

A Negative Answer to a Question of Praeger and Seress on Minimal Subdegrees of Primitive Groups of Twisted Wreath Type

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Abstract

Let G be a finite primitive permutation group of twisted wreath type, with socle T^k and point stabiliser P , and let H be its primitive component: the group induced on one coordinate of the product decomposition Δ^k , where $|\Delta| = |T|$. Praeger and Seress (Oberwolfach Report 12/2005, on joint work with Giudici, Li and Trofimov) state that the least non-trivial subdegree satisfies $\text{MinSubDeg}(G) \leq k \cdot \text{MinSubDeg}(H)$, and they ask in their Question 4 (p. 692) whether this inequality is always strict. We give a negative answer to Question 4. Let $P = \text{PGL}(3, 4)$ act on the 21 points of the projective plane of order 4, let Q be the stabiliser of a point y , let $T = \text{PSL}(2, 4) \cong A_5$, and let Q act on T through its action on the quotient space $\mathbb{F}_4^3/y$. The twisted wreath product $G = T \text{ twr } P$, acting on its base group, is primitive of twisted wreath type of degree 60^{21} , with $k = 21$, $H = A_5 \times A_5$, $\text{MinSubDeg}(H) = 12$ and $\text{MinSubDeg}(G) = 252 = 21 \cdot 12$. By a reduction which is known in substance, and which we prove again, the last equality is a statement about the subgroups of $\text{PGL}(3, 4)$ of order greater than 240. We prove this statement in two ways: computer-assisted, by exhaustive exact computations with independently written programs; and by a written argument from the list of the maximal subgroups of $\text{PSL}(3, 4)$ and $\text{PGL}(3, 4)$ in the tables of Connor and Leemans. Computer-assisted, we also find equality for the top groups $\text{PGL}(3, 4)$ ($315 = 21 \cdot 15$) and $\text{PSL}(3, 5)$ ($465 = 31 \cdot 15$), where H is the full holomorph of A_5 , and strict inequality for $\text{PSL}(3, 4)$ and $\text{PGL}(3, 4)$. The full paper of Giudici, Li, Praeger, Seress and Trofimov (2006) and its companion paper were not accessible to us; we make no statement that depends on their content. No answer to Question 4 was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. We claim neither that the example is the smallest one nor a classification of the cases of equality, and the question remains open under additional hypotheses. This is an unrefereed note.

1 Introduction

1.1 The question

All groups in this note are finite. Let G be a transitive permutation group on a set Ω and let $\alpha \in \Omega$. A *suborbit* of G is an orbit Γ of the stabiliser G_α ; its size is a *subdegree*; both are *non-trivial* if $\Gamma \neq \{\alpha\}$. Following Praeger and Seress [PS05, p. 691], we write $\text{MinSubDeg}(G)$ for the minimum of the non-trivial subdegrees.

For a set Δ and an integer $k \geq 2$, the wreath product $\text{Sym}(\Delta) \wr S_k$ acts on Δ^k in *product action*: its elements are the permutations

$$(\delta_1, \dots, \delta_k) \longmapsto (\delta_{1\pi}^{h_1}, \dots, \delta_{k\pi}^{h_k}), \quad \pi \in S_k, \quad h_1, \dots, h_k \in \text{Sym}(\Delta). \quad (1)$$

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If $G \leq \text{Sym}(\Delta) \wr S_k$, the elements of G with $1\pi = 1$ form a subgroup, the stabiliser of the first coordinate, and $g \mapsto h_1$ maps it homomorphically onto a subgroup H of $\text{Sym}(\Delta)$, the group *induced by G on Δ* . The abstract [PS05, p. 691] considers primitive groups $G \leq H \wr S_k$ on Δ^k , where H is a primitive group on Δ which is induced by G , and calls H the *primitive component* of G relative to the Cartesian decomposition Δ^k ; we read this in the standard way just described. Theorem 1 of [PS05], quoted there from [GLP⁺06, Theorem 1.1], says that then $\text{MinSubDeg}(G) = k \cdot \text{MinSubDeg}(H)$, provided that $\text{Soc}(G) = \text{Soc}(H \wr S_k)$ is non-abelian; here Soc denotes the socle, the product of the minimal normal subgroups.

A primitive permutation group is of *twisted wreath type* (O’Nan–Scott type TW) if it has a unique minimal normal subgroup and this subgroup is non-abelian, non-simple and regular; see for instance [Faw22, §1]. For such a group G the abstract [PS05, p. 692] records the following: $G \leq H \wr S_k$ acting on Δ^k , with primitive component H and $k \geq 2$; $\text{Soc}(G) = T^k$ for a non-abelian simple group T , while $\text{Soc}(H) \cong T \times T$, so that Theorem 1 does not apply; the socle of G is regular on Δ^k ; and $G = \text{Soc}(G) \rtimes P$ for a transitive subgroup P of S_k . For a group P let $\text{MinDeg}(P)$ be the least n such that P acts faithfully and transitively on a set of size n .

Theorem 3 of [PS05]. *Let G be a finite primitive permutation group of type TW, and suppose that $G = \text{Soc}(G) \rtimes P \leq H \wr S_k$ acting on a set Δ^k ($k \geq 2$) with primitive component $H \leq \text{Sym}(\Delta)$. Then*

$$\max\{\text{MinSubDeg}(H), \text{MinDeg}(P)\} \leq \text{MinSubDeg}(G) \leq k \cdot \text{MinSubDeg}(H). \quad (2)$$

The theorem adds that the lower bound is attained by infinitely many groups. The authors then write that they wonder whether a group of type TW can ever achieve the bound of Theorem 1, and ask:

Question 4 of [PS05, p. 692]. *Is it true that, for all finite primitive permutation groups $G = \text{Soc}(G) \rtimes P \leq H \wr S_k$ of type TW, with P and H as in Theorem 3, “the strict inequality $\text{MinSubDeg}(G) < k \cdot \text{MinSubDeg}(H)$ holds”?*

The question is record OWR-793-006 of the UnsolvedMath collection [ula26], listed as open.

We read the question as follows: k is the number of simple factors of $\text{Soc}(G) = T^k$ (it is also the degree of P and the exponent in Δ^k , and $|\Delta| = |T|$ because the socle is regular); H is the group induced on one coordinate by the stabiliser in G of this coordinate; G acts on Δ^k with a regular socle, which for a twisted wreath product is its action on the base group (Section 2); and the non-strict inequality is Theorem 3 of [PS05]. A negative answer is therefore a primitive group of type TW with $\text{MinSubDeg}(G) = k \cdot \text{MinSubDeg}(H)$.

1.2 Results

The twisted wreath product $T \text{twr}_\varphi P$ of two groups with respect to a homomorphism $\varphi: Q \rightarrow \text{Aut}(T)$ of a subgroup $Q \leq P$, and its action on the base group, are recalled in Section 2. Let $V = \mathbb{F}_4^3$, and let $\text{PG}(2, 4)$ be the projective plane of V , with 21 points (the subspaces of dimension 1) and 21 lines (the subspaces of dimension 2); five lines pass through every point.

Theorem 1.1. *Let $P = \text{PGL}(3, 4)$ act on the 21 points of $\text{PG}(2, 4)$, let y be a point and $Q = P_y$ its stabiliser. Let $T = \text{PGL}(V/y) = \text{PSL}(2, 4) \cong A_5$, and let $\varphi: Q \rightarrow \text{Aut}(T)$ map $q \in Q$ to the conjugation by the transformation which q induces on the 2-dimensional space V/y . Let $G = T \text{twr}_\varphi P$ act on its base group, a set of size 60^{21} .*

- (a) *G is a primitive permutation group of type TW. Its socle is the base group, isomorphic to T^{21} , and its point stabiliser P permutes the 21 simple factors as it permutes the 21 points. Moreover $G = \text{Soc}(G) \rtimes P \leq H \wr S_{21}$ in product action on Δ^{21} , $\Delta = T$, with primitive component $H = T \rtimes \text{Inn}(T) \cong A_5 \times A_5$.*

(b) $\text{MinSubDeg}(H) = 12$. So $k = 21$ and $k \cdot \text{MinSubDeg}(H) = 252$.

(c) $\text{MinSubDeg}(G) = 252$.

Hence $\text{MinSubDeg}(G) = k \cdot \text{MinSubDeg}(H)$, and the answer to Question 4 of [PS05] is negative.

Parts (a) and (b) are proved in Section 3 without computation. By the reduction of Section 2, part (c) is equivalent to a statement about the group $\text{PGL}(3, 4)$ of order 60480: *if S is a subgroup of order greater than 240, then for every point d the stabiliser S_d induces a group of order at least 6 on the five lines through d* (Proposition 3.3). We prove this statement in two ways. The first proof is *computer-assisted*: exhaustive exact computations with seven programs, written at four independent stages and using four methods (Section 6); the programs and their recorded outputs are in the reproducibility package. The second is a *written argument* (Section 4) from the maximal subgroups of $\text{PSL}(3, 4)$ and $\text{PGL}(3, 4)$ listed in the table of Connor and Leemans [CL12]; no computation of ours enters it.

The same programs give further examples. For a group P with $\text{PSL}(3, q) \leq P \leq \text{PTL}(3, q)$ let $Q = P_y$ be the stabiliser of a point y , let $T = \text{PSL}(V/y) \cong \text{PSL}(2, q)$, and let $\varphi(q')$ be the conjugation by the semilinear transformation which $q' \in Q$ induces on V/y . Note that $\text{PSL}(2, 4) \cong \text{PSL}(2, 5) \cong A_5$.

Theorem 1.2 (computer-assisted). *For the five groups P of Table 1, the group $G = T \text{ twr}_\varphi P$ is primitive of type TW, with primitive component $H = T \rtimes \varphi(Q)$, and the values of $\text{MinSubDeg}(H)$ and $\text{MinSubDeg}(G)$ are those of the table. In particular $\text{MinSubDeg}(G) = k \cdot \text{MinSubDeg}(H)$ for $P = \text{PTL}(3, 4)$ and for $P = \text{PSL}(3, 5)$, and in these two cases $\varphi(Q) = \text{Aut}(T)$, so that H is the holomorph $T \rtimes \text{Aut}(T)$ of A_5 .*

P	$ P $	k	$\varphi(Q)$	M	M_G	$k \cdot \text{MinSubDeg}(H)$	$\text{MinSubDeg}(G)$	
$\text{PSL}(3, 4)$	20160	21	A_5	80	160	$21 \cdot 12 = 252$	126	strict
$\text{PSL}(3, 4)$	40320	21	S_5	128	320	$21 \cdot 15 = 315$	126	strict
$\text{PGL}(3, 4)$	60480	21	A_5	240	240	$21 \cdot 12 = 252$	252	equality
$\text{PTL}(3, 4)$	120960	21	S_5	384	384	$21 \cdot 15 = 315$	315	equality
$\text{PSL}(3, 5)$	372000	31	S_5	800	800	$31 \cdot 15 = 465$	465	equality

Table 1: The groups $A_5 \text{ twr}_\varphi P$, with P acting on the points of $\text{PG}(2, q)$. For M and M_G see (5); $k \cdot \text{MinSubDeg}(H) = |P|/M$ and $\text{MinSubDeg}(G) = |P|/M_G$.

Section 5 also contains computed values for five classical groups of type TW, where the inequality is strict (Table 3), and three sufficient conditions for strict inequality (Proposition 5.2).

1.3 What is known, what is new, and what remains open

The full paper was not read. The abstract [PS05] announces the full version [GLP⁺06] and cites a companion paper [GLP⁺07] on limit graphs. Neither was accessible to us. What we say about [GLP⁺06] is taken from the abstract [PS05], from the review of the paper in zbMATH (Zbl 1111.20003) and from the paper [CGM19], which quotes it; we make no statement that depends on its content, in particular none about what it proves on the attainment of the upper bound in (2). No answer to Question 4 was found in the literature accessible to us; see the paragraph “Scope and priority” at the end. A search that finds nothing is not a proof of novelty, and we claim no priority.

The reduction is known in substance. The lemmas of Section 2 are proved here to make the note self-contained. None of them is claimed as new; the sources are given in Remark 2.5.

New, to our knowledge, are the example $G = A_5 \text{ twr } \text{PGL}(3, 4)$ with $\text{MinSubDeg}(G) = 252 = 21 \cdot 12$ and the two further examples with equality, with top groups $\text{PTL}(3, 4)$ and $\text{PSL}(3, 5)$. In these two

the group $\varphi(Q)$ is all of $\text{Aut}(T)$, so that the primitive component is the full holomorph of A_5 , and the negative answer does not depend on the convention for the primitive component (Remark 5.1). We do not claim that $A_5 \text{ twr } \text{PGL}(3, 4)$ is the smallest primitive group of type TW with equality, and we do not classify the groups with equality. In all three examples φ has a non-trivial kernel; whether the strict inequality holds under additional hypotheses, for instance for all primitive groups of type TW with $\ker \varphi = 1$, remains open.

2 Twisted wreath products and the reduction to the top group

2.1 The construction

Twisted wreath products go back to B. H. Neumann (see [CGM19, §2]); we use the conventions of [CGM19, §2], [DGPS13, Example 4.4] and [Faw22, §4]. Groups act on the right, and $t^{ab} = (t^a)^b$. Let T be a group, P a group, $Q \leq P$, and $\varphi: Q \rightarrow \text{Aut}(T)$ a homomorphism. The *base group* is

$$B = \{f: P \rightarrow T \mid f(pq) = f(p)^{\varphi(q)} \text{ for all } p \in P, q \in Q\}$$

with pointwise multiplication. For $f \in B$ and $x \in P$ let $f^x(p) := f(xp)$. Then $f^x \in B$, because $f^x(pq) = f(xpq) = f(xp)^{\varphi(q)}$, and $(f^x)^{x'} = f^{xx'}$; so P acts on B on the right, by automorphisms. The *twisted wreath product* is the semidirect product $G = T \text{ twr}_\varphi P := B \rtimes P$. It acts on the set $\Omega = B$ by

$$\omega^{bx} := (\omega b)^x \quad (\omega, b \in B, x \in P). \quad (3)$$

In this action B is regular, and the stabiliser of the point $1 \in B$ is P , the *top group*.

Let $k = |P : Q|$. For a left coset cQ let T_c be the set of all $f \in B$ which are 1 outside cQ . A function on a left transversal of Q extends uniquely to an element of B . Hence $f \mapsto f(c)$ is an isomorphism $T_c \rightarrow T$, the group B is the direct product of the k subgroups T_c , and $B \cong T^k$. From $f^x(p) = f(xp)$ we get

$$(T_c)^x = T_{x^{-1}c} \quad (x \in P). \quad (4)$$

If $f \in B$ and $q \in Q$, then $f^q(1) = f(q) = f(1)^{\varphi(q)}$; so Q normalises T_1 and acts on $T_1 \cong T$ via φ .

For $t \in T$ and $b \in B$ let

$$\begin{aligned} Q_t &:= \{q \in Q : t^{\varphi(q)} = t\}, & C_P(b) &:= \{x \in P : b^x = b\}, \\ M &:= \max\{|Q_t| : t \in T, t \neq 1\}, & M_G &:= \max\{|C_P(b)| : b \in B, b \neq 1\}. \end{aligned} \quad (5)$$

2.2 Subdegrees and the primitive component

Lemma 2.1. *Let T be a non-abelian simple group and $\text{Inn}(T) \leq \varphi(Q)$.*

- (a) *The non-trivial subdegrees of G on B are the numbers $|P : C_P(b)|$, $b \in B \setminus \{1\}$. Hence $\text{MinSubDeg}(G) = |P|/M_G$.*
- (b) *Let $c_1 = 1, c_2, \dots, c_k$ be a left transversal of Q in P , and identify B with Δ^k , $\Delta := T$, by $f \mapsto (f(c_1), \dots, f(c_k))$. Then $G \leq H \wr S_k$ in product action on Δ^k , where*

$$H := \{t \mapsto (ts)^a : s \in T, a \in \varphi(Q)\} = T \rtimes \varphi(Q) \leq \text{Sym}(T)$$

is the group induced by G on Δ . The group H is primitive, $\text{Soc}(H) \cong T \times T$, the stabiliser of $1 \in T$ in H is $\varphi(Q)$, and the non-trivial subdegrees of H are the numbers $|Q : Q_t|$, $t \in T \setminus \{1\}$. Hence

$$\text{MinSubDeg}(H) = |Q|/M \quad \text{and} \quad k \cdot \text{MinSubDeg}(H) = |P|/M.$$

(c) $M_G \geq M$, that is, $\text{MinSubDeg}(G) \leq k \cdot \text{MinSubDeg}(H)$.

Proof. (a) By (3) the stabiliser P of $1 \in B$ acts on B by $f \mapsto f^x$; so the suborbits are its orbits on B .

(b) Let $\omega, b \in B$, $x \in P$ and $1 \leq i \leq k$. Write $xc_i = c_j q$ with $q \in Q$; then $i \mapsto j$ is a permutation π_x of $\{1, \dots, k\}$, and

$$(\omega b)^x(c_i) = (\omega b)(c_j q) = (\omega(c_j) b(c_j))^{\varphi(q)}.$$

So bx acts on Δ^k as in (1), with $\pi = \pi_x$ and $h_i: t \mapsto (tb(c_j))^{\varphi(q)}$, and all h_i lie in H . Hence $G \leq H \wr S_k$. The element bx fixes the first coordinate if and only if $xQ = Q$, that is $x \in Q$, and then h_1 is the map $t \mapsto (tb(1))^{\varphi(x)}$. Since $b(1)$ runs through T , the group induced by G on Δ is H .

A map $t \mapsto (ts)^a$ fixes 1 if and only if $s = 1$. So the stabiliser of 1 in H is $\varphi(Q)$, the group R of right multiplications is a regular normal subgroup of H , and $H = R \rtimes \varphi(Q)$. The subgroups of H containing $\varphi(Q)$ correspond to the $\varphi(Q)$ -invariant subgroups of T . Such a subgroup is invariant under $\text{Inn}(T)$, hence normal in T , hence 1 or T . So $\varphi(Q)$ is a maximal subgroup of H , and H is primitive. For $g \in T$ the left multiplication $t \mapsto g^{-1}t$ is the composition of the inner automorphism $t \mapsto g^{-1}tg$ and of the right multiplication by g^{-1} ; so H contains the group L of left multiplications. The groups R and L are normal in H , isomorphic to T , and L is the centraliser of R in $\text{Sym}(T)$. A minimal normal subgroup of H different from R meets R trivially, centralises R , and so equals L . Hence $\text{Soc}(H) = R \times L \cong T \times T$. The orbit of t under $\varphi(Q)$ has length $|Q : Q_t|$, and $k|Q|/M = |P|/M$.

(c) Let $t \neq 1$, and let $b \in T_1$ with $b(1) = t$. If $x \in C_P(b)$, then $1 \neq b \in T_1 \cap (T_1)^x$, so $x \in Q$ by (4). For $x \in Q$ the elements b and b^x of T_1 are equal if and only if $b^x(1) = b(1)$, and $b^x(1) = t^{\varphi(x)}$. Hence $C_P(b) = Q_t$, and $M_G \geq M$. $\square$

Definition 2.2. A subgroup S of P is *good* if $S \cap Q \leq Q_t$ for some $t \in T \setminus \{1\}$, that is, if $\varphi(S \cap Q)$ fixes a non-identity element of T .

Lemma 2.3 (criterion). *Let T , P , Q and φ be as in the construction above.*

(a) *Let $b \in B \setminus \{1\}$, let $p_0 \in P$ with $t := b(p_0) \neq 1$, and let $S := p_0^{-1}C_P(b)p_0$. Then $S \cap Q \leq Q_t$; so S is a good subgroup of order $|C_P(b)|$.*

(b) *Let $S \leq P$ and $S \cap Q \leq Q_t$ with $t \neq 1$. Then*

$$b(sq) := t^{\varphi(q)} \quad (s \in S, q \in Q), \quad b(p) := 1 \quad (p \in P \setminus SQ)$$

is a well-defined element of $B \setminus \{1\}$, and $S \leq C_P(b)$.

(c) $M_G = \max\{|S| : S \leq P \text{ good}\}$. *Under the hypotheses of Lemma 2.1, $\text{MinSubDeg}(G) = k \cdot \text{MinSubDeg}(H)$ if and only if no good subgroup of P has order greater than M .*

Proof. (a) Let $c \in S \cap Q$, and write $c = p_0^{-1}sp_0$ with $s \in C_P(b)$. Then $p_0c = sp_0$, and, since $c \in Q$,

$$t^{\varphi(c)} = b(p_0)^{\varphi(c)} = b(p_0c) = b(sp_0) = b^s(p_0) = b(p_0) = t.$$

(b) If $sq = s_1q_1$ with $s, s_1 \in S$ and $q, q_1 \in Q$, then $q_1q^{-1} = s_1^{-1}s \in S \cap Q \leq Q_t$, and therefore $t^{\varphi(q_1)} = (t^{\varphi(q_1q^{-1})})^{\varphi(q)} = t^{\varphi(q)}$. So b is well defined. The set SQ is a union of left cosets of Q , and $b(sq \cdot q') = t^{\varphi(q)\varphi(q')} = b(sq)^{\varphi(q')}$; hence $b \in B$, and $b(1) = t \neq 1$. The set SQ is invariant under left multiplication by S , and $b(s'sq) = t^{\varphi(q)} = b(sq)$ for $s' \in S$; hence $b^{s'} = b$.

(c) By (a) every number $|C_P(b)|$, $b \neq 1$, is the order of a good subgroup, and by (b) every good subgroup lies in some $C_P(b)$ with $b \neq 1$. The second statement follows with Lemma 2.1. $\square$

For $g \in P$ we have $C_P(b^g) = g^{-1}C_P(b)g$. So M_G is also the largest order of a subgroup of P which has a good conjugate; a conjugate of a good subgroup need not be good.

2.3 Primitivity

Lemma 2.4 (primitivity). *Let T be a non-abelian simple group, let Q be a maximal subgroup of P with $\text{core}_P(Q) = 1$, and let $\varphi: Q \rightarrow \text{Aut}(T)$ be a homomorphism with $\text{Inn}(T) \leq \varphi(Q)$ which does not extend to a homomorphism $P \rightarrow \text{Aut}(T)$. Then $G = T \text{ twr}_\varphi P$ acts faithfully and primitively on B , and $B \cong T^k$, $k = |P : Q| \geq 2$, is its unique minimal normal subgroup. So G is a primitive permutation group of type TW with socle B and point stabiliser P , and P permutes the k simple factors T_c of the socle faithfully and transitively, as it permutes the left cosets of Q .*

Proof. For $f \in B$ let $\text{supp}(f)$ be the set of left cosets cQ with $f(c) \neq 1$; then $\text{supp}(f^x) = x^{-1} \text{supp}(f)$. For a left coset cQ let $\pi_c: B \rightarrow T$, $f \mapsto f(c)$; this is a homomorphism.

Step 1: the only P -invariant subgroups of B are 1 and B . Let $X \neq 1$ be a P -invariant subgroup. For $f \in X$ and $q \in Q$ we have $\pi_1(f^q) = f(q) = \pi_1(f)^{\varphi(q)}$. So $\pi_1(X)$ is $\varphi(Q)$ -invariant, hence normal in T . Moreover $\pi_c(f) = \pi_1(f^c)$, so $\pi_c(X) = \pi_1(X)$. As $X \neq 1$, we get $\pi_c(X) = T$ for every c .

For a set J of left cosets let $X_J := \{f \in X : \text{supp}(f) \subseteq J\}$, a normal subgroup of X , and let J be minimal among the non-empty sets with $X_J \neq 1$. Let $cQ \in J$. Then $\pi_c(X_J)$ is a normal subgroup of $\pi_c(X) = T$, and it is not trivial, because otherwise $X_J = X_{J \setminus \{cQ\}}$, against the minimality of J . So $\pi_c(X_J) = T$; the kernel of π_c on X_J is $X_{J \setminus \{cQ\}} = 1$. Thus $\pi_c: X_J \rightarrow T$ is an isomorphism for every $cQ \in J$. Since $(X_J)^x = X_{x^{-1}J}$, every translate $x^{-1}J$ is again minimal. Let $J \neq J'$ be minimal sets with a common element cQ . Then $[X_J, X_{J'}] \leq X_{J \cap J'}$ and $\pi_c([X_J, X_{J'}]) = [T, T] = T$, so $X_{J \cap J'} \neq 1$, although $J \cap J'$ is a non-empty proper subset of J or of J' : a contradiction. Hence the translates of J are pairwise equal or disjoint. Their union is P -invariant, so they form a P -invariant partition of the set of left cosets of Q . As Q is maximal, P is primitive on this set, and $|J| = 1$ or J is the whole set. If $J = \{cQ\}$, then $X_J = T_c \leq X$, and $X = B$ by (4). If J is the whole set, then $X = X_J$, the map $\pi_1: X \rightarrow T$ is an isomorphism, and

$$\psi(x): t \mapsto \pi_1((\pi_1^{-1}(t))^x) \quad (x \in P)$$

defines a homomorphism $\psi: P \rightarrow \text{Aut}(T)$ with $\psi(q) = \varphi(q)$ for $q \in Q$, by the first formula of this step. This is excluded by hypothesis. Hence $X = B$.

Step 2: P is a maximal subgroup of G . If $P \leq Y \leq G$, then $Y = (Y \cap B)P$, and $Y \cap B$ is P -invariant. By Step 1, $Y = P$ or $Y = G$. So G is primitive on the cosets of P , that is, on B .

Step 3: $C_G(B) = 1$. Let $bx \in C_G(B)$ with $b \in B$, $x \in P$. Then x induces on B an inner automorphism of B . Inner automorphisms of $B = \prod_c T_c$ normalise every T_c . By (4), $x^{-1}cQ = cQ$ for all c , so $x \in \text{core}_P(Q) = 1$, and then b lies in the centre of B , which is trivial. In particular the action on B is faithful: its kernel N is normal in G and contained in P , so $[N, B] \leq N \cap B = 1$ and $N \leq C_G(B) = 1$.

Step 4. A normal subgroup of G contained in B is P -invariant; by Step 1, B is a minimal normal subgroup. Another minimal normal subgroup N would satisfy $N \cap B = 1$, so $N \leq C_G(B) = 1$. So B is the unique minimal normal subgroup; it is regular, non-abelian and not simple. By (4) the group P permutes the factors T_c as it permutes the left cosets of Q , faithfully because $\text{core}_P(Q) = 1$. $\square$

Remark 2.5 (sources). None of the lemmas of this section is new. Lemma 2.1(a) is [CGM19, Lemma 2.2], and Lemma 2.1(c) is the upper bound of Theorem 3 of [PS05]. The function of Lemma 2.3(b) is the one used in [DGPS13, Example 4.4] and in [CGM19, Lemma 2.6]; the authors of [CGM19] describe theirs as very similar to the functions used in [DGPS13] and in [GLP⁺06], and the abstract [PS05, p. 692] says that $\text{MinSubDeg}(G)$ is always attained on a suborbit containing one of certain special points constructed in [GLP⁺06]. For primitivity, [CGM19, Theorem 2.3] and [DGPS13, Example 4.4] quote a criterion from the book of Dixon and Mortimer [DM96] (as Lemma 4.7A and as Lemma 4.7B, respectively): P is a primitive permutation group with point

stabiliser Q , $\text{Inn}(T) \leq \varphi(Q)$, and $\varphi(Q)$ is not a homomorphic image of P . These hypotheses imply those of Lemma 2.4. Indeed, let $\psi: P \rightarrow \text{Aut}(T)$ extend φ . If $\ker \psi \neq 1$, then $\ker \psi$ is transitive on the cosets of Q , so $P = Q \ker \psi$ and $\psi(P) = \varphi(Q)$ is a homomorphic image of P . If ψ is injective, then $\psi^{-1}(\text{Inn}(T))$ is a non-trivial normal subgroup of P contained in Q , against $\text{core}_P(Q) = 1$. Primitive twisted wreath products are characterised by Baddeley [Bad93, Theorem 3.5], in the form stated in [Faw22, Theorem 4.6]. We did not consult [DM96, Bad93]; they are cited through [CGM19, DGPS13, Faw22].

3 The example

Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ with $\omega^2 = \omega + 1$, and let $V = \mathbb{F}_4^3$ be the space of row vectors, on which matrices act on the right. The group $P = \text{PGL}(3, 4) = \text{GL}(3, 4)/\mathbb{F}_4^\times$ has order $60480 = 2^6 \cdot 3^3 \cdot 5 \cdot 7$ and acts faithfully and 2-transitively on the 21 points of $\text{PG}(2, 4)$; its subgroup $\text{PSL}(3, 4)$, of index 3, is simple. For a point d let $\mathcal{L}_d$ be the set of the five lines through d , the points of the projective line of V/d . The stabiliser P_d , of order 2880, acts on V/d , giving a surjective homomorphism

$$P_d \longrightarrow \text{PGL}(V/d) = \text{PGL}(2, 4) = \text{PSL}(2, 4).$$

The group $\text{PGL}(V/d)$ has order 60 and acts faithfully on the five elements of $\mathcal{L}_d$; so it is the alternating group of $\mathcal{L}_d$, and $\text{PGL}(V/d) \cong A_5$. For a subgroup $S \leq P$ we write

$$S^{(d)} := \text{the image of } S_d = S \cap P_d \text{ in } \text{PGL}(V/d),$$

the group induced by S_d on $\mathcal{L}_d$. Let K_d be the kernel of $P_d \rightarrow \text{PGL}(V/d)$; it has order $2880/60 = 48$.

Lemma 3.1. *K_d is the group of all 48 perspectivities with centre d , that is, of all elements of P which fix every line through d . It consists of the 16 elations with centre d (the identity included), which form the elementary abelian normal subgroup $U_d = K_d \cap \text{PSL}(3, 4)$, and of the 32 homologies with centre d ; each of the 16 lines not through d is the axis of two of these homologies.*

Proof. Let $d = \langle e_1 \rangle$. Each element of K_d has a unique representative inducing the identity on V/d , namely

$$m_{a,b}: \quad e_1 \mapsto ae_1, \quad e_i \mapsto e_i + b_i e_1 \quad (i = 2, 3), \quad a \in \mathbb{F}_4^\times, \quad b = (b_2, b_3) \in \mathbb{F}_4^2,$$

and these 48 matrices represent the elements of K_d . A vector $v = (v_1, v_2, v_3)$ is fixed by $m_{a,b}$ if and only if $(a - 1)v_1 + b_2 v_2 + b_3 v_3 = 0$. For $a = 1$ the fixed vectors form a line through d (or all of V): $m_{1,b}$ is an elation with centre d , and $b \mapsto m_{1,b}$ is an isomorphism of $(\mathbb{F}_4^2, +)$ onto U_d . For $a \neq 1$ they form a line not through d : $m_{a,b}$ is a homology with centre d and this axis, and for each of the two values of a the 16 vectors b give the 16 lines not through d . The determinant of $m_{a,b}$ is a . All scalars of $\text{GL}(3, 4)$ have determinant 1, so $m_{a,b}$ represents an element of $\text{PSL}(3, 4)$ if and only if $a = 1$. $\square$

Now fix a point y , let $Q = P_y$ and $T = \text{PGL}(V/y)$, let $\bar{\varphi}: Q \rightarrow T$ be the homomorphism above, and let $\varphi(q) \in \text{Aut}(T)$ be the conjugation by $\bar{\varphi}(q)$: $t^{\varphi(q)} = \bar{\varphi}(q)^{-1} t \bar{\varphi}(q)$. Then $\varphi(Q) = \text{Inn}(T)$ and $\ker \varphi = K_y$. For $t \in T$ we have $Q_t = \bar{\varphi}^{-1}(C_T(t))$, a group of order $48|C_T(t)|$. The centralisers in A_5 of elements of order 2, 3, 5 have order 4, 3, 5. Hence

$$|Q_t| = 192, 144, 240 \quad \text{for } t \text{ of order } 2, 3, 5; \quad M = 240. \quad (6)$$

Let $G = T \text{twr}_\varphi P$; its degree is $|T|^{21} = 60^{21}$.

Lemma 3.2. *A subgroup S of P is good (Definition 2.2) if and only if $|S^{(y)}| \leq 5$. More generally, if $g \in P$ and $d = y^{g^{-1}}$, then $g^{-1}Sg$ is good if and only if $|S^{(d)}| \leq 5$.*

Proof. The group $S \cap Q$ fixes an element t of T under φ if and only if $\bar{\varphi}(S \cap Q) = S^{(y)}$ centralises t . The subgroups of A_5 have order 1, 2, 3, 4, 5, 6, 10, 12 or 60. Those of order at most 5 are abelian, so they centralise a non-identity element of A_5 . Those of order 6, 10, 12, 60 are isomorphic to S_3 , D_{10} , A_4 , A_5 , and their centralisers in A_5 are trivial. This proves the first statement. For the second, g maps d to y and $\mathcal{L}_d$ onto $\mathcal{L}_y$, and $(g^{-1}Sg)_y = g^{-1}S_dg$; so $(g^{-1}Sg)^{(y)} \cong S^{(d)}$. $\square$

Proof of Theorem 1.1(a) and (b). We verify the hypotheses of Lemma 2.4. The group $T \cong A_5$ is non-abelian and simple, and $\varphi(Q) = \text{Inn}(T)$. As P is 2-transitive on the points, Q is a maximal subgroup, and $\text{core}_P(Q) = 1$ because P acts faithfully. Suppose that $\psi: P \rightarrow \text{Aut}(T) \cong S_5$ extends φ . Then $\psi(P) \supseteq \text{Inn}(T)$, so $N := \ker \psi$ has index 60 or 120 in P and order 1008 or 504. But $N \cap \text{PSL}(3, 4)$ is a normal subgroup of the simple group $\text{PSL}(3, 4)$: either $N \supseteq \text{PSL}(3, 4)$ and $|P : N| \leq 3$, or $N \cap \text{PSL}(3, 4) = 1$ and $|N| \leq 3$. This is a contradiction, so φ does not extend. By Lemma 2.4, G is primitive of type TW, with socle $B \cong T^{21}$ and point stabiliser P , and P permutes the simple factors as it permutes the left cosets of Q , that is, the 21 points. By Lemma 2.1(b), $G \leq H \wr S_{21}$ in product action on T^{21} , with primitive component $H = T \rtimes \varphi(Q) = T \rtimes \text{Inn}(T)$. This group is generated by the right and the left multiplications of T , so $H = \text{Soc}(H) \cong T \times T$. This proves (a). By Lemma 2.1(b) and (6), $\text{MinSubDeg}(H) = |Q|/M = 2880/240 = 12$; indeed the non-trivial subdegrees of H are the sizes 12, 12, 15, 20 of the non-trivial conjugacy classes of A_5 . This proves (b). $\square$

Proposition 3.3. *Let $S \leq P = \text{PGL}(3, 4)$ with $|S| > 240$. Then $|S^{(d)}| \geq 6$ for every point d .*

Proposition 3.3 is proved in Section 4 by a written argument, and, independently of that argument, by the computations of Section 6.

Proof of Theorem 1.1(c). By Lemma 2.1(a) and Lemma 2.3(c), $\text{MinSubDeg}(G) = |P|/M_G$, where M_G is the largest order of a good subgroup of P . If $t \in T$ has order 5, then Q_t is a good subgroup of order 240 by (6). By Lemma 3.2 and Proposition 3.3 (for $d = y$), no good subgroup has order greater than 240. Hence $M_G = 240 = M$ and $\text{MinSubDeg}(G) = 60480/240 = 252$. $\square$

4 The subgroups of order greater than 240: a written argument

In this section we prove Proposition 3.3 without a computation of our own. The proof rests on three facts, which we take from the table [CL12] of the subgroup lattice of $\text{PGL}_3(4)$ in the atlas of Connor and Leemans [CL15]. The table lists the 100 conjugacy classes of subgroups (Nr. 1–100) with their orders, numbers of conjugates, structure and maximal subgroups.

- (F1) The maximal subgroups of $P = \text{PGL}(3, 4)$ are $\text{PSL}(3, 4)$ (Nr. 2), the subgroups of order 2880 (Nr. 3 and 4), and subgroups of order 216 (Nr. 15) and 63 (Nr. 34).
- (F2) The maximal subgroups of $\text{PSL}(3, 4)$ have order 960 (Nr. 5 and 6), 360 (Nr. 10), 168 (Nr. 21) and 72 (Nr. 32); those of order 360 are isomorphic to A_6 .
- (F3) P has exactly two conjugacy classes of subgroups of order 2880, and exactly two of order 960; each consists of 21 subgroups.

The stabilisers P_x of the points form a conjugacy class of 21 subgroups of order 2880, and so do the stabilisers P_l of the lines. A line stabiliser fixes no point. By (F3), every subgroup of order 2880 is therefore a point stabiliser or a line stabiliser. As $\text{PSL}(3, 4)$ is transitive on the points and on the lines, the groups $P_x \cap \text{PSL}(3, 4)$ and the groups $P_l \cap \text{PSL}(3, 4)$ form two classes of 21 subgroups of order 960, and the latter fix no point. By (F3), every subgroup of order 960 is one of these.

Lemma 4.1. *Let $S \leq P$ with $|S| > 240$. Then $S = P$, or $S = \text{PSL}(3, 4)$, or $S \cong A_6$ and $S \leq \text{PSL}(3, 4)$, or S lies in the stabiliser of a point or of a line.*

Proof. Let $S \neq P$. Then S lies in a maximal subgroup X of P with $|X| > 240$. By (F1), X is a point stabiliser or a line stabiliser, or $X = \text{PSL}(3, 4)$. In the last case let $S \neq X$. Then S lies in a maximal subgroup Y of $\text{PSL}(3, 4)$ with $|Y| > 240$. By (F2), $|Y| = 960$, and then Y lies in a point stabiliser or a line stabiliser; or $|Y| = 360$, $Y \cong A_6$, and $S = Y$ because $|Y : S| < 2$. $\square$

Fix a decomposition $V = \langle e \rangle \oplus W$ with $\dim W = 2$, let $x = \langle e \rangle$, and let l be the line of W . Every element of P_x has a unique representative g with $eg = e$, and every element of P_l has a unique representative g with $eg \in e + W$. Let

$$\begin{aligned} L &:= \{g : eg = e, Wg = W\} \cong \text{GL}(W), & \text{the stabiliser of } x \text{ and } l; \\ U_x &:= \{u_\lambda : e \mapsto e, w \mapsto w + \lambda(w)e \mid \lambda \in W^*\}, & \text{the elations with centre } x; \\ U_l &:= \{u^v : e \mapsto e + v, w \mapsto w \mid v \in W\}, & \text{the elations with axis } l. \end{aligned}$$

Then $P_x = U_x \rtimes L$ and $P_l = U_l \rtimes L$, and for $h \in L$

$$h^{-1}u_\lambda h = u_{\lambda \circ h^{-1}}, \quad h^{-1}u^v h = u^{vh}. \quad (7)$$

Let $Z \leq L$ be the group of order 3 of the elements acting on W as scalars, and let $L_0 \cong \text{SL}(W) \cong A_5$ be the group of the elements of determinant 1 on W . Then $L = Z \times L_0$, because $c \mapsto c^2$ is a bijection of $\mathbb{F}_4^\times$. So both groups have the form $X = E \rtimes L$, where E ($= U_x$ or U_l) is elementary abelian of order 16, and by (7) E is a 2-dimensional vector space over $\mathbb{F}_4$ (W^* or W) on which Z acts by scalar multiplications and L_0 acts transitively on the 15 non-zero vectors.

Lemma 4.2. *Let $X = E \rtimes L$ be P_x or P_l , and let $S \leq X$ with $|S| > 240$. Then $E \leq S$ and $S = E \rtimes R$, where $R := S \cap L$ is one of the groups*

$$L, \quad L_0, \quad Z \times A_0 \quad \text{with } A_0 \leq L_0, A_0 \cong A_4, D_{10} \text{ or } S_3,$$

of order 180, 60, 36, 30, 18. In particular $Z \leq R$ or $R = L_0$.

Proof. Let $\bar{S}$ be the image of S in $X/E \cong L$. From $|S| = |S \cap E| |\bar{S}| > 240$ we get $|\bar{S}| > 15$. If $\bar{S} \cap Z = 1$, then $\bar{S}$ is isomorphic to its projection to $L_0 \cong A_5$, which has order greater than 15 and is therefore L_0 ; so $\bar{S}$ is the graph of a homomorphism $L_0 \rightarrow Z$, which is trivial, and $\bar{S} = L_0$. If $Z \leq \bar{S}$, then $\bar{S} = Z \times A_0$ with $A_0 = \bar{S} \cap L_0$ of order greater than 5, so $A_0 = L_0$ or $A_0 \cong A_4, D_{10}$ or S_3 .

The group $S \cap E$ is normalised by S , hence invariant under $\bar{S}$. If $Z \leq \bar{S}$, it is an $\mathbb{F}_4$ -subspace of E . It is not trivial, because $|\bar{S}| \leq 180$. If it had order 4, then $|\bar{S}| > 60$, so $\bar{S} = L$, which leaves no 1-dimensional subspace invariant. So $S \cap E = E$. If $\bar{S} = L_0$, then $S \cap E$ is invariant under a group which is transitive on the non-zero vectors, and it is not trivial because $|S| > 60$; again $S \cap E = E$. Finally $S = E(S \cap L)$ and $S \cap L \cong \bar{S}$. $\square$

Proof of Proposition 3.3. Let $S \leq P$ with $|S| > 240$, and let d be a point. We go through the cases of Lemma 4.1. Recall that $\text{PGL}(V/d)$ acts on $\mathcal{L}_d$ as the alternating group; the stabiliser in it of a line $n \in \mathcal{L}_d$ is isomorphic to A_4 , and its normal subgroup of order 4 consists of the elations of the projective line $\mathcal{L}_d$ with centre n .

Case $S = P$ or $S = \text{PSL}(3, 4)$. The stabiliser of d in $\text{PSL}(3, 4)$ contains all elements which fix d , preserve a 2-dimensional complement W' of d , and have determinant 1 on W' ; they induce $\text{PSL}(V/d) = \text{PGL}(V/d)$. So $|S^{(d)}| = 60$.

order	conjugates	description	orbits on the points: (length, $ S_d $, $ S^{(d)} $)
60480	1	P	(21, 2880, 60)
20160	1	$\text{PSL}(3, 4)$	(21, 960, 60)
2880	21	$P_x = U_x \rtimes L$	(1, 2880, 60), (20, 144, 12)
2880	21	$P_l = U_l \rtimes L$	(5, 576, 12), (16, 180, 60)
960	21	$U_x \rtimes L_0$	(1, 960, 60), (20, 48, 12)
960	21	$U_l \rtimes L_0$	(5, 192, 12), (16, 60, 60)
576	105	$U_x \rtimes (Z \times A_4)$, a flag stabiliser	(1, 576, 12), (4, 144, 12), (16, 36, 12)
480	126	$U_x \rtimes (Z \times D_{10})$	(1, 480, 10), (20, 24, 12)
480	126	$U_l \rtimes (Z \times D_{10})$	(5, 96, 12), (16, 30, 10)
360	168	A_6	(6, 60, 60), (15, 24, 6)
288	210	$U_x \rtimes (Z \times S_3)$	(1, 288, 6), (8, 36, 12), (12, 24, 12)
288	210	$U_l \rtimes (Z \times S_3)$	(2, 144, 12), (3, 96, 12), (16, 18, 6)
240	126	$U_x \rtimes (Z \times C_5)$, conjugate to Q_t , t of order 5	(1, 240, 5), (20, 12, 12)
240	126	$U_l \rtimes (Z \times C_5) = U_l \rtimes C_{15}$	(5, 48, 12), (16, 15, 5)

Table 2: The 14 conjugacy classes of subgroups S of $\text{PGL}(3, 4)$ of order at least 240. By Lemma 3.2, S has a good conjugate if and only if $|S^{(d)}| \leq 5$ for some d : only in the last two rows.

Case $S \cong A_6$, $S \leq \text{PSL}(3, 4)$. The kernel of the action of S_d on $\mathcal{L}_d$ is $S_d \cap K_d \leq K_d \cap \text{PSL}(3, 4) = U_d$ (Lemma 3.1). So it is an elementary abelian 2-subgroup of S . The Sylow 2-subgroups of A_6 are dihedral of order 8; hence this kernel has order at most 4. The length of an orbit of S on the points divides 360, so it is not 19 or 21. If it were 18 or 20, then S would permute the remaining three points or fix the remaining point; as A_6 has no subgroup of index 2 or 3, it would fix a point x' and lie in $P_{x'} \cap \text{PSL}(3, 4)$, a group of order 960, although 360 does not divide 960. Hence all orbits of S have length at most 17, so $|S_d| \geq 360/17 > 21$ and $|S^{(d)}| \geq |S_d|/4 > 5$.

Case $S \leq P_l$ for a line l . We use the notation above; by Lemma 4.2, $S = U_l \rtimes R$.

Let first $d \notin l$. The group U_l is regular on the 16 points off l ; so we may assume that $d = x$. Then $S_x = R$. Every line through x meets l in one point, so R acts on $\mathcal{L}_x$ as on the five points of l , and $S^{(x)} \cong RZ/Z$ has order 60, 60, 12, 10 or 6.

Let now $d \in l$, say $d = \langle w_0 \rangle$, and let w_0, w_1 be a basis of W . Then S_d fixes the line $l \in \mathcal{L}_d$, so $S^{(d)}$ lies in the stabiliser of l in $\text{PGL}(V/d)$, a group isomorphic to A_4 . The group $U_l \leq S_d$ induces on $V/d = \langle \bar{w}_1, \bar{e} \rangle$ the maps $\bar{w}_1 \mapsto \bar{w}_1$, $\bar{e} \mapsto \bar{e} + c\bar{w}_1$ ($c \in \mathbb{F}_4$), that is, the four elations with centre l . Moreover R contains an element h with $w_0h \in \langle w_0 \rangle$ and $w_1h \in bw_1 + \langle w_0 \rangle$ for some $b \neq 1$: if $Z \leq R$, take a non-trivial element of Z , which fixes every point of l ; if $R = L_0$, take the element with $w_0h = b^{-1}w_0$, $w_1h = bw_1$. This h lies in S_d and induces on V/d the map $\bar{w}_1 \mapsto b\bar{w}_1$, $\bar{e} \mapsto \bar{e}$, of order 3 in $\text{PGL}(V/d)$. A subgroup of A_4 which contains the normal subgroup of order 4 and an element of order 3 is A_4 . Hence $|S^{(d)}| = 12$ for every point d of l .

Case $S \leq P_x$ for a point x . By Lemma 4.2, $S = U_x \rtimes R$.

Let first $d = x$. The group U_x acts trivially on $\mathcal{L}_x$, and R acts on $V/x \cong W$ as on W . So $S^{(x)} \cong RZ/Z$ has order 60, 60, 12, 10 or 6.

Let now $d \neq x$, and let n be the line through x and d . Let $\langle w_0 \rangle$ be the point of n on l , and let w_0, w_1 be a basis of W ; then $d = \langle w_0 + \alpha e \rangle$ for some $\alpha \in \mathbb{F}_4$. The group S_d fixes x and d , hence the line $n \in \mathcal{L}_d$, so $S^{(d)}$ lies in a group isomorphic to A_4 . We have $(w_0 + \alpha e)u_\lambda = w_0 + (\alpha + \lambda(w_0))e$. So the four elations u_λ with $\lambda(w_0) = 0$ fix d ; they induce on $V/d = \langle \bar{e}, \bar{w}_1 \rangle$ the maps $\bar{e} \mapsto \bar{e}$, $\bar{w}_1 \mapsto \bar{w}_1 + \lambda(w_1)\bar{e}$, the four elations with centre n . The same formula shows that U_x is transitive on the four points of n other than x . As before, R contains an element h with $w_0h \in \langle w_0 \rangle$ and $w_1h \in bw_1 + \langle w_0 \rangle$, $b \neq 1$. This h fixes x and n ; so there is $u \in U_x$ such that $s := hu$ fixes d . We have $es = e$, hence $\bar{e}s = \bar{e}$ in V/d ,

and s acts on $V/\langle e, w_0 \rangle$ as h does, by the scalar b . So s induces on V/d a map $\bar{e} \mapsto \bar{e}$, $\bar{w}_1 \mapsto b\bar{w}_1 + c\bar{e}$; since $1 + b + b^2 = 0$, its cube is the identity, and it has order 3 in $\text{PGL}(V/d)$. Hence again $|S^{(d)}| = 12$ for every point $d \neq x$. In all cases $|S^{(d)}| \geq 6$. $\square$

By Lemmas 4.1 and 4.2 the subgroups of order greater than 240 are, up to conjugacy, the twelve groups in the upper part of Table 2; the stabiliser of a point x and a line through x occurs for both types. The data of the table are the output of each of four programs for methods (2) and (3) in Section 6; they agree with the orders $|S^{(d)}|$ found in the proof, and the orders and numbers of conjugates are those of Nr. 1–14 of [CL12].

5 Further examples, and conditions for strict inequality

Proof of Theorem 1.2. Let P be one of the five groups, so $\text{PSL}(3, q) \leq P \leq \text{P}\Gamma\text{L}(3, q)$ with $q \in \{4, 5\}$. It is 2-transitive and faithful on the points, so Q is maximal and $\text{core}_P(Q) = 1$. The image $\bar{\varphi}(Q)$ of Q in $\text{P}\Gamma\text{L}(V/y)$ contains $T = \text{PSL}(V/y) \cong A_5$, so $\text{Inn}(T) \leq \varphi(Q)$. Let $\psi: P \rightarrow \text{Aut}(T) \cong S_5$ extend φ , and let $N = \ker \psi$; then $|P : N|$ is 60 or 120. But $N \cap \text{PSL}(3, q)$ is a normal subgroup of the simple group $\text{PSL}(3, q)$, of order 20160 or 372000. If it is trivial, then N embeds into $P/\text{PSL}(3, q)$, of order at most 6; otherwise $|P : N| \leq 6$. Both are impossible. So Lemma 2.4 and Lemma 2.1(b) apply: G is primitive of type TW with primitive component $H = T \rtimes \varphi(Q)$, and $k \cdot \text{MinSubDeg}(H) = |P|/M$, $\text{MinSubDeg}(G) = |P|/M_G$. The group $\bar{\varphi}(Q)$ is $\text{PGL}(2, 4) \cong A_5$ for $P = \text{PSL}(3, 4)$ and $\text{PGL}(3, 4)$, and it is $\text{P}\Gamma\text{L}(2, 4) \cong S_5$ or $\text{PGL}(2, 5) \cong S_5$ in the other three cases, where $\varphi(Q) = \text{Aut}(T)$. The numbers M and M_G were computed: M from its definition, and M_G , which by Lemma 2.3 is the largest order of a good subgroup, by the complete enumerations described in Section 6. $\square$

Remark 5.1 (the convention for the primitive component). In Theorem 1.1 the induced group is $H = T \rtimes \text{Inn}(T)$. If one replaced it by the holomorph $T \rtimes \text{Aut}(T)$ of A_5 , whose non-trivial subdegrees are 15, 20, 24, the product would be $21 \cdot 15 = 315 > 252$; this is not what the source asks, since there H is induced by G . For $P = \text{P}\Gamma\text{L}(3, 4)$ and $P = \text{PSL}(3, 5)$ the induced group is the holomorph itself, the normaliser in $\text{Sym}(T)$ of the group of right multiplications and so the largest group that could be meant. These two examples answer the question under either convention.

For $P = \text{PSL}(3, 4)$, where $M = 80$, a witness for the strict inequality is $S = U_l \rtimes D$ with $D \leq L_0$ dihedral of order 10 (Section 4): $|S| = 160$ and $|S^{(d)}| = 4$ for $d \in l$.

Proposition 5.2 (sufficient conditions for strict inequality). *In the situation of Lemma 2.1, each of the following conditions implies $\text{MinSubDeg}(G) < k \cdot \text{MinSubDeg}(H)$.*

- (a) P has a p -subgroup of order greater than M , for a prime p dividing $|T|$.
- (b) There is $x \in Q$ such that $\varphi(x)$ is a non-trivial inner automorphism of T and $|C_P(x)| > M$.
- (c) There are a subgroup $S \leq P$ with $|S| > M$ and $g \in P$ with $S \cap gQg^{-1} = 1$.

Proof. By Lemma 2.3(c) it suffices to find a good subgroup of order greater than M .

(a) Let S be the p -subgroup and $A := \varphi(S \cap Q)$, a p -group of automorphisms of T . Let Σ be a Sylow p -subgroup of $T \rtimes A$ containing A . Then $\Sigma \cap T$ is a Sylow p -subgroup of T , hence a non-trivial normal subgroup of the p -group Σ , and it contains an element $t \neq 1$ of the centre of Σ . As A fixes t , S is good.

(b) Let $\varphi(x)$ be the conjugation by $t \neq 1$. For $q \in C_Q(x)$ the automorphism $\varphi(q)$ commutes with $\varphi(x)$, so the conjugations by $t^{\varphi(q)}$ and by t coincide, and $t^{\varphi(q)} = t$ because the centre of T is trivial. So $C_P(x) \cap Q \leq Q_t$, and $C_P(x)$ is good.

(c) The group $g^{-1}Sg$ meets Q trivially; so it is good. $\square$

Remark 5.3. The example of Theorem 1.1 satisfies none of these conditions, since $M = 240 = |\ker \varphi| \cdot 5$ is large: the Sylow subgroups of $\mathrm{PGL}(3, 4)$ have order 64, 27, 5, 7; the centralisers of its non-identity elements have order at most 192 (computed); and a subgroup meeting a point stabiliser trivially has order at most 21.

Classical examples. Table 3 lists five classical primitive groups of type TW. Here P is a 2-transitive group of degree k , Q is a point stabiliser, T is the derived subgroup of Q , and $\varphi(q)$ is the conjugation by q on T ; so $\ker \varphi = 1$. Lemma 2.4 applies: a homomorphism $P \rightarrow \mathrm{Aut}(T)$ extending φ would be injective on the simple group P' (which is P , or A_6 for $P = S_6$), but $|P'| > |\mathrm{Aut}(T)|$. The values were computed by three independent programs (Section 6); all five inequalities are strict.

P, Q	T	k	M	M_G	$k \cdot \mathrm{MinSubDeg}(H)$	$\mathrm{MinSubDeg}(G)$
A_6, A_5	A_5	6	5	24	72	15
S_6, S_5	A_5	6	8	48	90	15
A_7, A_6	A_6	7	9	36	280	70
$\mathrm{PSL}(2, 11), A_5$	A_5	11	5	55	132	12
M_{11}, M_{10}	A_6	11	16	144	495	55

Table 3: Computed values for five classical groups $T \mathrm{ twr}_\varphi P$; φ is the conjugation action of Q on T .

6 Computations and reproducibility

Proposition 3.3 was also verified by exhaustive exact computations with permutations of 21 points. The reproducibility package contains the programs of four stages with their recorded outputs: those with which the results were first obtained (`original/`) and those of the three verification runs described below (`verification_run_A/`, `verification_run_B/`, `independent_run_2/`). Each stage constructs P in its own way and checks the datum (Lemma 3.1, (6)). There are seven complete enumerations, by four methods. Times are for one laptop (Python 3 with numpy).

(1) *Search over the orbit of the base point* (`original/`, `maxstab.py`; 30 seconds). Let S be good with $|S| > 240$, and $R = S \cap Q \leq Q_t$. After conjugation in Q , t is one of four representatives of the Q -orbits on $T \setminus \{1\}$, and $|R| \geq 12$ because $|S| = |R| |y^S| \leq 21 |R|$. For each of the 192 such subgroups R the program finds all subgroups S with $S \cap Q = R$ and $|S| > 240$ by a depth-first search over the orbit of y . For the smallest point d^* not yet decided, either d^* is excluded from the orbit, or the current group is enlarged by an element mapping y to d^* , one from each coset of R (a group with $S \cap Q = R$ contains exactly one such coset), provided that the stabiliser of y remains R . A branch ends when $|R|$ times the number of points not excluded is at most 240. Every S in question dictates a path which is not cut, and the group at its end has the same orbit and the same stabiliser of y as S , so it is S . Result: no such S exists. With the bound 239 the search finds 102 good subgroups, all of order 240.

(2) *All subgroups up to conjugacy* (`original/`, `lattice.py`, 6 minutes; `verification_run_B/`, `b1_lattice.py`, 80 seconds). Starting with the trivial group, each class representative S is extended by the elements $z \notin S$ of prime power order, one for each orbit of $N_P(S)$ on the cyclic subgroups, and $\langle S, z \rangle$ is compared with the representatives found so far by an exact conjugacy test. This is complete, because every non-trivial subgroup is generated by one of its maximal subgroups and an element of prime power order. Both programs find 100 classes with 81220 subgroups and the data of Table 2; the two lists were matched class by class. A third program (`independent_run_2/`, 11 seconds) agrees.

(3) *Overgroups of Sylow subgroups* (`verification_run_A/`, `t2_main.py`; 30 seconds). Let $|S| > 240$. If 7 or 5 divides $|S|$, a conjugate of S contains a fixed subgroup of that order. Otherwise

S is a $\{2, 3\}$ -group, hence soluble, with an elementary abelian minimal normal subgroup N . If N is a 3-group, then $|S| \leq |N_P(N)| \leq 216$ (computed for all subgroups of order 3 and 9; there is no elementary abelian subgroup of order 27). If N is a 2-group, its fixed points are one point or the points of one line, so S lies in P_x or P_l , $|S| \in \{288, 576\}$, and a conjugate of S contains a fixed Sylow 3-subgroup of P_x or P_l . The program lists all overgroups of these four subgroups (the last two inside P_x and P_l ; 11, 29, 14 and 14 groups), by adjoining to each group U one element of each double coset UgU , and finds the 14 classes of Table 2. The third run (`independent_run_2/`, 20 seconds) does not use solubility: the divisors of 60480 which are at least 240 and prime to 35 are multiples of 9, so it lists the overgroups of a subgroup of order 5, of a subgroup of order 7 and of the four subgroups of order 9 of a Sylow 3-subgroup, with the same result.

(4) *Extension of the stabiliser of the base point* (`verification_run_B/`, `b1_ext.py`; 10 seconds). For each of the 60 subgroups R of Q with $21 \mid |R| \geq 240$, up to conjugacy in Q , all subgroups S with $S \cap Q = R$ are found by a closure search which adjoins elements g with $\langle R, g \rangle \cap Q = R$, one for each double coset; every such S is reached from R through a chain of such extensions. The numbers of subgroups of each order ≥ 240 are those of Table 2, and the good ones are 102 subgroups of order 240.

A certificate without the reduction. For each of the twelve classes of order greater than 240 the second and the third run computed, each with its own program, the set of all elements of the base group A_5^{21} fixed by S , from the equations $b^s = b$ in the coordinates of Lemma 2.1(b): it is $\{1\}$ in every case. Since $C_P(b^g) = g^{-1}C_P(b)g$, this proves $M_G \leq 240$ from the definition of the action and the list of subgroups alone, without Lemma 2.3. Each of the 252 subgroups S of order 240 fixes exactly four elements $b \neq 1$, and $C_P(b) = S$ for them, since no larger subgroup fixes such an element. These $252 \cdot 4$ elements lie in orbits of length 252; hence G has exactly four suborbits of length 252: two consist of elements of a single factor T_c , and two of elements supported on the 16 points off a line.

Tests in the base group; validation. With stabilisers computed by testing all elements of P : the $21 \cdot 59$ elements supported on one coordinate have stabilisers of order 192, 144 or 240; the $210 \cdot 59^2$ elements supported on two coordinates, of order at most 32; the $2 \cdot 59^3$ elements supported on a collinear triple or a triangle, at most 96; about 330000 further elements, at most 240. Lemma 2.3 was confirmed on small twisted wreath products with completely listed base groups (ten, eleven and seven examples at three of the stages). The first two lattice programs reproduce the numbers of conjugacy classes of subgroups and of subgroups of S_n ($n \leq 7$) and A_n ($n \leq 8$) in the OEIS [OEI26] (A000638, A005432, A029726, A029725); for M_{11} all three give 39 and 8651. For $\text{PGL}(3, 4)$ the orders and numbers of conjugates of all 100 classes are those of the table [CL12].

The other rows. For $\text{PSL}(3, 4)$, $\text{P}\Sigma\text{L}(3, 4)$ and $\text{PTL}(3, 4)$ the numbers M_G were obtained by methods (1), (2) (second program; 95, 152 and 226 classes) and (4); for $\text{PSL}(3, 5)$ by (1) and (4), and the numbers of subgroups of each order ≥ 800 found by (4) are those of the table of $L_3(5)$ in [CL12]. For $\text{PTL}(3, 4)$ and $\text{PSL}(3, 5)$ the certificate without the reduction was computed as well. Table 3 was computed by method (1) and with the second lattice program. The third run computed all rows of Tables 1 and 3 again from lists of all subgroups up to conjugacy (140 classes for $\text{PSL}(3, 5)$, with the orders and numbers of conjugates of the table of $L_3(5)$). The package has a `README.md` with all command lines; `run_main_certificate.sh` repeats the main computations in about three minutes.

Verification. The results were first obtained with the lemmas of Section 2, methods (1) and (2), and a first form of the argument of Section 4. Two independent verification runs, both AI-assisted, followed; neither saw the other, and each wrote its own programs. The first run compared the statement with [PS05], derived Lemmas 2.1, 2.3 and 2.4 before reading their proofs, proved Proposition 3.3 by method (3), checked the written argument, and compared (F1)–(F3) and Table 2 with [CL12]; it required a source which had been read for the lists of maximal subgroups, the description of K_d as the group of all perspectivities, and an explicit treatment of the points $d \in l$ and $d \neq x$ in Section 4.

The second run recomputed everything with its own programs, read and re-ran the first programs, computed the other rows of Tables 1 and 3, and examined the literature; it required the attributions of Remark 2.5, the statements about the papers which were not read, and the removal of sample code never run. Neither found a mathematical error, and all corrections were made.

Written after these two runs were the case $S \cong A_6$ in the proof of Proposition 3.3, the arrangement of Section 4 around (F1)–(F3), the count of the suborbits of length 252, and the comparisons of complete lists with [CL12]. A third run, also AI-assisted, examined the final text, and these four things first. It read [PS05] again, went through Sections 2 to 5 line by line, and compared Remark 2.5 and (F1)–(F3) with the texts of [CGM19, DGPS13, Faw22, CL12]. With its own programs, written from the statements of this note, it proved Proposition 3.3 by the two enumerations described above and recomputed the datum, Tables 1 to 3, the certificate and the four suborbits; it confirmed the case $S \cong A_6$ on the stabiliser of a hyperoval, and compared all 100 classes of subgroups of $\mathrm{PGL}(3, 4)$ and all 140 of $\mathrm{PSL}(3, 5)$ with [CL12]. It ran the package again from the archive, with the recorded outputs. It found no error; it required that this paragraph and Section 6 describe the state after its checks, and a justification of the count of the suborbits.

Scope and priority. The question is Question 4 of C. E. Praeger and Á. Seress in Oberwolfach Report 12/2005 [PS05, p. 692], an abstract on joint work with M. Giudici, C. H. Li and V. Trofimov. The authors ask a question and write that they wonder whether the bound can be attained; this note gives a negative answer to Question 4, in the reading of Section 1.1: k is the number of simple factors of the socle, H is the group induced on one coordinate (the primitive component), the action is the action on the base group, and the non-strict inequality is Theorem 3 of [PS05].

The full paper [GLP⁺06] and the companion paper [GLP⁺07] were not read: they were not accessible to us (the third run tried again), and no open copy was found. What we know about [GLP⁺06] comes from the abstract [PS05] (Theorems 1 and 3, a construction of special points, results on minimum-valency graphs), from the zbMATH review Zbl 1111.20003 (the equality for the non-twisted types, and infinitely many examples of type TW with $\mathrm{MinSubDeg}(G) = \mathrm{MinSubDeg}(H)$), and from [CGM19] (the paper proves bounds on the minimal subdegrees and constructs a minimal suborbit). None of them mentions an answer to Question 4. We make no claim about the content of the two papers beyond this; we do not know whether the question is repeated or restricted there, nor which examples are treated. A reader with access to them should compare.

The reduction of Section 2 is known in substance (Remark 2.5), and (F1)–(F3) are taken from [CL12]. New, to our knowledge, are the three examples with equality (Section 1.3). We do not claim that the first one is the smallest example, the cases of equality are not classified, and the question remains open under additional hypotheses such as $\ker \varphi = 1$.

What was read: [PS05] completely, in the publisher’s file; of [CGM19] (arXiv version 1) Sections 1 and 2, of [Faw22] (arXiv version 2) the definition in Section 1 and Section 4 up to Lemma 4.7, of [DGPS13] (arXiv version 2) Example 4.4 (numbering of these versions; the journal versions were not compared); the table of $\mathrm{PGL}_3(4)$ in [CL12] completely, and that of $L_3(5)$ for the orders and lengths of its classes; of [CL15] the abstract. The searches were made on 11 October 2026, at each stage and again by the third run: queries to arXiv, zbMATH, Crossref, OpenAlex and, at the earlier stages, Semantic Scholar for minimal subdegrees, suborbits of primitive groups and twisted wreath products; the works citing [GLP⁺06] (five: [PS05, GLP⁺07, CGM19] and two papers on other subjects, of which only passages were read); and nine web searches. Full texts were not searched, and the authors of the question were not contacted. No answer to Question 4 was found in the literature accessible to us. A search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has

reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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