the count in one explicit case; the general behaviour remains open.

1.2 Results

The *Möbius–Kantor configuration* MK has the points $\mathbb{Z}/8\mathbb{Z}$ and the lines $L_i = \{i, i+1, i+3\}$, $i \in \mathbb{Z}/8\mathbb{Z}$:

$$L_0, \dots, L_7 = \{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{0, 5, 6\}, \{1, 6, 7\}, \{0, 2, 7\}.$$

Let Γ_{MK} be the bipartite graph on the 8 points and 8 lines of MK in which a point is adjacent to a line if and only if it does *not* lie on it; it has 40 edges, and all degrees are 5. For a prime power q let $c(q)$ be the number of roots of $t^2 - t + 1$ in $\mathbb{F}_q$; thus $c(q) = 2, 0, 1$ for $q \equiv 1, 2, 0 \pmod{3}$ (Proposition 3.2), and $c(p) = 1 + (\frac{-3}{p})$ for odd primes p , with $(\frac{-3}{3}) = 0$.

Theorem 1.1. *Let*

$$\begin{aligned} h(t) = & 1064t^{15} + 5048t^{14} - 30096t^{13} + 54308t^{12} - 49458t^{11} + 26090t^{10} - 10996t^9 + 10198t^8 \\ & - 11192t^7 + 6500t^6 - 956t^5 - 1056t^4 + 714t^3 - 190t^2 + 24t - 2. \end{aligned}$$

For every odd prime p ,

$$c_{\Gamma_{\text{MK}},3}(p) = p^{10}(h(p) + c(p)(p-1)^{15}). \quad (1)$$

Thus $c_{\Gamma_{\text{MK}},3}(p)$ equals $p^{10}(h(p) + 2(p-1)^{15})$ if $p \equiv 1 \pmod{3}$, $p^{10}h(p)$ if $p \equiv 2 \pmod{3}$, and $3^{10}(h(3) + 2^{15})$ if $p = 3$. No $f \in \mathbb{Q}[t]$ satisfies $c_{\Gamma_{\text{MK}},3}(p) = f(p)$ for all p in a set of primes that contains infinitely many primes $p \equiv 1$ and infinitely many primes $p \equiv 2 \pmod{3}$; in particular, not for all large p .

The leading terms are $(1065 + (\frac{-3}{p}))p^{25} + (5033 - 15(\frac{-3}{p}))p^{24}$; values are in Table 1. By Section 2, (1) amounts to $c_{\Gamma_{\text{MK}},3}(p) = p^{10}N(p)$ and

$$N(q) := \#\{B \in \mathbb{F}_q^{8 \times 8} : \text{rank } B = 3, B_{PL} = 0 \text{ whenever } P \in L\} = h(q) + c(q)(q-1)^{15}, \quad (2)$$

rows being indexed by points and columns by lines. We prove (2) for every prime power q , including $q = 2$, where $N(2) = h(2) = 14,818,566$ (Theorem 5.7); the interpretation by characters needs q odd (Remark 2.2).

The rank-3 matrices with support exactly the non-incidences correspond to the realisations of MK over $\mathbb{F}_q$; there are $c(q)(q-1)^{15}$ such matrices (Corollary 3.3). This alone gives a short proof of a weaker statement (Section 4):

Theorem 1.2. *For some set T of edges of Γ_{MK} , the graph Γ_T with the same 16 vertices and the edge set T has the following property: there are no $f \in \mathbb{Q}[t]$ and p_0 with $c_{\Gamma_T,3}(p) = f(p)$ for all primes $p > p_0$.*

For Theorem 1.1 we show, without a computer, that the matrices with smaller support contribute a polynomial (Section 5). The polynomial h is the output of an exact computation in integer arithmetic implementing this proof; it was recomputed by an independent implementation and confirmed by direct counts of $N(p)$ for $p = 2, 3, 5, 7$ with programs of different design, by values of an intermediate quantity for $p \leq 13$, and for $q = 4, 8, 9$ (Section 8).

Theorem 1.3. (a) *For every graph Γ , $c_{\Gamma,1}(p)$ is given by a polynomial in p .* (b) *If Γ is bipartite, $c_{\Gamma,2}(p)$ is given by a polynomial in p as well.* (c) *At least one of $c_{\Gamma_{\text{MK}},4}, \dots, c_{\Gamma_{\text{MK}},8}$ is not given by a polynomial in p .*

So $i = 3$ is the smallest exponent with non-polynomial counts among bipartite graphs. Assuming the universality theorem of Mnëv and Sturmfels in the form of [BB03, Theorem 11.1], whose proof Belkale and Brosnan only sketch, we show that some $c_{\Gamma,3}$ is not even a polynomial on residue classes (Proposition 7.1 and Corollary 7.2); this gives existence only.

1.3 Related work

For a set T of positions, the number of matrices of rank r over $\mathbb{F}_q$ supported in T is a q -rook number in the sense of Lewis, Liu, Morales, Panova, Sam and Zhang, who ask when it is a polynomial in q [LLM⁺11, Question 5.3]. It is one for Ferrers boards (Haglund [Hag98]), for further boards such as skew Young diagrams (Klein, Lewis and Morales [KLM14]), and for all boards with at most six rows and columns [CS26, Remark 2.3]. The known exception is Stembridge’s Fano board, whose number of invertible matrices depends on the parity of q [Ste98, §7], [LLM⁺11, §5.1], [LM20, Example 6.4], [CS26, Example 2.2]. By Lemma 2.3 these numbers are, up to a power of p , character counts of bipartite graphs; the Fano board is polynomial on odd q , whereas (2) depends on q modulo 3. For symmetric matrices with prescribed zeros, Belkale and Brosnan [BB03] showed that the numbers of invertible ones are as complicated as point counts of arbitrary schemes. Rossmann and Voll note that it is “unknown whether the same conclusion holds for spaces of arbitrary or antisymmetric matrices” [RV24, §1.1]; their uniformity results for average sizes of kernels show that the class numbers $k(G_\Gamma(\mathbb{F}_q))$ are polynomials [RV24, Corollary 1.3], [Ros22, Theorem 1.2]. Rossmann also noted non-PORC (not polynomial on residue classes) numbers of orbits of a given size of unipotent groups, and wild examples from [BB03, RV24] [EHNO21, p. 2064]; these are linear orbits, not characters. Using the work of Belkale and Brosnan, Rossmann showed that the class numbers of Baer group schemes (unipotent group schemes of class at most two) and the numbers of linear orbits of unipotent groups are wild modulo powers of q : for every scheme Y of finite type over $\mathbb{Z}$ and every n , $\#Y(\mathbb{F}_q)$ modulo q^n is given, uniformly in q , by finitely many of them and Laurent polynomials in q [Ros25, Theorem A]. For class-two groups of exponent p that are not graphical, O’Brien and Voll [OV15, Theorem 4.1], generalising work of Boston and Isaacs (see [OV15, Example 4.3]), express the numbers of characters of each degree through the number of rational points of a Pfaffian hypersurface, for the groups of Boston and Isaacs a plane cubic curve. Halasi and Pálfi [HP11] proved that the class numbers of some pattern groups of class two are not polynomials. None of this answers the question: for graphical groups the class number, the total number of characters, is a polynomial, their commutator matrices have prescribed zeros and otherwise independent entries, and the question concerns the individual degrees. The configuration 8_3 is classical; as a matroid it is known as the MacLane matroid, and it can be realised over $\mathbb{C}$ but not over $\mathbb{R}$, with two realisations up to projectivities [BGS24, Lemma 3.22]; over $\mathbb{Q}$ its space of realisations is $\text{Spec } \mathbb{Q}[z]/(z^2 - z + 1)$ [IKK⁺26, §3.1.2]. The same computation

gives the criterion over an arbitrary field; we include it (Proposition 3.2) because we need the number of realisations, and we do not claim it as new. We also include a proof of Theorem 2.1, which is the case of graphical groups of a theorem of O'Brien and Voll [OV15, Theorem B] (Remark 2.2).

2 Characters and ranks

Let Γ have the vertices $v_1, \dots, v_n$ and the edge set E , and put $m = |E|$. For a field F define $M_\Gamma(F)$ as in the introduction. Since $v_i \not\sim v_i$, the diagonal of a matrix in $M_\Gamma(F)$ is zero, so $M_\Gamma(F)$ consists of alternating matrices, and their ranks are even.

Theorem 2.1 (O'Brien–Voll). *For every graph Γ with n vertices, every odd prime p and every $i \geq 0$,*

$$c_{\Gamma,i}(p) = p^{n-2i} \#\{A \in M_\Gamma(\mathbb{F}_p) : \text{rank } A = 2i\}.$$

Proof. The group. Let $V = \mathbb{F}_p^n$ with basis $e_1, \dots, e_n$, let W be the $\mathbb{F}_p$ -vector space with basis (e_{jk}) indexed by the edges $\{v_j, v_k\}$, $j < k$, and let $\beta : V \times V \rightarrow W$ be the alternating bilinear map with $\beta(e_j, e_k) = -\beta(e_k, e_j) = e_{jk}$ for edges with $j < k$ and $\beta(e_j, e_k) = 0$ if $v_j \not\sim v_k$. On $V \times W$ put $(v, w)(v', w') = (v + v', w + w' + \frac{1}{2}\beta(v, v'))$. This is a group (associative by bilinearity; $(-v, -w)$ is inverse to (v, w) as $\beta(v, v) = 0$), $(v, w)^k = (kv, kw)$, and for $g = (v, w)$, $h = (v', w')$ the commutator $[g, h] = g^{-1}h^{-1}gh = (0, \beta(v, v'))$ lies in the central subgroup $Z = 0 \times W$. So $V \times W$ has class at most 2 and exponent p , and the elements $(e_i, 0)$ satisfy the defining relations of $G_\Gamma(\mathbb{F}_p)$; they generate $V \times W$, as their commutators generate Z . Conversely, in $G = G_\Gamma(\mathbb{F}_p)$ the commutator map is bilinear, so G' is generated by the central elements $[x_j, x_k]$ with $v_j \sim v_k$, of order dividing p ; hence $|G'| \leq p^m$, $|G/G'| \leq p^n$, and $G_\Gamma(\mathbb{F}_p) \cong V \times W$ by comparing orders.

Characters. The characters of Z are $\lambda_\omega(0, w) = \zeta^{\omega(w)}$ with $\omega \in W^* = \text{Hom}(W, \mathbb{F}_p)$ and $\zeta = e^{2\pi i/p}$, and the Gram matrices A_ω of the alternating forms $B_\omega = \omega \circ \beta$ in the basis (e_i) are exactly the elements of $M_\Gamma(\mathbb{F}_p)$, each once. Let R_ω be the radical of B_ω ; then $\hat{Z}_\omega = R_\omega \times W$ has index $p^{\text{rank } A_\omega}$. Let $\chi \in \text{Irr}(G)$ lie over $\lambda = \lambda_\omega$ and be afforded by ρ , so that $\rho(z) = \lambda(z)$ for $z \in Z$. If $g = (v, w)$ with $v \notin R_\omega$, choose v' with $B_\omega(v, v') \neq 0$ and put $h = (v', 0)$; then $h^{-1}gh = g[g, h]$ gives $\chi(g) = \zeta^{B_\omega(v, v')} \chi(g)$, so $\chi(g) = 0$. If $g \in \hat{Z}_\omega$, then $\lambda([g, h]) = 1$ for all h , so $\rho(g)$ is a scalar by Schur's lemma and $|\chi(g)| = \chi(1)$. Hence $1 = \langle \chi, \chi \rangle = |\hat{Z}_\omega| \chi(1)^2 / |G|$, that is, $\chi(1) = p^{\text{rank } A_\omega / 2}$, and $p^n = (\text{Ind}_Z^G \lambda)(1) = \sum_{\chi \in \text{Irr}(G|\lambda)} \chi(1)^2$ gives $\#\text{Irr}(G|\lambda) = p^{n - \text{rank } A_\omega}$. Summing over the ω with $\text{rank } A_\omega = 2i$ gives the claim. $\square$

Remark 2.2 (prime powers). For an odd prime power q , the graphical group $G_\Gamma(\mathbb{F}_q)$ of [Ros22, §1.1] is isomorphic, by the Lazard correspondence [Ros22, §2.4], to $V \times W$ as above with $V = \mathbb{F}_q^n$, $W = \mathbb{F}_q^E$. With the characters $\psi \circ \omega$ of W ($\psi \neq 1$ a character of $(\mathbb{F}_q, +)$, $\omega \in \text{Hom}_{\mathbb{F}_q}(W, \mathbb{F}_q)$), and v' chosen with $\psi(B_\omega(v, v')) \neq 1$ (possible, as $v' \mapsto B_\omega(v, v')$ is onto $\mathbb{F}_q$), the proof gives $\#\{\chi \in \text{Irr}(G_\Gamma(\mathbb{F}_q)) : \chi(1) = q^i\} = q^{n-2i} \#\{A \in M_\Gamma(\mathbb{F}_q) : \text{rank } A = 2i\}$. This is the case of graphical groups of [OV15, Theorem B]: there, for a finite nilpotent Lie algebra $\mathfrak{g}$ of class less than p over a compact discrete valuation ring of characteristic 0 with residue field of characteristic p , such that $\mathfrak{g}'$ and $\mathfrak{g}/\mathfrak{z}$ are vector spaces over the residue field, and for an unramified extension with residue field $\mathbb{F}_q$, the number of characters of degree q^i of the associated group G is $q^{-2i}|G : G'|$ times the number of y for which the commutator matrix $B(y)$ has rank $2i$; for the graphical Lie algebra over $\mathbb{F}_p$ the matrices $B(y)$ are the elements of $M_\Gamma(\mathbb{F}_q)$ with the rows and columns of the isolated vertices removed. The arguments of Sections 3 to 6 apply verbatim to prime powers q , with this remark in place of Theorem 2.1, so Theorems 1.1 to 1.3 hold for odd prime powers q in place of p , with the same h .

Lemma 2.3. *Let Γ be bipartite with parts U and U' of sizes a and b and edge set $T \subseteq U \times U'$. For every field F and every $r \geq 0$, $\#\{A \in M_\Gamma(F) : \text{rank } A = 2r\}$ is the number $M_r(T; F)$ of matrices*

$B \in F^{U \times U'}$ of rank r with $B_{uu'} = 0$ for $(u, u') \notin T$. Hence $c_{\Gamma, i}(p) = p^{a+b-2i} M_i(T; \mathbb{F}_p)$ for odd primes p .

Proof. List the vertices of U first. There are no edges inside U or inside U' , so the matrices in $M_{\Gamma}(F)$ are the matrices $\begin{pmatrix} 0 & B \\ -B^{\top} & 0 \end{pmatrix}$ with B supported in T , and their rank is $2 \operatorname{rank} B$. The second statement follows from Theorem 2.1. $\square$

For Γ_{MK} we take U the points and U' the lines; T is the set S of the 40 non-incident pairs, so that $c_{\Gamma_{\text{MK}}, 3}(p) = p^{10} N(p)$ with N as in (2).

3 The Möbius–Kantor configuration

Lemma 3.1. (F1) Every point of MK lies on exactly three lines; two distinct points P, Q lie on a common line if and only if $Q - P \neq 4$, and then on exactly one; two distinct lines meet in at most one point. (F2) Every set of three points that is not a line contains two points of a common line. (F3) If L is a line and $P \notin L$, then the only set of points that contains $L \cup \{P\}$ and is closed under the rule “if two points of a line belong to the set, so does the third” is the set of all points. (F4) No line contains three of the points $0, 1, 2, 5$.

Proof. The differences of the elements of $\{0, 1, 3\}$ are $\pm 1, \pm 2, \pm 3$, each once; this gives (F1), and two lines with two common points would give two points on two common lines. Three points no two of which are on a common line would have pairwise differences 4, which is impossible; this gives (F2). For (F3) we may assume $L = L_0 = \{0, 1, 3\}$, by the symmetry $i \mapsto i + 1$. A closed set containing $\{0, 1, 2, 3\}$ contains $4, 5, 7$ (from L_1, L_2, L_7) and then 6 (from L_3). For $P = 2$ this applies directly; for $P = 4, 5, 7$ the lines L_1, L_2, L_7 give the point 2; for $P = 6$ the line L_3 gives 4 and then L_1 gives 2. For (F4), the lines $L_0, \dots, L_7$ meet $\{0, 1, 2, 5\}$ in $\{0, 1\}, \{1, 2\}, \{2, 5\}, \emptyset, \{5\}, \{0, 5\}, \{1\}, \{0, 2\}$. $\square$

Let F be a field. An *exact realisation* of MK over F consists of points $x_0, \dots, x_7$ and lines $l_0, \dots, l_7$ of the projective plane $\mathbb{P}^2(F)$ such that $x_P \in l_j$ if and only if $P \in L_j$. Points are written $[x : y : z]$, and the line with normal vector $u \in F^3 \setminus 0$ is $\{[w] : u \cdot w = 0\}$.

Proposition 3.2. Let F be a field.

- (a) In an exact realisation the points x_P are distinct, the lines l_j are distinct, and three distinct points x_a, x_b, x_c are collinear if and only if $\{a, b, c\}$ is a line of MK. In particular x_0, x_1, x_2, x_5 form a projective frame.
- (b) The exact realisations with $x_0 = [1 : 0 : 0]$, $x_1 = [0 : 1 : 0]$, $x_2 = [0 : 0 : 1]$ and $x_5 = [1 : 1 : 1]$ correspond bijectively to the roots $c \in F$ of $t^2 - t + 1$; the realisation belonging to c has

$$x_3 = [1 : 1 : 0], \quad x_4 = [0 : 1 - c : 1], \quad x_6 = [c : 1 : 1], \quad x_7 = [c : 0 : 1].$$

- (c) For $F = \mathbb{F}_q$ there are exactly $c(q) |\operatorname{PGL}_3(q)|$ exact realisations, and $c(q) = 2, 0, 1$ for $q \equiv 1, 2, 0 \pmod{3}$.

Proof. (a) If $x_P = x_Q$ with $P \neq Q$, then P and Q would lie on the same lines, but by (F1) they share at most one of their three lines. Dually for the lines. If $\{a, b, c\}$ is a line L_j , the three points lie on l_j . If not, by (F2) two of them, say a and b , lie on a common line L_j , and $c \notin L_j$; the line through $x_a \neq x_b$ is l_j , and $x_c \in l_j$ would contradict exactness. With (F4) it follows that no three of x_0, x_1, x_2, x_5 are collinear.

(b) Each line l_j is the line through two of its points. Thus $l_0 = \{z = 0\}$, $l_1 = \{x = 0\}$, $l_7 = \{y = 0\}$, $l_2 = \{x = y\}$ (through x_2, x_5) and $l_5 = \{y = z\}$ (through x_0, x_5). Hence $x_3 \in l_0 \cap l_2$ is $[1 : 1 : 0]$, $x_4 \in l_1$

is $[0 : a : b]$, $x_7 \in l_7$ is $[e : 0 : f]$, and $x_6 \in l_5$, $x_6 \neq x_0$, is $[c : 1 : 1]$. Collinearity on $L_6 = \{1, 6, 7\}$ gives $\det(x_1, x_6, x_7) = e - cf = 0$; here $f \neq 0$, as otherwise $x_7 = 0$, so $x_7 = [c : 0 : 1]$. On $L_3 = \{3, 4, 6\}$ it gives $\det(x_3, x_4, x_6) = a - b + bc = 0$; here $b \neq 0$, as otherwise $a = 0$, so $x_4 = [0 : 1 - c : 1]$. On $L_4 = \{4, 5, 7\}$ it gives $\det((0, 1 - c, 1), (1, 1, 1), (c, 0, 1)) = -(c^2 - c + 1) = 0$.

Conversely, let $c \in F$ be a root of $t^2 - t + 1$, define x_P as above (with x_0, x_1, x_2, x_5 as in (b)) and let l_j be the line with normal vector y_j , where

$$y_0, \dots, y_7 = (0, 0, 1), (1, 0, 0), (1, -1, 0), (1, -1, 1 - c), (c, -1, 1 - c), (0, 1, -1), (1, 0, -c), (0, 1, 0).$$

With the representatives of (b), the matrix $(x_P \cdot y_j)_{P,j}$ is, after reduction by $c^2 = c - 1$,

$$B(c) = \begin{pmatrix} 0 & 1 & 1 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 - c & 1 - c & -1 & -c & 0 \\ 0 & 1 & 0 & 0 & c - 1 & 1 & 1 & 1 \\ 1 & 0 & c - 1 & 0 & 0 & -c & -c & 1 - c \\ 1 & 1 & 0 & 1 - c & 0 & 0 & 1 - c & 1 \\ 1 & c & c - 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & c & c & 1 & 0 & -1 & 0 & 0 \end{pmatrix} \quad (3)$$

(rows $P = 0, \dots, 7$, columns $j = 0, \dots, 7$). Its zero entries are exactly the 24 incident pairs, and its other entries are ± 1 , $\pm c$ and $\pm(1 - c)$, which are units because $c(1 - c) = 1$. So $x_P \in l_j$ if and only if $P \in L_j$, and we have an exact realisation. Distinct roots give distinct realisations, as they differ in x_6 .

(c) $\text{PGL}_3(F)$ acts on the exact realisations and simply transitively on ordered projective frames, so by (a) and (b) there are $|\text{PGL}_3(q)|$ times as many exact realisations as roots. If $3 \mid q$, then $t^2 - t + 1 = (t + 1)^2$. Otherwise, as $(t + 1)(t^2 - t + 1) = t^3 + 1$, the roots are the elements of order 6 (if q is odd) or 3 (if q is even) of the cyclic group $\mathbb{F}_q^\times$: two if $q \equiv 1 \pmod{3}$, none otherwise. $\square$

Corollary 3.3. *For every prime power q , the number $E(q)$ of matrices $B \in \mathbb{F}_q^{8 \times 8}$ of rank 3 with $B_{PL} \neq 0$ if and only if $P \notin L$ is $c(q)(q - 1)^{15}$.*

Proof. Every B of rank 3 is $XY^\top$ with $X, Y \in \mathbb{F}_q^{8 \times 3}$ of rank 3, and these pairs form one free orbit of $\text{GL}_3(q)$, acting by $(X, Y) \mapsto (Xg, Yg^{-\top})$ (the columns of X are a basis of the column space of B , and then Y is determined). Let x_P and y_L be the rows of X and Y , so that $B_{PL} = x_P \cdot y_L$. If the support of B is the set of non-incident pairs, then no row or column of B vanishes, so $x_P \neq 0 \neq y_L$, and the points $[x_P]$ with the lines with normal vectors y_L form an exact realisation. Conversely, take non-zero representatives x_P of the points and normal vectors y_L of the lines of an exact realisation; $XY^\top$ has the required support. X has rank 3: otherwise all points $[x_P]$ would lie on one line, which would contain three distinct points of each l_j and hence equal every l_j , contradicting Proposition 3.2(a); dually, Y has rank 3, since otherwise all lines would pass through one point, which would then equal every x_P , as x_P lies on two distinct lines. Hence $E(q) |\text{GL}_3(q)| = (q - 1)^{16} c(q) |\text{PGL}_3(q)|$, and $|\text{GL}_3(q)| = (q - 1) |\text{PGL}_3(q)|$. $\square$

4 A short existence proof

Proof of Theorem 1.2. Let S be the set of non-incident pairs (the edge set of Γ_{MK}). For $T \subseteq S$ let $N_T(q)$ be the number of matrices of rank 3 in $\mathbb{F}_q^{8 \times 8}$ supported in T , and $E_T(q)$ the number of those with support exactly T . Then $N_T = \sum_{T' \subseteq T} E_{T'}$, so by Möbius inversion over the subsets of S ,

$$E_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} N_T.$$

By Lemma 2.3, $c_{\Gamma_T,3}(p) = p^{10}N_T(p)$ for odd p . Suppose that for every T there are $f_T \in \mathbb{Q}[t]$ and p_T with $c_{\Gamma_T,3}(p) = f_T(p)$ for all primes $p > p_T$. Put $f = \sum_T (-1)^{|S \setminus T|} f_T$ and $p_0 = \max_T p_T$. By Corollary 3.3, $f(p) = p^{10}c(p)(p-1)^{15}$ for all primes $p > p_0$. There are infinitely many primes $p \equiv 2 \pmod{3}$, at which f vanishes, so $f = 0$; and there are infinitely many primes $p \equiv 1 \pmod{3}$, at which $f(p) = 2p^{10}(p-1)^{15} \neq 0$. (Both facts are Dirichlet's theorem for the modulus 3, which has elementary proofs.) This contradiction proves the theorem. $\square$

This argument uses only Sections 2 and 3; it does not tell which T is non-polynomial, and Theorem 1.1 shows that $T = S$ is.

5 The explicit count

Throughout this section q is a prime power and $F = \mathbb{F}_q$. An *incidence structure* I consists of a set of m points, a set of n lines and an incidence relation between them; its *incidence graph* H_I is the bipartite graph on the points and the lines with an edge for each incident pair. We keep track of which vertices are points and which are lines. For $r, s \geq 0$ let

$$N_I^{(r)}(q) = \#\{B \in F^{\text{points} \times \text{lines}} : \text{rank } B = r, B_{PL} = 0 \text{ whenever } P \in L\},$$

$$Z_s(q) = Z_s^I(q) = \#\{(x_P)_P, (y_L)_L \in (F^s)^m \times (F^s)^n : x_P \cdot y_L = 0 \text{ whenever } P \in L\}.$$

For $I = \text{MK}$ we have $N_{\text{MK}}^{(3)} = N$, and H_{MK} denotes the incidence graph of MK (cubic, 16 vertices, 24 edges). We shall use an elementary fact: (*) if $D \in \mathbb{Z}[t]$ is monic, $A \in \mathbb{Z}[t]$, and $A(q)/D(q) \in \mathbb{Z}$ for infinitely many prime powers q , then D divides A in $\mathbb{Z}[t]$. Indeed, $A = gD + r$ with $g, r \in \mathbb{Z}[t]$ and $\deg r < \deg D$, and $r(q)/D(q)$ is an integer tending to 0, so r has infinitely many roots.

Lemma 5.1. *For every incidence structure I and every $r \geq 0$,*

$$N_I^{(r)}(q) |\text{GL}_r(q)| = \sum_{U, U' \leq F^r} \mu(U) \mu(U') q^{m(\dim U - s) + n(\dim U' - s)} Z_s(q),$$

where $s = s(U, U')$ is the rank of the dot product on $U \times U'$ and $\mu(U) = (-1)^k q^{k(k-1)/2}$ with $k = r - \dim U$. The number of pairs (U, U') with $\dim U = u$, $\dim U' = u'$ and $s(U, U') = s$ is $\begin{bmatrix} r \\ u' \end{bmatrix}_q \begin{bmatrix} r-u' \\ u-s \end{bmatrix}_q \begin{bmatrix} u' \\ s \end{bmatrix}_q q^{(r-u'-u+s)s}$. Consequently $N_I^{(r)} |\text{GL}_r| = Z_r + \sum_{s < r} a_{r,s} Z_s$ with $a_{r,s} \in \mathbb{Z}[q]$ depending only on r, m and n .

Proof. As in the proof of Corollary 3.3, $N_I^{(r)}(q) |\text{GL}_r(q)|$ is the number of pairs $(X, Y) \in F^{m \times r} \times F^{n \times r}$ of rank r with $x_P \cdot y_L = 0$ whenever $P \in L$. For a subspace $U_* \leq F^r$ we have $\sum_{U_* \leq U \leq F^r} \mu(U) = 1$ if $U_* = F^r$ and $= 0$ otherwise; this is the q -binomial theorem $\sum_j \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j q^{j(j-1)/2} = \prod_{i < k} (1 - q^i)$ for $k = r - \dim U_*$. Applying this to the spans of the rows of X and of Y gives the sum over (U, U') of $\mu(U) \mu(U')$ times the number of $(X, Y) \in U^m \times U'^n$ with $x_P \cdot y_L = 0$ on the incidences. Let $U_0 = U \cap U'^\perp$ and $U'_0 = U' \cap U^\perp$, of dimensions $u - s$ and $u' - s$. The dot product induces a perfect pairing between complements U_1 of U_0 in U and U'_1 of U'_0 in U' , which in dual bases is the standard dot product of F^s , and $x \cdot y$ depends only on the components in U_1 and U'_1 . So the term of (U, U') is $q^{m(u-s) + n(u'-s)} Z_s$. For the number of pairs, choose U' , then $K = U \cap U'^\perp$ in $\begin{bmatrix} r-u' \\ u-s \end{bmatrix}_q$ ways, and then U/K , an s -dimensional subspace of F^r/K meeting $U'^\perp/K$ trivially, in $q^{(r-u'-u+s)s} \begin{bmatrix} u' \\ s \end{bmatrix}_q$ ways. Finally $s = r$ only for $U = U' = F^r$. $\square$

For a bipartite graph G whose vertices are labelled as points and lines (every edge joining a point to a line), let $\text{hom}(G)$ be the number of maps sending the points of G to points and the lines of G

to lines of $\mathbb{P}^2(F)$ such that the image of a point lies on the image of every adjacent line. For sets R of points and C of lines of I , $H_I[R \cup C]$ denotes the induced subgraph, and $\kappa(R, C)$ its number of connected components.

Lemma 5.2. *Summing over all sets R of points and C of lines of I ,*

$$\begin{aligned} Z_0 &= 1, & Z_1 &= \sum_{\substack{(R,C) \\ \text{no edge between } R \text{ and } C}} (q-1)^{|R|+|C|}, & Z_2 &= \sum_{(R,C)} (q-1)^{|R|+|C|} (q+1)^{\kappa(R,C)}, \\ Z_3 &= \sum_{(R,C)} (q-1)^{|R|+|C|} \text{hom}(H_I[R \cup C]). \end{aligned}$$

In particular Z_0, Z_1, Z_2 are given by integer polynomials in q .

Proof. Group the pairs (X, Y) by $R = \{P : x_P \neq 0\}$ and $C = \{L : y_L \neq 0\}$. For non-zero x, y the condition $x \cdot y = 0$ depends only on the classes $[x], [y]$, and scalars contribute $(q-1)^{|R|+|C|}$. For $s = 1$, $xy \neq 0$. For $s = 2$ the non-zero y with $x \cdot y = 0$ form one class $[x]^\perp$, and $\perp$ is an involution of $\mathbb{P}^1(F)$; so along each edge the class at one end determines the class at the other, and each connected component of $H_I[R \cup C]$ admits exactly $q+1$ consistent choices. For $s = 3$ the classes of non-zero vectors are the points of $\mathbb{P}^2(F)$ and, as normal vectors, its lines, and $x \cdot y = 0$ means incidence. $\square$

For vertices $u \neq w$ of the same kind, $G/\{u, w\}$ denotes the graph in which u and w are replaced by one vertex adjacent to the union of their neighbourhoods.

Lemma 5.3. *For every finite bipartite graph G with points and lines:*

- (R0) *hom is multiplicative on disjoint unions; the empty graph gives 1 and a single vertex gives $q^2 + q + 1$;*
- (R1) *if v has degree 1, then $\text{hom}(G) = (q+1) \text{hom}(G-v)$;*
- (R2) *if v has degree 2 with neighbours u, w , then $\text{hom}(G) = \text{hom}(G-v) + q \text{hom}((G-v)/\{u, w\})$;*
- (R3) *if $a \neq a'$ are of the same kind and have two common neighbours $b \neq b'$, then $\text{hom}(G) = \text{hom}(G/\{a, a'\}) + \text{hom}(G/\{b, b'\}) - \text{hom}(G/\{a, a'\}/\{b, b'\})$.*

The same holds for homomorphisms into the incidence graph of any projective plane of order q .

Proof. (R0) and (R1) hold because a plane of order q has $q^2 + q + 1$ points and lines, and every line has $q+1$ points and every point lies on $q+1$ lines. (R2) Given the images of $G-v$, the images of v are the elements incident with the images of u and w : one if these images are distinct, $q+1$ if they are equal; and the maps of $G-v$ that agree on u and w are the maps of $(G-v)/\{u, w\}$. (R3) If the images of a and a' are distinct, then the images of b and b' are both incident with both of them and hence equal. So every map identifies a with a' or b with b' , and (R3) is inclusion and exclusion. $\square$

A *minor* of H_{MK} is given by a set of deleted vertices, a partition of the remaining points into blocks and a partition of the remaining lines into blocks; its vertices are the blocks, and a block of points is adjacent to a block of lines if some point of the first lies on some line of the second. It is *proper* if it has fewer than 16 vertices, that is, unless nothing is deleted and all blocks are singletons. Deleting a vertex of a minor, merging two of its vertices of the same kind as above, and passing to a connected component all give minors again. The induced subgraphs $H_{\text{MK}}[R \cup C]$ and the graphs H_{MK}/σ , in which each block of a partition σ of the points is merged into one vertex, are minors.

Lemma 5.4. *Every non-empty proper minor of H_{MK} in which every vertex has degree at least 3 contains a cycle of length 4.*

Proof. (a) Let G be a non-empty bipartite graph with parts A and B , in which every vertex has degree at least 3 and which has no 4-cycle. A vertex of G has at least three neighbours, so $|A|, |B| \geq 3$. Two vertices of A have at most one common neighbour, so counting pairs of vertices of A with a common neighbour gives $3|B| \leq \sum_{b \in B} \binom{d(b)}{2} \leq \binom{|A|}{2}$, and symmetrically $3|A| \leq \binom{|B|}{2}$. If $|A| \leq 6$, then $|B| \leq 5$, hence $|A| \leq 3$ and $|B| \leq 1$, a contradiction; so $|A| \geq 7$ and likewise $|B| \geq 7$. For a proper minor, $|A|, |B| \leq 8$ and $|A| + |B| \leq 15$; the cases $(7, 8)$ and $(8, 7)$ violate $3 \cdot 8 \leq \binom{7}{2} = 21$. So $|A| = |B| = 7$, and $21 \leq \sum_b \binom{d(b)}{2} \leq 21$ forces $d(b) = 3$ for all b , and likewise for A : G is cubic with $7 + 7$ vertices.

(b) No proper minor G of H_{MK} is cubic with $7 + 7$ vertices. Seven blocks of points arise either by deleting one point, all blocks being singletons, or by deleting nothing and merging two points; the same holds for lines. A block $\{P, P'\}$ of two points is adjacent to the blocks containing the at least five lines through P or P' (by (F1) they share at most one line); as at most one line is deleted or one pair of lines merged, these lines lie in at least four blocks, and the degree of $\{P, P'\}$ is at least 4. Dually for a block of two lines. So one point P_0 and one line L_0 are deleted and all blocks are singletons. But P_0 lies on at least two lines different from L_0 , and in G such a line has degree 2. With (a) this proves the lemma. $\square$

Corollary 5.5. *For every proper minor G of H_{MK} there is $f_G \in \mathbb{Z}[t]$ such that $\text{hom}(G) = f_G(q)$ for every prime power q ; more generally, $f_G(q)$ is the number of homomorphisms from G into the incidence graph of any projective plane of order q .*

Proof. Induction on the number of vertices. The empty graph and the single vertex are covered by (R0). If G is disconnected, (R0) reduces it to its components. If G is connected with at least two vertices and has a vertex of degree 1 or 2, apply (R1) or (R2); otherwise all degrees are at least 3, and by Lemma 5.4 G has a 4-cycle a, b, a', b' , so that (R3) applies. In each case the graphs on the right are minors with fewer vertices. $\square$

Lemma 5.6. *Let σ run through the partitions of the set of points of MK , let $\hat{0}$ be the partition into singletons, and let $\mu(\hat{0}, \sigma) = \prod_{\beta \in \sigma} (-1)^{|\beta|-1} (|\beta| - 1)!$. Then*

$$\text{hom}(H_{\text{MK}}) = c(q) |\text{PGL}_3(q)| + (q^2 + q + 1)(q + 1)q(q - 1) \cdots (q - 6) - \sum_{\sigma \neq \hat{0}} \mu(\hat{0}, \sigma) \text{hom}(H_{\text{MK}}/\sigma).$$

Proof. For a homomorphism φ let $\pi(\varphi)$ be the partition of the points into the fibres of φ . The φ that are constant on the blocks of σ are the homomorphisms of H_{MK}/σ , so by Möbius inversion in the lattice of partitions, whose Möbius function is $\mu(\hat{0}, \sigma)$, the number of φ injective on the points is $\sum_{\sigma} \mu(\hat{0}, \sigma) \text{hom}(H_{\text{MK}}/\sigma)$. Let φ be injective on the points. Then the image of each line L is the line through the images of any two points of L . If all images of points lie on one line ℓ , then all lines map to ℓ ; there are $(q^2 + q + 1)(q + 1)q \cdots (q - 6)$ such φ . Otherwise φ is an exact realisation. Indeed, suppose that $\varphi(P) \in \varphi(L)$ for some $P \notin L$, and let K be the set of points Q with $\varphi(Q) \in \varphi(L)$; it contains $L \cup \{P\}$. If two points of a line L'' are in K , their distinct images lie on $\varphi(L)$ and on $\varphi(L'')$, so $\varphi(L'') = \varphi(L)$ and the third point of L'' is in K . By (F3), K contains all points, so all images lie on $\varphi(L)$, a contradiction. Conversely, every exact realisation is a homomorphism, injective on the points by Proposition 3.2(a), whose points are not collinear (otherwise all its lines would coincide). By Proposition 3.2(c) there are $c(q)|\text{PGL}_3(q)|$ exact realisations, and the formula follows, the term $\sigma = \hat{0}$ being $\text{hom}(H_{\text{MK}})$. $\square$

Theorem 5.7. *For every prime power q , $N(q) = h(q) + c(q)(q - 1)^{15}$, with h as in Theorem 1.1.*

Proof. By Lemma 5.1 with $r = 3$ and by Lemma 5.2,

$$N(q) |\mathrm{GL}_3(q)| = \sum_{(R,C)} (q-1)^{|R|+|C|} \mathrm{hom}(H_{\mathrm{MK}}[R \cup C]) + \sum_{s \leq 2} a_{3,s}(q) Z_s(q),$$

and the last sum is an integer polynomial in q . For (R, C) different from (all points, all lines), $H_{\mathrm{MK}}[R \cup C]$ is a proper minor, so its term is a polynomial by Corollary 5.5. The remaining term is $(q-1)^{16} \mathrm{hom}(H_{\mathrm{MK}})$; by Lemma 5.6 and Corollary 5.5, $\mathrm{hom}(H_{\mathrm{MK}}) = c(q) |\mathrm{PGL}_3(q)| + P_0(q)$ with $P_0 \in \mathbb{Z}[t]$, all H_{MK}/σ with $\sigma \neq \hat{0}$ being proper minors. Hence there is $A \in \mathbb{Z}[t]$ with $N(q) |\mathrm{GL}_3(q)| = A(q) + c(q)(q-1)^{16} |\mathrm{PGL}_3(q)|$ for every prime power q . As $|\mathrm{GL}_3(q)| = (q-1) |\mathrm{PGL}_3(q)|$, the number $N(q) - c(q)(q-1)^{15} = A(q)/|\mathrm{GL}_3(q)|$ is an integer, and $|\mathrm{GL}_3(q)|$ is the value of the monic polynomial $G_3(t) = (t^3-1)(t^3-t)(t^3-t^2)$. By (*), $A = gG_3$ with $g \in \mathbb{Z}[t]$, so $N(q) = g(q) + c(q)(q-1)^{15}$. The computation described below gives $g = h$. $\square$

Proof of Theorem 1.1. By Lemma 2.3, $c_{\Gamma_{\mathrm{MK}},3}(p) = p^{10}N(p)$, so (1) follows from Theorem 5.7, and the three cases from Proposition 3.2(c). Let $f \in \mathbb{Q}[t]$ agree with $c_{\Gamma_{\mathrm{MK}},3}$ on a set Π of primes as in the theorem. Then $f - t^{10}h$ vanishes at the infinitely many primes $p \in \Pi$ with $p \equiv 2 \pmod{3}$, so $f = t^{10}h$; but then $2p^{10}(p-1)^{15} = 0$ for the primes $p \in \Pi$ with $p \equiv 1 \pmod{3}$, which is absurd. $\square$

This argument uses only that g is a polynomial, not its value; so the non-polynomiality in Theorem 1.1 does not depend on a computer.

The computation of g . The program `rank3_formula.py` (with `hompoly.py`) evaluates the identity in the proof of Theorem 5.7 in exact integer arithmetic: Z_1, Z_2 by Lemma 5.2; hom of the $2^{16} - 1$ proper induced subgraphs (up to the 48 collineations of MK) and of the 4,139 graphs H_{MK}/σ by the rules of Lemma 5.3 (it stops if no rule applies, which does not happen); and the sum of Lemma 5.1, whose numbers of pairs it checks by enumeration over $\mathbb{F}_2$ and $\mathbb{F}_3$. In about 7 seconds it finds that G_3 divides A , with quotient h , and

$$P_0(q) = 2q^{10} + 34q^9 + 258q^8 + 820q^7 + 1404q^6 + 1508q^5 + 1040q^4 + 439q^3 + 112q^2 + 16q + 1.$$

So h is the result of an exact computation; it was not derived without a computer, and Section 8 lists the independent checks.

Remark 5.8. (a) $h = N - E$ counts the rank-3 matrices supported in S with smaller support. As the congruence of Lewis et al. [LLM⁺11, Proposition 5.1] for N predicts (note that $(q-1)^4$ divides E), $(q-1)^3$ divides h , and $h(q)/(q-1)^3$ is 4760 at $q = 1$, the number of placements of three non-attacking rooks on the 40 cells of S ; this checks the computation. (b) Sections 3 and 5 do not use that q is prime; Theorem 5.7 was also confirmed for $q = 4, 8, 9$ (Section 8).

6 Small degrees

Proof of Theorem 1.3. (a) A matrix $A \in M_\Gamma(F)$ of rank 2 is alternating, so $A = ZJZ^\top$ with $Z \in F^{n \times 2}$ of rank 2 and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The column space of Z is that of A , and $ZgJg^\top Z^\top = \det(g) ZJZ^\top$ for $g \in \mathrm{GL}_2(F)$; hence Z is determined by A up to the free action of $\mathrm{SL}_2(F)$. Let $z_v \in F^2$ be the rows of Z ; then $A_{uv} = \det(z_u, z_v)$. So $A \in M_\Gamma(F)$ means that z_u and z_v are linearly dependent whenever $u \neq v$ are not adjacent, and rank 2 means that the z_v span F^2 . Grouping by $R = \{v : z_v \neq 0\}$ and by the classes $[z_v] \in \mathbb{P}^1(F)$: the classes must be constant on the connected components of the complement of $\Gamma[R]$, and not constant on R . Hence

$$\#\{A \in M_\Gamma(\mathbb{F}_q) : \mathrm{rank} A = 2\} \cdot |\mathrm{SL}_2(q)| = \sum_{\emptyset \neq R} (q-1)^{|R|} ((q+1)^{\kappa'(R)} - (q+1)),$$

where $\kappa'(R)$ is the number of connected components of the complement of $\Gamma[R]$. As $|\mathrm{SL}_2(q)| = q^3 - q$, by (*) of Section 5 the number of matrices of rank 2 is a polynomial in q , and $c_{\Gamma,1}(p) = p^{n-2} \#\{A \in M_{\Gamma}(\mathbb{F}_p) : \text{rank } A = 2\}$ by Theorem 2.1.

(b) Let Γ have parts U, U' and edge set T , and let I be the incidence structure with points U , lines U' and incidence relation $(U \times U') \setminus T$. By Lemma 2.3, $c_{\Gamma,2}(p) = p^{a+b-4} N_I^{(2)}(p)$. By Lemma 5.1 with $r = 2$, $N_I^{(2)}|\mathrm{GL}_2|$ is an integer combination of Z_0, Z_1, Z_2 with polynomial coefficients, hence a polynomial by Lemma 5.2, and $|\mathrm{GL}_2(q)| = (q^2 - 1)(q^2 - q)$ is monic; apply (*).

(c) For odd p the degrees of the irreducible characters of $G_{\Gamma_{\mathrm{MK}}}(\mathbb{F}_p)$ are p^i with $0 \leq i \leq 8$, so $\sum_{i=0}^8 c_{\Gamma_{\mathrm{MK}},i}(p) = k(G_{\Gamma_{\mathrm{MK}}}(\mathbb{F}_p))$, the number of conjugacy classes. By Rossmann [Ros22, Theorem 1.2] (quoted there from [RV24, Corollary 1.3]; an explicit formula is [Ros22, Corollary B]) it is a polynomial in p . By Theorem 2.1, $c_{\Gamma_{\mathrm{MK}},0}(p) = p^{16}$; by (b), $c_{\Gamma_{\mathrm{MK}},1}$ and $c_{\Gamma_{\mathrm{MK}},2}$ are polynomials, and $c_{\Gamma_{\mathrm{MK}},3}$ is not, by Theorem 1.1. So $\sum_{i=4}^8 c_{\Gamma_{\mathrm{MK}},i}$ is not a polynomial in p , and hence at least one of its summands is not. $\square$

7 A conditional universality statement

Following [BB03, §1.2], let $\mathcal{S} \subset \mathbb{Z}[q]$ be the saturated multiplicative set generated by the polynomials $q^n - q$ ($n > 1$), and let $\mathcal{R} = \mathcal{S}^{-1}\mathbb{Z}[q]$. Its elements are functions on the prime powers, which we restrict to the odd primes. For a scheme X of finite type over $\mathbb{Z}$ let $[X](q) = \#X(\mathbb{F}_q)$. For a finite set V , subsets $H_1, \dots, H_m$ of V and $\pi \in \mathbb{Z}^m$, $[L(3, \pi)](q)$ is the number of maps $f: V \rightarrow \mathbb{F}_q^3$ such that $f(H_j)$ spans a space of dimension π_j for every j ; it is the point count of a scheme $L(3, \pi)$.

Hypothesis (BB) [BB03, Theorem 11.1]. *For every quasi-projective scheme X of finite type over $\mathbb{Z}$ there are $V, H_1, \dots, H_m, \pi$ and $\sigma \in \mathcal{S}$ with $\sigma[X] = [L(3, \pi)]$.*

Belkale and Brosnan attribute this to Mnëv and Sturmfels; unable to extract this exact statement from the literature, they sketch its proof [BB03, §11]. We use it only as a hypothesis. As every scheme of finite type over $\mathbb{Z}$ is a finite disjoint union of quasi-projective ones [BB03, Remark 11.2], it expresses every $[X]$ through the $[L(3, \pi)]$.

Proposition 7.1. *Assume Hypothesis (BB). For every scheme X of finite type over $\mathbb{Z}$, the function $p \mapsto \#X(\mathbb{F}_p)$ on the odd primes is an $\mathcal{R}$ -linear combination of finitely many functions $c_{\Gamma,3}$ with Γ bipartite, and also of finitely many functions $c_{\Gamma,N}$ with Γ bipartite with two parts of size N (characters of the maximal degree p^N).*

Proof. Step 1. It suffices to treat $[L(3, \pi)]$. For $t \in \{0, 1, 2, 3\}^m$ let I^t be the incidence structure with point set V and with t_j copies of the line H_j for each j . For $X = (x_v) \in (\mathbb{F}_q^3)^V$, the vectors orthogonal to all x_v , $v \in H_j$, form a space of dimension $3 - \dim\langle x_v : v \in H_j \rangle$, so

$$Z_3^{I^t} = \sum_{\rho \in \{0,1,2,3\}^m} [L(3, \rho)] \prod_{j=1}^m (q^{3-\rho_j})^{t_j}.$$

The matrix $((q^{3-\rho})^t)_{t, \rho \in \{0,1,2,3\}^m}$ is a Vandermonde matrix with determinant $\pm \prod_{a < b} q^a (q^{b-a} - 1)$, which lies in $\mathcal{S}$ since $\mathcal{S}$ is saturated and contains $q^2 - q$ and $q(q^k - 1)$. So its m -th tensor power is invertible over $\mathcal{R}$, and each $[L(3, \rho)]$ is an $\mathcal{R}$ -linear combination of the $Z_3^{I^t}$. By Lemmas 5.1 and 5.2, $Z_3^{I^t} = |\mathrm{GL}_3| N_{I^t}^{(3)} - f_t$ with $f_t \in \mathbb{Z}[q]$, and by Lemma 2.3, $N_{I^t}^{(3)}(p) = p^{-(|V|+|t|-6)} c_{\Gamma_t,3}(p)$, where Γ_t is the bipartite graph of the non-incident pairs of I^t and $|t| = \sum_j t_j$. Finally q and $|\mathrm{GL}_3(q)| = q^3(q^3 - 1)(q^2 - 1)(q - 1)$ lie in $\mathcal{S}$, and polynomials are multiples of $1 = c_{K_{3,3,3}}/|\mathrm{GL}_3|$.

Step 2. For $T \subseteq [n] \times [n]$ and $k \geq 0$ let T_k be the set of positions of the matrices $\begin{pmatrix} B & U \\ V & 0 \end{pmatrix}$ with B supported in T and $U \in \mathbb{F}_q^{n \times k}$, $V \in \mathbb{F}_q^{k \times n}$ arbitrary. The number $f_k(r)$ of pairs (U, V) that make

such a matrix invertible depends only on $r = \text{rank } B$ (replace B by PBQ). Let $B = \text{diag}(I_r, 0)$, $m = n - r$, and split U into U_1 (the first r rows) and U_2 , and V into V_1 (the first r columns) and V_2 . Elimination with I_r gives the determinant $\pm \det \begin{pmatrix} 0 & U_2 \\ V_2 & -V_1'U_1 \end{pmatrix}$. It vanishes unless $\text{rank } U_2 = \text{rank } V_2 = m$. Then multiplying this matrix on the left by $\text{diag}(g, h')$ and on the right by $\text{diag}(g', h)$, with $g, g' \in \text{GL}_m(q)$ and $h, h' \in \text{GL}_k(q)$ suitably chosen, moves U_2 to $(I_m \ 0)$ and V_2 to its transpose, replaces (U_1, V_1) by $(U_1h, h'V_1)$, a bijection, and multiplies the determinant by a unit; the determinant becomes $\pm \det(V_1''U_1'')$, with V_1'' the last $k - m$ rows of V_1 and U_1'' the last $k - m$ columns of U_1 . Hence $f_k(r) = 0$ unless $n - r \leq k \leq n$, and then

$$f_k(r) = \nu(m, k)^2 q^{2rm} \prod_{i < k-m} (q^r - q^i) \cdot |\text{GL}_{k-m}(q)| q^{(r-k+m)(k-m)}, \quad \nu(m, k) = \prod_{i < m} (q^k - q^i).$$

This is a polynomial in q ; $f_{n-r}(r) = |\text{GL}_{n-r}(q)|^2 q^{2r(n-r)}$ is a unit of $\mathcal{R}$, and $f_{n-r}(r') = 0$ for $r' < r$. So the numbers of invertible matrices supported in T_{n-r} , $r = n, \dots, 0$, form a triangular system in the numbers $M_{r'}(T)$ whose diagonal coefficients are units of $\mathcal{R}$, and each $M_r(T)$ is an $\mathcal{R}$ -linear combination of them; rectangular boards are padded with forbidden rows or columns. By Lemma 2.3, the number of invertible matrices supported in an $N \times N$ board is the number of characters of degree p^N of a bipartite graph on $N + N$ vertices. $\square$

A function on the odd primes is *PORC* if there are $M \geq 1$ and polynomials $f_0, \dots, f_{M-1}$ such that it agrees with f_a on the primes $p \equiv a \pmod{M}$; $c_{\Gamma_{\text{MK}}, 3}$ is PORC with $M = 3$.

Corollary 7.2. *Assume Hypothesis (BB). Then there is a bipartite graph Γ for which $c_{\Gamma, 3}$ is not PORC, and likewise for characters of maximal degree.*

Proof. Let $X = \{y^2 = x^3 - x\} \subseteq \mathbb{A}^2$. By a classical formula (see [IR90, Chapter 18]), for $p \equiv 1 \pmod{4}$ we have $\#X(\mathbb{F}_p) = p - 2a$ for integers a, b with $p = a^2 + b^2$. Write $[X] = \sum_j r_j c_{\Gamma_j, 3}$ with $r_j \in \mathcal{R}$ (Proposition 7.1) and suppose that all $c_{\Gamma_j, 3}$ are PORC, with a common M . On the primes $p \equiv 1 \pmod{4M}$, $[X]$ agrees with some $\rho \in \mathbb{Q}(t)$, and $|\rho(p) - p| = |2a| \leq 2\sqrt{p}$. So $\rho(t) - t$ has degree at most 0 and tends to a limit k , and the integers $\rho(p) - p$ equal k for all large such p . Then $p = k^2/4 + b^2$ for all large primes $p \equiv 1 \pmod{4M}$. But at most $\sqrt{x} + 1$ integers up to x have this form, whereas, by the prime number theorem for arithmetic progressions, at least a constant times $x/\log x$ primes up to x are $\equiv 1 \pmod{4M}$. So some $c_{\Gamma_j, 3}$ is not PORC; the same argument applies to characters of maximal degree. $\square$

Both statements give existence only: we do not know an explicit graph Γ for which $c_{\Gamma, 3}$ is not PORC.

8 Computations, verification and scope

All results are proved in the text, except that the coefficients of h and P_0 come from the computation of Section 5, and that Proposition 7.1 and Corollary 7.2 assume Hypothesis (BB), whose source gives only a sketch of proof. The source archive contains the programs, their outputs, a README with commands and running times, `check_note.py`, which checks the statements as printed, and the programs and outputs of the three independent verification runs.

Verification. The results were first obtained, with proofs, together with the programs of the package, which include direct counts of $N(p)$ by two programs of different design (both for $p = 2, 3$, one of them also for $p = 5, 7$), $Z_3(p)$ for $p \leq 13$, and a comparison of Theorem 2.1 with GAP for 11 groups. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote

its own programs, and neither found a gap in a proof. The first run reproduced the outputs of the Python programs and of `zcount.c`, and counted $N(p)$ and Z_1, Z_2, Z_3 for $p = 2, 3, 5, 7$ by enumerating column spaces. It counted homomorphisms by brute force for 20 proper minors of H_{MK} at $q = 2, 3, 5$ (60 comparisons with Corollary 5.5) and for H_{MK} at $q = 2, 3, 5, 7$, obtaining $c(q)|\text{PGL}_3(q)| + P_0(q)$ each time, and it enumerated all 447,153,316 minors of H_{MK} with at least one point and one line: 9,193,080 have minimum degree at least 3, and only H_{MK} itself among them has no 4-cycle, which confirms Lemma 5.4 independently of its proof. The second run counted $N(p)$ by a dynamic programme over row spaces ($p = 2, 3$) and by evaluating the sum of Lemma 5.1 as a partition function ($p = 2, 3, 5, 7$ and $q = 4, 8, 9$, with $N(4) = 1,206,524,165,076$, $N(8) = 46,425,923,430,737,214$ and $N(9) = 271,958,641,065,406,464$ as in (2)); it counted $E(p)$ for $p = 2, 3, 5$ and the exact realisations for all primes $p \leq 31$, repeated the enumeration of minors, and compared Theorem 2.1 with GAP for 12 further graphs. These counts test the computation of h at seven field sizes; they do not determine a polynomial of degree 15. The corrections required by the two runs were made; for instance, Lemma 5.4 was first stated without excluding the empty minor, and Remark 2.2 was added.

A third independent verification run, also AI-assisted, examined the text of this note, first the parts written after the two runs: Remark 2.2, the proofs of Lemmas 5.1, 5.3, 5.4 and 5.6, Remark 5.8 and Section 7. It re-derived all proofs, compared the cited statements with the sources, read [OV15], and ran the package again from the source archive, with identical outputs. Its own programs recomputed h and P_0 by the route of Section 5, with a different implementation of the rules of Lemma 5.3 and the same coefficients; counted $N(2)$ in two ways and $N(3)$ directly, by row spaces; counted the exact realisations of MK over $\mathbb{F}_q$ for all 18 prime powers $q \leq 32$, using only that their points are distinct; compared Corollary 5.5 and Lemma 5.6 with brute-force counts of homomorphisms (183 comparisons, among them $\text{hom}(H_{\text{MK}})$ at $q = 2, 3, 4, 5$); compared Theorem 2.1 and Remark 2.2 with GAP for 10 further groups, three of them over $\mathbb{F}_9$ and one over $\mathbb{F}_4$; and checked the formula for $\#X(\mathbb{F}_p)$ used in Corollary 7.2 for all primes $p < 20,000$. It found no wrong statement and no gap. Its corrections concern attributions and wording: the realisability of 8_3 is now attributed to sources it read, [OV15, Theorem B] is cited directly, the results of [OV15, Ros25] on groups that are not graphical were added, and the version of the corpus was corrected.

Scope and priority. The note answers the precise core of Rossmann’s question negatively for one explicit graph and determines the count for it; the counts are polynomials for $i = 1$, and for $i \leq 2$ if Γ is bipartite; that they need not be PORC is shown only conditionally, and only as existence. Which functions occur, which graphs give polynomials (for instance, whether $c_{\Gamma,2}$ is a polynomial for every graph), which of $c_{\Gamma_{\text{MK}},4}, \dots, c_{\Gamma_{\text{MK}},8}$ are not polynomials, an explicit or unconditional non-PORC example, and the representation theory of $G_\Gamma(\mathbb{F}_p)$ beyond these numbers remain open. The question is Rossmann’s [EHNO21], [Ros22, Question 1.10], who suspected that these functions might be wild. We use [OV15, Theorem B] (with a proof), [Ros22, Theorem 1.2], [RV24, Corollary 1.3], the realisability of 8_3 , classical over $\mathbb{C}$ [BGS24, IKK⁺26] (with the computation over every field), [BB03, Theorem 11.1] as a hypothesis, and the classical point count of $y^2 = x^3 - x$. The argument of Section 4, inclusion and exclusion over supports with a configuration that needs a quadratic irrationality, is of the same kind as the argument of Belkale and Brosnan with the Fano matroid [BB03, §10] and may be regarded as known to specialists in this form. For groups that are not graphical, character counts that depend on the arithmetic of a hypersurface [OV15, Theorem 4.1] and class numbers that are wild modulo powers of q [Ros25, Theorem A] are known. To our knowledge, the negative answer to Rossmann’s question, Theorem 1.1 with its proof, Theorem 1.3 and Proposition 7.1 are not in print; we make no claim of priority.

What was read: [EHNO21] (pp. 2064–2066, 2081–2082); [Ros22] (arXiv v3), [RV24] (v2), [BB03] (v1), [OV15] (v2, Sections 1, 2 and 4), [Ros25] (v2, Sections 1 and 4), [LLM⁺11] (v3), [LM20] (v2),

[CS26] (v1), [BGS24] (v1, Section 3) and [IKK⁺26] (v1, Section 3), whose numbering we use; the abstracts of [HP11] and [KLM14]. We did not read [Ste98] (known through [LLM⁺11, LM20, CS26]), [Hag98] (through [LLM⁺11]), [Mar99] (cited after [Ros22]), or [IR90], from which the formula for $\#X(\mathbb{F}_p)$ in Corollary 7.2 is quoted; this formula was checked numerically for all primes below 20,000. Bibliographic data were checked with Crossref. The searches (27 October 2026) covered arXiv (all papers of T. Rossmann; queries on graphical groups, q -rook numbers, prescribed zeros, support constraints, Kontsevich’s conjecture and the Möbius–Kantor configuration), the works citing [Ros22], [LLM⁺11], [Ste98] and [OV15] in OpenAlex and Semantic Scholar, zbMATH Open, and four web searches. No answer to Rossmann’s question was found, nor an explicit count of matrices of given rank with prescribed zeros that fails to be a polynomial in odd characteristic. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [BB03] Prakash Belkale and Patrick Brosnan. Matroids, motives, and a conjecture of Kontsevich. *Duke Math. J.*, 116(1):147–188, 2003. [arXiv:math/0012198](#), [doi:10.1215/S0012-7094-03-11615-4](#).
- [BGS24] Taylor Brysiewicz, Fulvio Gesmundo, and Avi Steiner. Quatroids and rational plane cubics. *Beitr. Algebra Geom.*, 65(4):923–972, 2024. [arXiv:2309.07357](#), [doi:10.1007/s13366-024-00776-5](#).
- [CS26] Jeffrey Chen and Jesse Selover. Positivity properties of q -hit numbers in the finite general linear group. *Enumer. Comb. Appl.*, 6(3):Article S2R22, 2026. [arXiv:2504.09300](#), [doi:10.54550/ECA2026V6S3R22](#).
- [EHNO21] Bettina Eick, Derek F. Holt, Gabriele Nebe, and Eamonn A. O’Brien. Computational group theory (hybrid meeting). *Oberwolfach Rep.*, 18(3):2027–2087, 2021. Report No. 38/2021. [doi:10.4171/OWR/2021/38](#).
- [Hag98] James Haglund. q -rook polynomials and matrices over finite fields. *Adv. in Appl. Math.*, 20(4):450–487, 1998. [doi:10.1006/aama.1998.0582](#).
- [HP11] Zoltán Halasi and Péter P. Pálffy. The number of conjugacy classes in pattern groups is not a polynomial function. *J. Group Theory*, 14(6):841–854, 2011. [doi:10.1515/jgt.2010.081](#).
- [IKK⁺26] Kelly Isham, Nathan Kaplan, Sam Kimport, Rachel Lawrence, Luke Peilen, and Max Weinreich. Configurations of 10 points and their incidence varieties. *Discrete Comput. Geom.*, 2026. Published online 28 September 2026. [arXiv:2507.09829](#), [doi:10.1007/s00454-026-00876-0](#).
- [IR90] Kenneth Ireland and Michael Rosen. *A Classical Introduction to Modern Number Theory*, volume 84 of *Grad. Texts in Math.* Springer, New York, second edition, 1990. [doi:10.1007/978-1-4757-2103-4](#).

- [KLM14] Aaron J. Klein, Joel Brewster Lewis, and Alejandro H. Morales. Counting matrices over finite fields with support on skew Young diagrams and complements of Rothe diagrams. *J. Algebraic Combin.*, 39(2):429–456, 2014. [arXiv:1203.5804](#), [doi:10.1007/s10801-013-0453-x](#).
- [LLM⁺11] Joel Brewster Lewis, Ricky Ini Liu, Alejandro H. Morales, Greta Panova, Steven V. Sam, and Yan X. Zhang. Matrices with restricted entries and q -analogues of permutations. *J. Comb.*, 2(3):355–395, 2011. [arXiv:1011.4539](#), [doi:10.4310/JOC.2011.v2.n3.a2](#).
- [LM20] Joel Brewster Lewis and Alejandro H. Morales. Rook theory of the finite general linear group. *Exp. Math.*, 29(3):328–346, 2020. [arXiv:1707.08192](#), [doi:10.1080/10586458.2018.1470045](#).
- [Mar99] Martin Marjoram. Irreducible characters of small degree of the unitriangular group. *Irish Math. Soc. Bull.*, 42:21–31, 1999. [doi:10.33232/BIMS.0042.21.31](#).
- [OV15] E. A. O’Brien and C. Voll. Enumerating classes and characters of p -groups. *Trans. Amer. Math. Soc.*, 367(11):7775–7796, 2015. [arXiv:1203.3050](#), [doi:10.1090/tran/6276](#).
- [Ros22] Tobias Rossmann. Enumerating conjugacy classes of graphical groups over finite fields. *Bull. Lond. Math. Soc.*, 54(5):1923–1943, 2022. [arXiv:2107.05564](#), [doi:10.1112/blms.12665](#).
- [Ros25] Tobias Rossmann. On the enumeration of orbits of unipotent groups over finite fields. *Proc. Amer. Math. Soc.*, 153(2):479–495, 2025. [arXiv:2208.04646](#), [doi:10.1090/proc/17030](#).
- [RV24] Tobias Rossmann and Christopher Voll. Groups, graphs, and hypergraphs: average sizes of kernels of generic matrices with support constraints. *Mem. Amer. Math. Soc.*, 294(1465), 2024. [arXiv:1908.09589](#), [doi:10.1090/memo/1465](#).
- [Ste98] John R. Stembridge. Counting points on varieties over finite fields related to a conjecture of Kontsevich. *Ann. Comb.*, 2(4):365–385, 1998. [doi:10.1007/BF01608531](#).
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Release v1.8.0 (commit d371ee9); record OWR-8415341-012.