

An Infinite Family of Primitive Groups of Twisted Wreath Type Attaining the Upper Bound for the Minimal Subdegree

Alper Ferudun*

October 14, 2026

Abstract

Let G be a finite primitive permutation group of twisted wreath type, with socle T^k and primitive component H . Giudici, Li, Praeger, Seress and Trofimov (2006) proved that the least non-trivial subdegree of G satisfies $\text{MinSubDeg}(G) \leq k \cdot \text{MinSubDeg}(H)$, and asked whether this inequality is always strict (their Question 1.6; Question 4 of Praeger and Seress in Oberwolfach Report 12/2005). An earlier note of the author gave a negative answer with the group $\text{PSL}(2, 4) \text{ twr } \text{PGL}(3, 4)$. Here we show that the bound is attained by an infinite family. For a prime power $q \geq 4$ let $P = \text{PGL}(3, q)$ act on the $k = q^2 + q + 1$ points of the projective plane $\text{PG}(2, q)$, let Q be the stabiliser of a point y , and let Q act on $T = \text{PSL}(2, q)$ by conjugation, through its action on the quotient space $\mathbb{F}_q^3/y$. Then $G_q = T \text{ twr } P$ is primitive of twisted wreath type, and $\text{MinSubDeg}(G_q) = k \cdot \text{MinSubDeg}(H) = q(q^3 - 1)(q + 1)/c_q$, where c_q is $q + 1$, $2(q - 1)$ or $2(q + 1)$ according as q is even, $q \equiv 1$ or $q \equiv 3 \pmod{4}$. By the reduction of the five authors this is a statement about the subgroups of $\text{PGL}(3, q)$ of order greater than $q^2(q - 1)c_q$. We prove it for $q \geq 5$ by a written argument from the list of the maximal subgroups of $\text{SL}(3, q)$ (Mitchell; Hartley; tabulated by Bray, Holt and Roney-Dougall), Dickson's list of the subgroups of $\text{PGL}(2, q)$ and an inverse-transpose duality; for $q = 4$ it is the result of the earlier note. Exhaustive computations for small q , with independently written programs, are tests and not part of the proof. No infinite family attaining the bound was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The question

All groups in this note are finite, and groups act on the right. Let G be a transitive permutation group on a set Ω and $\alpha \in \Omega$. The orbits of the stabiliser G_α are the *suborbits*, their lengths the *subdegrees*, and $\text{MinSubDeg}(G)$ is the least length of a suborbit other than $\{\alpha\}$ [PS05, p. 691], [GLP⁺06, §1]. A primitive group is of *twisted wreath type* (O'Nan–Scott type TW) if it has a unique minimal normal subgroup and this subgroup is non-abelian, not simple and regular. Such a group has the form $G = \text{Soc}(G) \rtimes P$ with $\text{Soc}(G) \cong T^k$ for a non-abelian simple group T and $k \geq 2$, and $G \leq H \wr S_k$ in product action on a Cartesian power Δ^k , where the *primitive component* H is the primitive group induced on Δ by the stabiliser of a coordinate [GLP⁺06, §§1, 2, 4]. For a group P let $\text{MinDeg}(P)$ be the least degree of a faithful transitive action of P . Theorem 1.5(a) of Giudici, Li, Praeger, Seress and Trofimov [GLP⁺06] states that for every such G

$$\max\{\text{MinSubDeg}(H), \text{MinDeg}(P)\} \leq \text{MinSubDeg}(G) \leq k \cdot \text{MinSubDeg}(H), \quad (1)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

and that the lower bound is attained by infinitely many groups. The authors wonder whether the upper bound can ever be attained, and their Question 1.6 asks whether the strict inequality $\text{MinSubDeg}(G) < k \cdot \text{MinSubDeg}(H)$ holds for all primitive groups $G = \text{Soc}(G) \rtimes P \leq H \wr S_k$ of type TW. The same question is Question 4 of the abstract of Praeger and Seress [PS05, p. 692], and it is record OWR-793-006 of the UnsolvedMath collection [ula26].

The author's earlier note [Fer26] answered the question in the negative: for $P = \text{PGL}(3, 4)$ and $T = \text{PSL}(2, 4) \cong A_5$ (in the notation of Section 1.3 below) it shows $\text{MinSubDeg}(G) = 252 = 21 \cdot \text{MinSubDeg}(H)$, by a written argument and by exhaustive computation [Fer26, Theorem 1.1], and it reports equality, found by complete enumeration of subgroups, for two further top groups, among them $\text{PSL}(3, 5) = \text{PGL}(3, 5)$ [Fer26, Theorem 1.2]. After that note had appeared, M. Giudici asked the author whether infinitely many such groups exist. This note answers that question: the upper bound in (1) is attained for every prime power $q \geq 4$ by the group $\text{PSL}(2, q) \text{ twr } \text{PGL}(3, q)$ (Theorem B).

1.2 Twisted wreath products and the criterion for equality

We use the conventions of [GLP⁺06, §4] and [Fer26, §2]. Let T be a non-abelian simple group, P a group, $Q < P$ a subgroup and $\varphi: Q \rightarrow \text{Aut}(T)$ a homomorphism. The *base group*

$$B = \{f: P \rightarrow T \mid f(pq) = f(p)^{\varphi(q)} \text{ for all } p \in P, q \in Q\}$$

is isomorphic to T^k , $k = |P : Q|$, and P acts on it by $f^x(p) = f(xp)$. The *twisted wreath product* $G = T \text{ twr}_\varphi P = B \rtimes P$ acts on $\Omega = B$ by $\omega^{bx} = (\omega b)^x$ ($\omega, b \in B$, $x \in P$); here B is regular and P is the stabiliser of the identity of B . The group induced on one coordinate of $B \cong T^k$ is $H = T \rtimes \varphi(Q)$, acting on T by $t \mapsto (ts)^a$ ($s \in T$, $a \in \varphi(Q)$) [GLP⁺06, §4], [Fer26, Lemma 2.1(b)]. For $t \in T$ let $Q_t = \{x \in Q : t^{\varphi(x)} = t\}$, and let

$$M := \max\{|Q_t| : t \in T \setminus \{1\}\}. \quad (2)$$

Lemma 1.1 ([Fer26, Lemmas 2.1 and 2.4]). *Let T be non-abelian simple and $\text{Inn}(T) \leq \varphi(Q)$.*

- (a) $\text{MinSubDeg}(H) = |Q|/M$ and $k \cdot \text{MinSubDeg}(H) = |P|/M$.
- (b) *If moreover Q is a maximal subgroup of P with $\text{core}_P(Q) = 1$, and φ does not extend to a homomorphism $P \rightarrow \text{Aut}(T)$, then G is a primitive permutation group of type TW on B with socle $B \cong T^k$, and $G = B \rtimes P \leq H \wr S_k$ in product action on T^k , with primitive component H .*

Part (b) is the sufficient form, proved in [Fer26, Lemma 2.4], of the criterion for primitivity stated at the beginning of [GLP⁺06, §4].

Definition 1.2. A subgroup $S \leq P$ is *good* if $\varphi(S \cap Q)$ fixes some $t \in T \setminus \{1\}$, that is, if $S \cap Q \leq Q_t$ for some $t \neq 1$.

The following criterion is due to Giudici, Li, Praeger, Seress and Trofimov: it is Construction 4.1 with Lemmas 4.2 and 4.3 of [GLP⁺06], and it is stated in this form in [Fer26, Lemma 2.3]. We repeat the short proof.

Lemma 1.3 (criterion). *Under the hypotheses of Lemma 1.1(a) let M_G be the largest order of a good subgroup of P . Then $\text{MinSubDeg}(G) = |P|/M_G$ and $M_G \geq M$. In particular $\text{MinSubDeg}(G) = k \cdot \text{MinSubDeg}(H)$ if and only if no good subgroup of P has order greater than M .*

Proof. The suborbits of G are the orbits of P on B , so the non-trivial subdegrees are the indices $|P : C_P(b)|$, $b \in B \setminus \{1\}$. Let $b \neq 1$, choose $p_0 \in P$ with $t := b(p_0) \neq 1$, and let $S = p_0^{-1}C_P(b)p_0$. If $c = p_0^{-1}sp_0 \in S \cap Q$ with $s \in C_P(b)$, then $t^{\varphi(c)} = b(p_0c) = b(sp_0) = b^s(p_0) = t$; so S is a good

subgroup of order $|C_P(b)|$. Conversely, let S be good, $S \cap Q \leq Q_t$ with $t \neq 1$. Put $b(sq) := t^{\varphi(q)}$ for $s \in S$, $q \in Q$, and $b := 1$ on $P \setminus SQ$. If $sq = s_1q_1$, then $q_1q^{-1} = s_1^{-1}s \in S \cap Q$ fixes t , so b is well defined; clearly $b \in B$, $b(1) = t \neq 1$, and $b^{s'} = b$ for $s' \in S$, because SQ is invariant under left multiplication by S and $b(s'sq) = b(sq)$. (This b is the element $f_{S,t}$ of [GLP⁺06, Construction 4.1].) Hence $M_G = \max\{|C_P(b)| : b \neq 1\}$ and $\text{MinSubDeg}(G) = |P|/M_G$. Every Q_t with $t \neq 1$ is good, so $M_G \geq M$, and the last statement follows from Lemma 1.1(a). $\square$

1.3 The family and the results

Let $q = p^f \geq 4$ be a prime power and $V = \mathbb{F}_q^3$, the space of row vectors, on which matrices act on the right. Let $P = \text{PGL}(3, q)$ act on the $k = q^2 + q + 1$ points (1-spaces) of $\text{PG}(2, q)$. For a point d let P_d be its stabiliser, $[d]$ the set of the $q + 1$ lines through d (the points of the projective line of V/d), and $\varphi_d: P_d \rightarrow \text{PGL}(V/d) \cong \text{PGL}(2, q)$ the induced action. We write $\text{PSL}(V/d) \cong \text{PSL}(2, q)$ for the image of $\text{SL}(V/d)$, and $S_d = S \cap P_d$ for $S \leq P$. Fix a point y and let

$$Q = P_y, \quad T = \text{PSL}(V/y), \quad t^{\varphi(x)} = \varphi_y(x)^{-1} t \varphi_y(x) \quad (x \in Q, t \in T),$$

and $G_q = T \text{twr}_\varphi P$. Its degree is $|T|^k = |\text{PSL}(2, q)|^{q^2+q+1}$.

Definition 1.4. A subgroup $S \leq P$ is *good at the point d* if $\varphi_d(S_d)$ centralises some non-identity element of $\text{PSL}(V/d)$.

Lemma 1.5. (a) φ_y maps Q onto $\text{PGL}(V/y)$, $\varphi(Q) \cong \text{PGL}(2, q)$ contains $\text{Inn}(T)$, and $\ker \varphi = \ker \varphi_y$ has order $q^2(q-1)$. For $t \in T$ we have $Q_t = \varphi_y^{-1}(C_{\text{PGL}(V/y)}(t))$.

(b) A subgroup $S \leq P$ is good if and only if it is good at y . For $g \in P$, the subgroup $g^{-1}Sg$ is good if and only if S is good at the point $y^{g^{-1}}$.

Proof. (a) In a basis e_1, e_2, e_3 with $y = \langle e_1 \rangle$, the matrices $\text{diag}(1, A)$, $A \in \text{GL}(2, q)$, induce all of $\text{GL}(V/y)$. If $g \in \text{PGL}(V/y)$ centralises T , it commutes with every non-identity element of order p of T , so it fixes the unique point of the projective line fixed by such an element (Lemma 2.1(i)); every point is of this kind, so $g = 1$. Hence $x \mapsto \varphi(x)$ has the same kernel as φ_y , and $\varphi(Q) \cong \text{PGL}(V/y)$; it contains the conjugations by elements of T . The order of the kernel is $|Q|/|\text{PGL}(2, q)| = q^3(q-1)^2(q+1)/(q(q^2-1))$. The last statement is the definition of Q_t .

(b) The first statement is (a). Let $d = y^{g^{-1}}$. Then $(g^{-1}Sg) \cap Q = g^{-1}S_dg$, and a matrix of g induces a linear isomorphism $\bar{g}: V/d \rightarrow V/y$ with $\varphi_y(g^{-1}sg) = \bar{g}^{-1}\varphi_d(s)\bar{g}$ for $s \in S_d$; conjugation by $\bar{g}$ maps $\text{PSL}(V/d)$ onto $\text{PSL}(V/y)$. $\square$

Let

$$c_q := \max\{|C_{\text{PGL}(2, q)}(t)| : 1 \neq t \in \text{PSL}(2, q)\}.$$

By Lemma 1.5(a), $|Q_t| = q^2(q-1)|C_{\text{PGL}(V/y)}(t)|$, so

$$M = q^2(q-1)c_q, \tag{3}$$

and $c_q = q+1, 2(q-1), 2(q+1)$ for q even, $q \equiv 1 \pmod{4}$, $q \equiv 3 \pmod{4}$ (Lemma 2.1).

Theorem A. Let $q \geq 4$ be a prime power and $M = q^2(q-1)c_q$. No subgroup S of $\text{PGL}(3, q)$ with $|S| > M$ is good at any point of $\text{PG}(2, q)$.

Theorem B. For every prime power $q \geq 4$ the group $G_q = \text{PSL}(2, q) \text{twr}_\varphi \text{PGL}(3, q)$ is a primitive permutation group of type TW, with socle $\text{PSL}(2, q)^k$, $k = q^2 + q + 1$, and primitive component $H = T \rtimes \varphi(Q)$, $\varphi(Q) \cong \text{PGL}(2, q)$; and

$$\text{MinSubDeg}(G_q) = k \cdot \text{MinSubDeg}(H) = \frac{q(q^3-1)(q+1)}{c_q},$$

where

$$\text{MinSubDeg}(H) = \frac{q(q^2 - 1)}{c_q} = \begin{cases} q(q - 1) & (q \text{ even}), \\ q(q + 1)/2 & (q \equiv 1 \pmod{4}), \\ q(q - 1)/2 & (q \equiv 3 \pmod{4}). \end{cases}$$

In particular the upper bound in (1) is attained by infinitely many primitive groups of type TW, and each G_q gives a negative answer to Question 1.6 of [GLP⁺06] (Question 4 of [PS05]).

q	k	c_q	M	$\text{MinSubDeg}(H)$	$\text{MinSubDeg}(G_q) = k \cdot \text{MinSubDeg}(H)$
4	21	5	240	12	252
5	31	8	800	15	465
7	57	16	4704	21	1197
8	73	9	4032	56	4088
9	91	16	10368	45	4095
11	133	24	29040	55	7315
13	183	24	48672	91	16653
16	273	17	65280	240	65520
17	307	32	147968	153	46971
19	381	40	259920	171	65151
23	553	48	558624	253	139909
25	651	48	720000	325	211575

Table 1: The first values of Theorem B; $M = q^2(q - 1)c_q$, $\text{MinSubDeg}(H) = |Q|/M$ and $\text{MinSubDeg}(G_q) = |P|/M$. The degree of G_q is $|\text{PSL}(2, q)|^k$.

For $q = 4$, Theorem B is [Fer26, Theorem 1.1]. For $q = 5$ we have $\text{PGL}(3, 5) = \text{PSL}(3, 5)$, and G_5 is the group with top group $\text{PSL}(3, 5)$ in [Fer26, Theorem 1.2], where the equality $465 = 31 \cdot 15$ was found by complete enumeration; the proof below is a written one for $q = 5$ as well.

1.4 Outline, inputs, and what is new

By Lemmas 1.1, 1.3 and 1.5, Theorem B follows from Theorem A and the primitivity of G_q (Section 7). For $q \geq 5$ the proof of Theorem A has three steps. *Reduction* (Section 4): a subgroup of $\text{PGL}(3, q)$ of order greater than M contains $\text{PSL}(3, q)$, or fixes a point, or fixes a line. *Line stabilisers* (Section 5): a subgroup of order greater than M that fixes a line is good at no point; for points off the line and for the flag case this is a count, and otherwise a translation argument. *Point stabilisers* (Section 6): by an inverse-transpose duality (Section 3) the remaining case becomes a statement about the groups induced on lines by a subgroup of a line stabiliser that fixes no point of the line; such a subgroup contains all q^2 elations with this axis, and a translation argument concludes. Section 2 collects the facts on $\text{PGL}(2, q)$ that are used. For $q = 4$ the reduction fails, because $\text{PSL}(3, 4)$ contains subgroups A_6 of order $360 > M = 240$; there Theorem A is the result of [Fer26] (Section 7).

The proof uses three published classifications, which we cite as standard results and do not re-prove: (R1) the maximal subgroups of $\text{SL}(3, q)$ (Mitchell [Mit11] for odd q , Hartley [Har25] for even q), in the tabulation of Bray, Holt and Roney-Dougal [BHRD13], which we read only as transcribed by Liu and Yang [LY22, Lemma 2.2] (Input R1); (R2) the automorphism groups of $\text{PSL}(3, q_0)$ and $\text{PSU}(3, q_0)$ [SvdW28, Ste60] (Input R2); and (D) Dickson's list of the subgroups of $\text{PGL}(2, q)$ [Dic01] (Lemma 2.3). The computations of Section 8 are tests; no proof depends on them.

The bounds (1), the question and the reduction of Lemma 1.3 are due to Giudici, Li, Praeger, Seress and Trofimov [GLP⁺06]; the example $q = 4$ is that of [Fer26]. New, to our knowledge and

relative to the literature we found, are the infinite family of Theorem B and its proof; see “Scope and priority” in Section 8.

2 Subgroups of $\mathrm{PGL}(2, q)$

In this section $q = p^f \geq 4$. The group $\mathrm{PGL}(2, q) = \mathrm{GL}(2, q)/\mathbb{F}_q^\times$ acts on the projective line $\mathrm{PG}(1, q) = \mathbb{F}_q \cup \{\infty\}$: the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps the point $\langle(x, 1)\rangle$ to $\langle(ax + c, bx + d)\rangle$. The square class of the determinant of a representing matrix is well defined, and $\mathrm{PSL}(2, q)$, the image of $\mathrm{SL}(2, q)$, consists of the elements with square determinant; it has index $\gcd(2, q - 1)$, it is simple, and it is the derived subgroup of $\mathrm{PGL}(2, q)$ (as $q \geq 4$). The stabiliser of ∞ is

$$B_\infty = \{x \mapsto \beta x + b : \beta \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\}, \quad U = \{x \mapsto x + b\} \trianglelefteq B_\infty,$$

and $U \leq \mathrm{PSL}(2, q)$, because $x \mapsto x + b$ is induced by the transvection $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$. A non-identity element of $\mathrm{PGL}(2, q)$ fixes at most two points, since $\mathrm{PGL}(2, q)$ is sharply 3-transitive.

Lemma 2.1 (centralisers). *Let $1 \neq t \in \mathrm{PGL}(2, q)$, and let $C(t)$ be its centraliser in $\mathrm{PGL}(2, q)$.*

- (i) *t fixes exactly one point if and only if t has order p . Then $C(t)$ is the group of order q consisting of the identity and the elements of order p fixing this point; for this point ∞ it is U .*
- (ii) *If t fixes exactly two points, then $C(t)$ is the cyclic group of order $q - 1$ of all elements fixing both points if $t^2 \neq 1$, and the dihedral group of order $2(q - 1)$ stabilising the pair if $t^2 = 1$.*
- (iii) *If t fixes no point, then $C(t)$ is cyclic of order $q + 1$ if $t^2 \neq 1$, and dihedral of order $2(q + 1)$ if $t^2 = 1$.*
- (iv) *Let q be odd. The involutions of type (ii) lie in $\mathrm{PSL}(2, q)$ if and only if $q \equiv 1 \pmod{4}$, and those of type (iii) if and only if $q \equiv 3 \pmod{4}$.*

Consequently $c_q = q + 1$ for even q , $c_q = 2(q - 1)$ for $q \equiv 1 \pmod{4}$ and $c_q = 2(q + 1)$ for $q \equiv 3 \pmod{4}$. In all cases $c_q \geq q + 1$ and $M = q^2(q - 1)c_q \geq q^2(q^2 - 1)$.

Proof. Let $A \in \mathrm{GL}(2, q)$ represent t . An element $g \in \mathrm{GL}(2, q)$ represents an element of $C(t)$ if and only if $gAg^{-1} = cA$ for some $c \in \mathbb{F}_q^\times$; then A and cA have the same characteristic polynomial.

If A has a repeated eigenvalue, then $A = \lambda(I + N)$ with $N \neq 0$, $N^2 = 0$, since $t \neq 1$; so t has order p and fixes exactly one point, the image of N . Here $c = 1$, g commutes with $I + N$, so $g = aI + bN$, and $C(t)$ is the group $\{I + bN\}$ of order q , whose non-identity elements have order p and fix the same point. If A has two distinct eigenvalues in $\mathbb{F}_q$, we may take $A = \mathrm{diag}(a, 1)$ with $a \neq 1$; then t has order dividing $q - 1$ and fixes exactly two points. Comparing eigenvalues, $c = 1$, or $c = a = -1$ (so p is odd and $t^2 = 1$). For $c = 1$, g is diagonal; for $c = -1$, g is anti-diagonal. This gives (ii), the anti-diagonal elements inverting the diagonal ones. If A has no eigenvalue in $\mathbb{F}_q$, then t fixes no point, $K := \mathbb{F}_q[A]$ is a field of order q^2 , and the matrices commuting with A are the non-zero elements of K ; their images form a cyclic group of order $(q^2 - 1)/(q - 1) = q + 1$, so t has order dividing $q + 1$. If $c \neq 1$, then $c = -1$ and A has trace 0, that is, $t^2 = 1$ and p is odd; conversely A and $-A$ then have the same irreducible characteristic polynomial and are conjugate, and conjugation by such a g induces on K the non-trivial automorphism $z \mapsto z^q$, which inverts $K^\times$ modulo $\mathbb{F}_q^\times$. This gives (iii). The three cases exhaust $t \neq 1$, and the orders p , divisors of $q - 1$, divisors of $q + 1$ show (i).

(iv) An involution of type (ii) is represented by $\mathrm{diag}(-1, 1)$, of determinant -1 , a square if and only if $q \equiv 1 \pmod{4}$. An involution of type (iii) is represented by a matrix A with trace 0 whose characteristic polynomial $X^2 + \det A$ is irreducible, that is, $-\det A$ is a non-square; then $\det A$ is a square if and only if -1 is a non-square, that is, $q \equiv 3 \pmod{4}$.

For the values of c_q : by (i)–(iii) the centraliser orders of non-identity elements are q , $q - 1$, $q + 1$, and $2(q - 1)$, $2(q + 1)$ for the involutions of types (ii) and (iii), which exist only for odd q . For even q , $\text{PSL}(2, q) = \text{PGL}(2, q)$ contains elements of type (iii), and the maximum is $q + 1$. For $q \equiv 1 \pmod{4}$ the involutions of type (ii) lie in $\text{PSL}(2, q)$ and those of type (iii) do not, so the maximum over $\text{PSL}(2, q)$ is $2(q - 1) \geq q + 1$. For $q \equiv 3 \pmod{4}$ the involutions of type (iii) lie in $\text{PSL}(2, q)$, and the maximum is $2(q + 1)$. $\square$

Lemma 2.2 (the stabiliser of a point). *An element $x \mapsto \beta x + b$ of B_∞ is a p -element if $\beta = 1$, and an element of order equal to the order of β , prime to p , if $\beta \neq 1$. For a subgroup $Y \leq B_\infty$ the following are equivalent:*

- (a) *Y centralises some non-identity element of $\text{PSL}(2, q)$;*
- (b) *Y is a p -group or a p' -group.*

If they hold, then $|Y| \leq q$.

Proof. If $\beta \neq 1$, the element fixes $x_0 = b/(1 - \beta)$ and is conjugate in B_∞ to $x \mapsto \beta x$. So every element of B_∞ is a p -element or a p' -element, and Y is a p -group or a p' -group if and only if its non-identity elements are all of one kind.

(b) $\Rightarrow$ (a). A p -subgroup Y lies in the abelian group U , and centralises every element of $U \setminus \{1\} \subseteq \text{PSL}(2, q)$. If $Y \neq 1$ is a p' -group, then $Y \cap U = 1$, so Y embeds in $B_\infty/U \cong \mathbb{F}_q^\times$ and is cyclic, $Y = \langle y \rangle$ with $y: x \mapsto \beta x + b$, $\beta \neq 1$. Then Y lies in the group T_0 of all elements fixing ∞ and $x_0 = b/(1 - \beta)$, which is cyclic of order $q - 1$, and $T_0 \cap \text{PSL}(2, q)$ has order $(q - 1)/\gcd(2, q - 1) \geq 2$ as $q \geq 4$. So Y centralises a non-identity element. The trivial group centralises every element.

(a) $\Rightarrow$ (b). Let $Y \leq C(t)$ with $1 \neq t \in \text{PSL}(2, q)$, and suppose that Y contains a non-identity p -element $u: x \mapsto x + b$ and a non-identity p' -element $s: x \mapsto \beta x + c$, $\beta \neq 1$. By Lemma 2.1(i), $t \in C(u) = U$, say $t: x \mapsto x + e$ with $e \neq 0$. Then $ts = st$ gives $\beta e = e$, a contradiction.

Finally a p -subgroup of B_∞ lies in U , of order q , and a p' -subgroup embeds in B_∞/U , of order $q - 1$. $\square$

The next lemma is Dickson's classification [Dic01] of the subgroups of $\text{PGL}(2, q)$, in the coarse form in which we use it. We do not re-prove it.

Lemma 2.3 (Dickson). *Every subgroup Y of $\text{PGL}(2, q)$ has at least one of the following properties.*

- (D1) *Y fixes a point of $\text{PG}(1, q)$.*
- (D2) *Y is cyclic or dihedral of order at most $2(q + 1)$.*
- (D3) *Y is isomorphic to A_4 , S_4 or A_5 .*
- (D4) *$Y \cong \text{PSL}(2, q_0)$ or $Y \cong \text{PGL}(2, q_0)$ for some $q_0 = p^e$ with $e \mid f$; and if $q_0 = q$, then $Y \supseteq \text{PSL}(2, q)$.*

Sources and supplements. For odd q this is read off from Theorem 2 of Cameron, Omid and Tayfeh-Rezaie [COTR06], which lists the subgroups of $\text{PGL}(2, q)$: cyclic and dihedral groups of order dividing $2(q \pm 1)$ (its items (i)–(v); (D2)); A_4 , S_4 , A_5 (item (vi); (D3)); $\text{PSL}(2, p^m)$ and $\text{PGL}(2, p^m)$ with $m \mid f$ (items (vii), (viii); (D4)); elementary abelian p -groups and their extensions by cyclic groups of order dividing $q - 1$ (items (ix), (x)). For even q we have $\text{PGL}(2, q) = \text{PSL}(2, q)$, and the statement is Dickson's theorem on the subgroups of $\text{PSL}(2, 2^f)$, usually cited in the form of Huppert [Hup67, Hauptsatz II.8.27]; for this case we consulted neither [Dic01] nor [Hup67], and we use the statement as a standard result. (For $q = 4, 8, 16$ it is consistent with the complete lists of conjugacy classes of

subgroups computed in one of the tests of Section 8, compared with (D1)–(D4) by orders and element orders.)

The groups of the remaining kind fix a point: a p -subgroup $U_0 \neq 1$ lies in a Sylow p -subgroup, a conjugate of U , whose non-identity elements all fix the same single point (Lemma 2.1(i)); so U_0 fixes exactly one point, and so does every subgroup normalising U_0 . If $q_0 = q$ in (D4), then $|\mathrm{PGL}(2, q) : Y| \leq \gcd(2, q-1)$, and a subgroup of index 2 is normal and contains the derived subgroup $\mathrm{PSL}(2, q)$, which itself has index 2. The types overlap (for instance, cyclic groups of order dividing $q-1$ fix two points); we only use that every subgroup has at least one of them. $\square$

Lemma 2.4 (large subgroups). *Every subgroup Y of $\mathrm{PGL}(2, q)$ with $|Y| > q^2 - 1$ contains $\mathrm{PSL}(2, q)$.*

Proof. We go through Lemma 2.3. (D1): $|Y| \leq |B_\infty| = q(q-1) < q^2 - 1$. (D2): $|Y| \leq 2(q+1) \leq q^2 - 1$ as $q \geq 3$. (D3): $|Y| \leq 60 \leq q^2 - 1$ if $q \geq 8$. For $q = 7$, $5 \nmid |\mathrm{PGL}(2, 7)| = 336$, so $|Y| \leq 24 < 48$. For $q = 5$, $q^2 - 1 = 24 \geq |S_4|$, and a subgroup A_5 has index 2 in $\mathrm{PGL}(2, 5)$, so it is $\mathrm{PSL}(2, 5)$. For $q = 4$, $\mathrm{PGL}(2, 4) \cong A_5$ has no subgroup S_4 , $|A_4| = 12 < 15$, and a subgroup A_5 is all of $\mathrm{PGL}(2, 4)$. (D4) with $q_0 < q$: then $q_0 \leq q^{1/2}$ and $|Y| \leq q_0(q_0^2 - 1) \leq q^{1/2}(q-1) < q^2 - 1$; with $q_0 = q$, $Y \supseteq \mathrm{PSL}(2, q)$. $\square$

Let $W = \mathbb{F}_q^2$ (row vectors) and $Z = \mathbb{F}_q^\times I$, the scalars of $\mathrm{GL}(2, q)$. For $H \leq \mathrm{GL}(2, q)$ write $\bar{H} = HZ/Z \leq \mathrm{PGL}(2, q)$.

Lemma 2.5 (lifting). *Let $H \leq \mathrm{GL}(2, q)$.*

- (a) *If $\bar{H} \supseteq \mathrm{PSL}(2, q)$, then $H \supseteq \mathrm{SL}(2, q)$.*
- (b) *Let $q_0 \geq 4$ be a power of p , and suppose that the derived subgroup of $\bar{H}$ is isomorphic to $\mathrm{PSL}(2, q_0)$. Then every H -invariant additive subgroup $N \neq W$ of W has order at most q^2/q_0^2 .*

Proof. (a) The image of H' in $\mathrm{PGL}(2, q)$ is $\bar{H}' \supseteq \mathrm{PSL}(2, q)' = \mathrm{PSL}(2, q)$, so $H'Z \supseteq \mathrm{SL}(2, q)$, and $\mathrm{SL}(2, q) = \mathrm{SL}(2, q)' \leq (H'Z)' = H'' \leq H$.

(b) Let $L = H' \leq \mathrm{SL}(2, q)$. Its image in $\mathrm{PGL}(2, q)$ is $\bar{H}' \cong \mathrm{PSL}(2, q_0)$, and $L \cap Z \leq \mathrm{SL}(2, q) \cap Z = \{\pm I\}$. If $p = 2$, then $L \cap Z = 1$ and $|L| = q_0(q_0^2 - 1)$. If p is odd, L has even order and contains an involution; the only involution of $\mathrm{SL}(2, q)$ is $-I$ (an involution of $\mathrm{SL}(2, q)$ is diagonalisable with eigenvalues ± 1 of product 1), so $|L| = 2|\mathrm{PSL}(2, q_0)| = q_0(q_0^2 - 1)$. In both cases the p -part of $|L|$ is q_0 . If $g \in \mathrm{SL}(2, q)$ fixes a vector $v \neq 0$, then both eigenvalues of g are 1, so $(g - I)^2 = 0$ and $g^p = I$; hence the stabiliser L_v is a p -group, $|L_v| \leq q_0$, and every L -orbit on $W \setminus \{0\}$ has length at least $q_0^2 - 1$. Let $\beta(v, w) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(v_1w_2 - v_2w_1)$. This $\mathbb{F}_p$ -bilinear form is non-degenerate (if $\beta(v, \cdot) = 0$, then the $\mathbb{F}_q$ -linear form $w \mapsto v_1w_2 - v_2w_1$ has image in the kernel of the trace, hence is zero, and $v = 0$) and invariant under $\mathrm{SL}(2, q)$. For N as in (b), $N^\perp = \{w : \beta(N, w) = 0\}$ is an L -invariant subgroup with $|N||N^\perp| = q^2$, and $N^\perp \neq 0$ as $N \neq W$. It is the union of $\{0\}$ and of L -orbits, so $|N^\perp| \geq q_0^2$ and $|N| \leq q^2/q_0^2$. $\square$

3 Perspectivities and duality

Lines of $\mathrm{PG}(2, q)$ are the 2-dimensional subspaces of V ; for $0 \neq u \in V$ let $L_u = \{v \in V : vu^\top = 0\}$, so that every line is L_u for some u , unique up to a scalar. For a line l let P_l be its stabiliser and $S_l = S \cap P_l$. We call the elements of $K_d = \ker \varphi_d$, that is, the elements of P fixing every line through the point d , the *perspectivities with centre d* .

Lemma 3.1 (the kernel K_d ; for $q = 4$ this is [Fer26, Lemma 3.1]). *Let d be a point. Then $|K_d| = q^2(q-1)$, and every non-identity element of K_d is of one of two kinds:*

- (a) an elation: it fixes exactly the points of one line a through d (its axis) and exactly the lines through d ; for each line a through d there are $q - 1$ non-identity elations with centre d and axis a ;
- (b) a homology: it fixes exactly d and the points of one line b not through d (its axis), and exactly the lines through d and the line b ; for each line b not through d there are $q - 2$ non-identity homologies with centre d and axis b .

Proof. Let $d = \langle e_1 \rangle$. An element of K_d induces a scalar on V/d , so it has a unique representative

$$m_{\alpha,w}: \quad e_1 \mapsto \alpha e_1, \quad e_2 \mapsto e_2 + w_2 e_1, \quad e_3 \mapsto e_3 + w_3 e_1, \quad \alpha \in \mathbb{F}_q^\times, \quad w = (w_2, w_3) \in \mathbb{F}_q^2,$$

and all these matrices represent elements of K_d ; so $|K_d| = q^2(q - 1)$. We have $vm_{\alpha,w} = (\alpha v_1 + w_2 v_2 + w_3 v_3, v_2, v_3)$, so a point $\langle v \rangle \neq d$ is fixed if and only if $(\alpha - 1)v_1 + w_2 v_2 + w_3 v_3 = 0$. A line not through d is L_u with $u = (1, u_2, u_3)$, and substituting $v_1 = (v'_1 - w_2 v'_2 - w_3 v'_3)/\alpha$ for $v' = vm_{\alpha,w}$ shows $L_u m_{\alpha,w} = L_{u'}$ with $u' = (1, \alpha u_2 - w_2, \alpha u_3 - w_3)$; so L_u is fixed if and only if $(\alpha - 1)(u_2, u_3) = w$. For $\alpha = 1$ and $w \neq 0$ the fixed points are those of the line $w_2 v_2 + w_3 v_3 = 0$ through d , and no line off d is fixed: an elation, determined by its axis up to the $q - 1$ non-zero vectors w defining it. For $\alpha \neq 1$ the fixed points other than d form the line L_u with $u = (1, w_2/(\alpha - 1), w_3/(\alpha - 1))$, which is also the only fixed line off d : a homology with axis L_u ; given the axis, w is determined by $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$. $\square$

Corollary 3.2. (a) If d is a point not on the line m , then $|K_d \cap P_m| = q - 1$. (b) If $d \neq z$ are points, then $|K_d \cap P_z| = q(q - 1)$. (c) If d lies on the line m , then $K_d \leq P_m$.

Proof. (a) An elation with centre d fixes only lines through d , and a homology with centre d fixes m if and only if m is its axis: $1 + (q - 2)$ elements. (b) An elation with centre d fixes z if and only if its axis is the line dz ; a homology with centre d fixes z if and only if z lies on its axis, a line through z other than dz : $1 + (q - 1) + q(q - 2)$ elements. (c) holds by definition. $\square$

Lemma 3.3 (duality). Let π map the point $\langle u \rangle$ to the line L_u , and let $g^\tau = (g^{-1})^\tau$ for $g \in \text{GL}(3, q)$; this induces an automorphism τ of P of order 2. Then π is a bijection from the points onto the lines, $\langle v \rangle$ lies on $\pi(\langle u \rangle)$ if and only if $\langle u \rangle$ lies on $\pi(\langle v \rangle)$, and $\pi(x)g = \pi(xg^\tau)$ for every point x and every $g \in P$. Consequently, for $S \leq P$ and points d, z :

- (a) the bijection $\langle u \rangle \mapsto \pi(\langle u \rangle)$ from the points of the line $\pi(d)$ onto the pencil $[d]$ carries the group induced by $(S^\tau)_{\pi(d)}$ on the points of $\pi(d)$ onto the group $\varphi_d(S_d)$ induced by S_d on $[d]$, and the group $\text{PSL}(2, q)$ of the line $\pi(d)$ onto $\text{PSL}(V/d)$;
- (b) S fixes a line through z if and only if S^τ fixes a point of the line $\pi(z)$, and $S \leq P_z$ if and only if $S^\tau \leq P_{\pi(z)}$.

Here, for a line l , the group induced by P_l on the points of l is $\text{PGL}(l) \cong \text{PGL}(2, q)$, and the group $\text{PSL}(2, q)$ of the line l is the image of $\text{SL}(l)$ in it, that is, its derived subgroup.

Proof. If $vu^\tau = 0$, then $(vg)(ug^\tau)^\tau = vgg^{-1}u^\tau = 0$; so $L_u g = L_{ug^\tau}$. Incidence is symmetric because $vu^\tau = uv^\tau$. For (a), let $g \in S_d$; then $\pi(d)g^\tau = \pi(dg) = \pi(d)$, so $(S^\tau)_{\pi(d)} = (S_d)^\tau$, and for a point $\langle u \rangle$ of $\pi(d)$ we have $\pi(\langle u \rangle g^\tau) = \pi(\langle u \rangle)g$. So the bijection intertwines the two actions. Applied to $S = P$, it carries the group $\text{PGL}(2, q)$ induced on the points of $\pi(d)$ onto $\text{PGL}(V/d)$ acting on $[d]$, and hence the derived subgroups onto each other. (b) S fixes $\pi(\langle u \rangle)$ if and only if $\langle u \rangle g^\tau = \langle u \rangle$ for all $g \in S$, and $z \in \pi(\langle u \rangle)$ if and only if $\langle u \rangle \in \pi(z)$. The second statement follows in the same way. $\square$

Definition 3.4. A subgroup $S \leq P$ is good on the line l if the group induced by S_l on the $q + 1$ points of l centralises a non-identity element of the group $\text{PSL}(2, q)$ of l .

By Lemma 3.3(a), S is good at the point d if and only if S^τ is good on the line $\pi(d)$.

4 Reduction to point and line stabilisers

In this section $q \geq 5$, $d_q = \gcd(3, q-1)$, and $Z_0 = Z(\mathrm{SL}(3, q))$, of order d_q . Since $\mathrm{SL}(3, q)$ is perfect, every maximal subgroup $\tilde{Y}$ of $\mathrm{SL}(3, q)$ contains Z_0 (otherwise $\tilde{Y}Z_0 = \mathrm{SL}(3, q)$ and $\mathrm{SL}(3, q)/\tilde{Y}$ would be abelian and non-trivial); so the maximal subgroups Y of $\mathrm{PSL}(3, q)$ are the images of the maximal subgroups $\tilde{Y}$ of $\mathrm{SL}(3, q)$, and $|\tilde{Y}| = d_q|Y|$.

Input R1 (maximal subgroups of $\mathrm{SL}(3, q)$). Let $q = p^f \geq 5$. Every maximal subgroup of $\mathrm{SL}(3, q)$ is isomorphic to one of the following groups, under the conditions stated:

- (C1) $E_q^2 : \mathrm{GL}(2, q)$;
- (C2) $(q-1)^2 : S_3$;
- (C3) $(q^2 + q + 1) : 3$;
- (C5) $\mathrm{SL}(3, q_0) \cdot \gcd\left(\frac{q-1}{q_0-1}, 3\right)$, where $q = q_0^r$ with r prime;
- (C6) $3_+^{1+2} : Q_8 \cdot \frac{\gcd(q-1, 9)}{3}$, where $q = p \equiv 1 \pmod{3}$;
- (C8) $d_q \times \mathrm{SO}(3, q)$ for odd q , and $\gcd(q_0-1, 3) \times \mathrm{SU}(3, q_0)$, where $q = q_0^2$;
- (S) $d_q \times \mathrm{PSL}(2, 7)$, where $q = p \equiv 1, 2, 4 \pmod{7}$; and $3.A_6$, where $q = p \equiv 1, 4 \pmod{15}$, or $q = p^2$ with $p \equiv 2, 3 \pmod{5}$, $p \neq 3$.

The labels are those of Aschbacher's classes [Asc84], in the form of [KL90, Chapter 4]. The members of (C1) are the stabilisers of the points and of the lines of $\mathrm{PG}(2, q)$; the members of type $\mathrm{SL}(3, q_0).e$ in (C5), and of type $\gcd(q_0-1, 3) \times \mathrm{SU}(3, q_0)$ in (C8), are the normalisers in $\mathrm{SL}(3, q)$ of the subgroups conjugate in $\mathrm{GL}(3, q)$ to $\mathrm{SL}(3, q_0)$ (the matrices with entries in $\mathbb{F}_{q_0}$ and determinant 1) and to $\mathrm{SU}(3, q_0)$, the matrices of determinant 1 that preserve the Hermitian form $\sum_i x_i y_i^{q_0}$ on $\mathbb{F}_q^3$.

The list is due to Mitchell [Mit11] (odd q) and Hartley [Har25] (even q), and is tabulated in [BHRD13]. We did not consult [Mit11, Har25, BHRD13]. We read the list as transcribed by Liu and Yang [LY22, Lemma 2.2], who state that they take it from pages 378 and 379 of [BHRD13]; that lemma gives the isomorphism types and the conditions above, and not the class labels. The list agrees with Theorem 19 of Gill [Gil07], which lists the maximal subgroups of $\mathrm{PSL}(3, q)$ “among” ten types with necessary conditions, and, for $q = 5, 7, 8$, with the lists of maximal subgroups of $L_3(5)$, $L_3(7)$ and $L_3(8)$ in the online ATLAS [W⁺26]. The description of the classes C1, C5 and C8 by subspaces, subfields and forms is that of [AB15, Table 6]. The proof below uses only the orders of the members of (C2), (C3), (C6), (C8) and (S), and the structure of the members of (C1), (C5) and (C8); see also Remark 4.3. The rows (C5) and (C8), on which the two tight cases of the proof of Proposition 4.2 depend, have not been compared with [BHRD13] itself.

Input R2 (automorphisms). For $q_0 \geq 3$, every automorphism of $\mathrm{PSL}(3, q_0)$ is induced by the conjugation by an element of $\mathrm{GL}(3, q_0)$, composed with a field automorphism (applied to the entries) and possibly with the graph automorphism $s \mapsto (s^{-1})^T$ [SvdW28]; and every automorphism of $\mathrm{PSU}(3, q_0)$, for the form $\sum_i x_i y_i^{q_0}$, is induced by the conjugation by an element of $\mathrm{GU}(3, q_0)$ composed with a power of the automorphism $x \mapsto x^p$ of $\mathbb{F}_{q_0^2}$ applied to the entries [Ste60]. (Input R2 is used only in Lemma 4.1.)

Lemma 4.1 (two normalisers). *Let $q = q_0^2$ with $q_0 = p^{f_0}$, let Z be the group of scalar matrices of $\mathrm{GL}(3, q)$, and let $\tilde{L} \leq \mathrm{SL}(3, q)$ be*

- (a) $\text{SL}(3, q_0)$, where $q_0 \geq 5$ and $q_0 \equiv 2 \pmod{3}$; or
 (b) $\text{SU}(3, q_0)$, where $q_0 \equiv 1 \pmod{3}$ (so $q_0 \geq 4$).

Let $L = \tilde{L}Z/Z \leq P$. Then the normaliser $N_P(L)$ is the image of $Z\text{GL}(3, q_0)$ in case (a), and of $Z\text{GU}(3, q_0)$ in case (b); it is isomorphic to $\text{PGL}(3, q_0)$, resp. $\text{PGU}(3, q_0)$, and $|N_P(L)| = |\text{SL}(3, q_0)|$, resp. $|\text{SU}(3, q_0)|$. The same holds for every conjugate of $\tilde{L}$ under $\text{GL}(3, q)$.

Proof. The congruence conditions give $Z(\tilde{L}) = 1$, so $\tilde{L} \cong \text{PSL}(3, q_0)$, resp. $\text{PSU}(3, q_0)$, is simple and perfect. Let D be the group of the diagonal matrices $\text{diag}(a, b, (ab)^{-1})$ with $a, b \in \mathbb{F}_{q_0}^\times$, resp. with $a^{q_0+1} = b^{q_0+1} = 1$; then $D \leq \tilde{L}$, and the three characters $a, b, (ab)^{-1}$ of D are pairwise distinct. So the D -invariant subspaces of $V \otimes \overline{\mathbb{F}}_q$ are the sums of coordinate lines. The permutation matrix of the 3-cycle $e_1 \mapsto e_2 \mapsto e_3 \mapsto e_1$ lies in $\tilde{L}$ and permutes the coordinate lines transitively. Hence $\tilde{L}$ is absolutely irreducible, and its centraliser in $\text{GL}(3, q)$ is Z .

Let $x \in \text{GL}(3, q)$ with $xZ \in N_P(L)$. Then x normalises $\tilde{L}Z$ and hence $(\tilde{L}Z)' = \tilde{L}$, and $s \mapsto x^{-1}sx$ is an automorphism of $\tilde{L}$. By Input R2 there are $g \in \text{GL}(3, q_0)$, resp. $g \in \text{GU}(3, q_0)$, and an automorphism β as described there (field automorphisms, the graph automorphism, powers of $x \mapsto x^p$), such that $x^{-1}sx = g^{-1}\beta(s)g$ for all $s \in \tilde{L}$. Put $x_1 = xg^{-1}$; then $x_1^{-1}sx_1 = \beta(s)$ for all $s \in \tilde{L}$. We show $\beta = 1$.

(a) For $t, u \in \mathbb{F}_{q_0}$ let $C(t, u)$ be the matrix with rows $e_2, e_3, (1, -u, t)$. Its characteristic polynomial is $X^3 - tX^2 + uX - 1$, so $C(t, u) \in \text{SL}(3, q_0)$, $\text{tr } C(t, u) = t$ and $\text{tr } C(t, u)^{-1} = u$. Since $\beta(s)$ is conjugate to s , $\text{tr } \beta(s) = \text{tr } s$ for all $s \in \tilde{L}$. If β is a non-trivial field automorphism σ , choose t with $t^\sigma \neq t$: then $\text{tr } \beta(C(t, 0)) = t^\sigma \neq t$. If β is the graph automorphism composed with a field automorphism σ (possibly trivial), then $\text{tr } \beta(C(0, 1)) = (\text{tr } C(0, 1)^{-1})^\sigma = 1 \neq 0 = \text{tr } C(0, 1)$. So $\beta = 1$.

(b) Let β be the p^i -th power map on the entries, $0 < i < 2f_0$, and let $v = e_1x_1$. For $d = \text{diag}(a, b, (ab)^{-1}) \in D$ we get $v\beta(d) = e_1dx_1 = av$. The matrices $\beta(d) = \text{diag}(a^{p^i}, b^{p^i}, (ab)^{-p^i})$, $d \in D$, have three pairwise distinct characters (the p^i -th power map is a bijection of the group μ of (q_0+1) -th roots of unity, as $p \nmid q_0+1$), so v spans a coordinate line. Hence $a = a^{p^i}$ for all $a, b \in \mu$, or $a = b^{p^i}$ for all a, b , or $a = (ab)^{-p^i}$ for all a, b . The second fails for $b = 1 \neq a$, the third for $a = 1 \neq b$; so $p^i \equiv 1 \pmod{q_0+1}$. But for $0 < i \leq f_0$ we have $1 < p^i \leq q_0$, and for $f_0 < i < 2f_0$ we have $p^i = q_0p^{i-f_0} \equiv -p^{i-f_0} \pmod{q_0+1}$ with $1 < p^{i-f_0} + 1 < q_0 + 1$. So $i = 0$ and $\beta = 1$.

Hence x_1 centralises $\tilde{L}$, so $x_1 \in Z$ and $x \in Zg$. Conversely, $Z\text{GL}(3, q_0)$, resp. $Z\text{GU}(3, q_0)$, normalises L . The image of $Z\text{GL}(3, q_0)$ in P is $\text{GL}(3, q_0)/\mathbb{F}_{q_0}^\times \cong \text{PGL}(3, q_0)$, of order $|\text{SL}(3, q_0)|$; the image of $Z\text{GU}(3, q_0)$ is $\text{GU}(3, q_0)/Z(\text{GU}(3, q_0)) \cong \text{PGU}(3, q_0)$, of order $|\text{SU}(3, q_0)|$. For a conjugate $\tilde{L}^h$ the normaliser is $N_P(L)^h$. $\square$

Proposition 4.2 (reduction). *Let $q \geq 5$ and let $S \leq P = \text{PGL}(3, q)$ with $|S| > M$. Then S contains $\text{PSL}(3, q)$, or S fixes a point, or S fixes a line.*

Proof. Let $S_0 = S \cap \text{PSL}(3, q)$. It is normal in S , and $|S : S_0| \leq |P : \text{PSL}(3, q)| = d_q \leq 3$, so $|S_0| > M/3$. Suppose that $S_0 \neq \text{PSL}(3, q)$, and let Y be a maximal subgroup of $\text{PSL}(3, q)$ containing S_0 , with preimage $\tilde{Y}$ in $\text{SL}(3, q)$. Recall $M \geq q^2(q^2 - 1)$ (Lemma 2.1).

Case $\tilde{Y}$ in (C1). Then S_0 fixes a point or a line. Let F be the set of points fixed by S_0 . As S normalises S_0 , it permutes F and the set of lines fixed by S_0 . If F contained three non-collinear points, S_0 would lie in the group of elements fixing them, which has order $(q-1)^2 < M/3$; so the points of F are collinear. If $|F| = 1$, S fixes this point; if $|F| \geq 2$, S fixes the line containing F . If $F = \emptyset$, then S_0 fixes a line, and only one, because two fixed lines would meet in a fixed point; so S fixes this line.

Other cases. We show that they cannot occur. In each of them $|S| \leq d_q|S_0| \leq d_q|Y| = |\tilde{Y}|$.

- (C2), (C3): $|\tilde{Y}| = 6(q-1)^2$, resp. $3(q^2 + q + 1)$, both less than $q^2(q^2 - 1) \leq M$.

- (C6): $q = p \equiv 1 \pmod{3}$ forces $q \geq 7$, and $|\tilde{Y}| \leq 648 < 4704 \leq M$.
- (C8), orthogonal: $|\tilde{Y}| = d_q q(q^2 - 1) \leq 3q(q^2 - 1) < q^2(q^2 - 1) \leq M$.
- (S): $|\tilde{Y}| \leq \max\{3 \cdot 168, 1080\} = 1080$, and for $q \geq 5$ the conditions give $q \geq 11$, so $M \geq q^2(q^2 - 1) > 1080$.
- (C5) with $r \geq 3$: then $q \geq 8$, and $|\tilde{Y}| \leq 3|\mathrm{SL}(3, q_0)| < 3q_0^8 \leq 3q^{8/3} < q^4 - q^2 \leq M$.
- (C5) with $r = 2$, $q = q_0^2$: $|\tilde{Y}| = \gcd(q_0 + 1, 3)|\mathrm{SL}(3, q_0)|$, and

$$q^2(q^2 - 1) - |\mathrm{SL}(3, q_0)| = q_0^8 - q_0^4 - q_0^3(q_0^3 - 1)(q_0^2 - 1) = q_0^3(q_0 + 1)^2(q_0 - 1) > 0. \quad (4)$$

So $|\tilde{Y}| < M$ unless $q_0 \equiv 2 \pmod{3}$. In that case $q_0 \geq 5$ and $d_q = 3$. The group $\tilde{Y}$ has a normal subgroup $\tilde{L}$ conjugate in $\mathrm{GL}(3, q)$ to $\mathrm{SL}(3, q_0)$, of index 3, and it contains Z_0 ; as $\mathbb{F}_{q_0}^\times$ contains no element of order 3, $Z_0 \cap \tilde{L} = 1$, so $\tilde{Y} = \tilde{L} \times Z_0$ and $Y = L := \tilde{L}Z_0/Z_0 \cong \tilde{L}$, a simple group of order $|\mathrm{SL}(3, q_0)| < M$. Hence $|L : S_0| < 3|L|/M < 3$, and $S_0 = L$, since a simple group has no subgroup of index 2. So S normalises L , and $|S| \leq |\mathrm{SL}(3, q_0)| < M$ by Lemma 4.1(a) and (4): a contradiction.

- (C8), unitary, $q = q_0^2$: $|\tilde{Y}| = \gcd(q_0 - 1, 3)|\mathrm{SU}(3, q_0)|$, and

$$q^2(q^2 - 1) - |\mathrm{SU}(3, q_0)| = q_0^8 - q_0^4 - q_0^3(q_0^3 + 1)(q_0^2 - 1) = q_0^3(q_0 - 1)^2(q_0 + 1) > 0. \quad (5)$$

So $|\tilde{Y}| < M$ unless $q_0 \equiv 1 \pmod{3}$. In that case $q_0 \geq 4$, $d_q = 3$, and $Z(\mathrm{SU}(3, q_0)) = 1$; the scalars ωI with $\omega^3 = 1 \neq \omega$ do not lie in $\mathrm{SU}(3, q_0)$, since $\omega \in \mathbb{F}_{q_0}$ and so $\omega^{q_0+1} = \omega^2 \neq 1$. As before $\tilde{Y} = \tilde{L} \times Z_0$ with $\tilde{L}$ conjugate to $\mathrm{SU}(3, q_0)$, $Y = L \cong \tilde{L}$ is simple of order $|\mathrm{SU}(3, q_0)| < M$, $S_0 = L$, and $|S| \leq |\mathrm{SU}(3, q_0)| < M$ by Lemma 4.1(b) and (5): a contradiction.

This exhausts Input R1. □

Remark 4.3 (an independent source for the part of Input R1 that is used). The proof uses Input R1 only for maximal subgroups Y of $\mathrm{PSL}(3, q)$ with $|Y| > M/3 \geq q^2(q^2 - 1)/3$. For $q \geq 5$ such a Y is *large* in the sense of Alavi and Burness, $|Y|^3 \geq |\mathrm{PSL}(3, q)|$, because $(q^2 - 1)^3 \geq 27q^2$ and $|\mathrm{PSL}(3, q)| < q^8$. By Aschbacher's theorem [Asc84] and [AB15, Theorem 4, Proposition 4.7 and Table 7], a large maximal subgroup of $L_3(q)$, $q \geq 5$, lies in $\mathcal{C}_1$, $\mathcal{C}_5$ or $\mathcal{C}_8$, or in $\mathcal{C}_2$ (only for $q \in \{5, 8, 9, 11\}$), or in $\mathcal{C}_3$ (only for $q = 5$); $\mathcal{C}_6$ and $\mathcal{S}$ contain large maximal subgroups of $L_3(q)$ only for $q = 4$. The orders of the members of $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_5$ given in [AB15, Lemmas 4.10, 4.11 and 4.13] agree with Input R1. So the completeness that the proof needs does not depend on the rows (C6) and (S) of the transcription.

Lemma 4.4. *If $S \leq P$ contains $\mathrm{PSL}(3, q)$, then S is good at no point.*

Proof. For $d = \langle e_1 \rangle$ the matrices $\mathrm{diag}(\det A^{-1}, A)$, $A \in \mathrm{GL}(2, q)$, lie in $\mathrm{SL}(3, q)$, fix d and induce A on $V/d = \langle \bar{e}_2, \bar{e}_3 \rangle$. So $\varphi_d(S_d) = \mathrm{PGL}(V/d)$, whose centre is trivial; it centralises no non-identity element. As $\mathrm{PSL}(3, q)$ is transitive on the points, this holds for every point. □

Remark 4.5 ($q = 4$). For $q = 4$, Proposition 4.2 is false: $\mathrm{PSL}(3, 4)$ contains subgroups isomorphic to A_6 , of order $360 > M = 240$, which fix no point and no line. The first case of the proof, and Sections 5 and 6, are valid for $q = 4$ as well; the subgroups A_6 are treated in [Fer26, proof of Proposition 3.3].

5 Subgroups fixing a line

In Sections 5 and 6, $q \geq 4$. Let $m = \langle e_1, e_2 \rangle$ be the line $v_3 = 0$, and $W = \mathbb{F}_q^2$. Every element of P_m has a unique representative

$$g(A, b) = \begin{pmatrix} A & 0 \\ b & 1 \end{pmatrix} \quad (A \in \mathrm{GL}(2, q), b \in W),$$

that is, $e_1, e_2 \mapsto (e_i A, 0)$ and $e_3 \mapsto (b, 1)$. It maps the point $\langle (w, 1) \rangle$ off m (an *affine point*) to $\langle (wA + b, 1) \rangle$, and the point $\langle (w, 0) \rangle$ of m , which we identify with the 1-space $D = \langle w \rangle$ of W , to DA . So $P_m \cong W \rtimes \text{GL}(2, q)$ is the affine group of W . The elements fixing every point of m are the $g(\lambda I, b)$; they form a group of order $q^2(q-1)$, consisting of the translations $t_b = g(I, b)$, which are the elations with axis m , and of the homologies with axis m .

For $X \leq P_m$ let

$$N = \{b \in W : t_b \in X\}, \quad H = \{A \in \text{GL}(2, q) : g(A, b) \in X \text{ for some } b\}.$$

Then $|X| = |N||H|$; N is an H -invariant additive subgroup of W , because $g(A, b)^{-1}t_n g(A, b) = t_{nA}$; and $\bar{H} = HZ/Z \leq \text{PGL}(2, q)$ is the group induced by X on the points of m , where $Z = \mathbb{F}_q^\times I$ as in Section 2.

Lemma 5.1. *Let D be a point of m , e a non-zero vector of D , and (e, e') a basis of W . For $A \in \text{GL}(2, q)$ with $DA = D$ write $eA = \lambda_A e$ and $e'A = \gamma_A e + \beta_A e'$.*

- (a) *The lines through D are m and the q affine lines $a + D$ ($a \in W$). Labelling $a + D$ by the e' -coordinate of a , and m by ∞ , is a projective coordinate on $[D]$. An element $g(A, b)$ fixes D if and only if $DA = D$, and then it acts on $[D]$ as $c \mapsto \beta_A c + b'$, where b' is the e' -coordinate of b . So $\varphi_D(X_D)$ fixes m , $\varphi_D(t_n)$ is the translation by the e' -coordinate of n , and $\varphi_D(g(A, b))$ is a p -element if and only if $\beta_A = 1$.*
- (b) *Let $l = a + D$ be an affine line. Its points are $\langle (a + se, 1) \rangle$, $s \in \mathbb{F}_q$, and D , and s (with $D = \infty$) is a projective coordinate on l . An element $g(A, b)$ fixes l if and only if $DA = D$ and $aA + b - a \in D$, and then it acts on l as $s \mapsto \lambda_A s + \kappa$ for some $\kappa \in \mathbb{F}_q$. For $n = n_e e \in D$, t_n fixes l and acts as $s \mapsto s + n_e$; for A with $DA = D$, the element $g(A, a - aA)$ fixes l and acts as $s \mapsto \lambda_A s$. The element induced by $g(A, b)$ on l is a p -element if and only if $\lambda_A = 1$.*

Proof. (a) The line through $D = \langle (e, 0) \rangle$ and an affine point $\langle (a, 1) \rangle$ is $\langle (e, 0), (a, 1) \rangle$, whose affine points are the $\langle (a + se, 1) \rangle$. In V/D , with the basis formed by the images of $(e', 0)$ and $(0, 0, 1)$, the line $a + D$ is the 1-space spanned by the image $(a_2, 1)$ of $(a, 1)$, where $a = a_1 e + a_2 e'$, and m is spanned by $(1, 0)$; this is the standard coordinate. As $(a_1 e + a_2 e')A + b = (a_1 \lambda_A + a_2 \gamma_A)e + a_2 \beta_A e' + b$, the element $g(A, b)$ maps the line labelled a_2 to the line labelled $\beta_A a_2 + b'$. The last statement is Lemma 2.2. (b) In the basis $(e, 0), (a, 1)$ of l the point $\langle (a + se, 1) \rangle$ has coordinates $(s, 1)$ and D has $(1, 0)$. An element fixing l fixes $l \cap m = D$, and maps $(a, 1)$ to $(aA + b, 1)$, which lies on l if and only if $aA + b = a + \kappa e$ for some κ ; then $(a + se)A + b = a + (\lambda_A s + \kappa)e$. The two special cases are $\kappa = n_e$, $\lambda_A = 1$, and $\kappa = 0$. The last statement is Lemma 2.2 again. $\square$

Proposition 5.2. *Let $X \leq P_m$ with $|X| > M$. Then X is good at no point.*

Proof. (i) *Affine points.* Let $d \notin m$. If X were good at d , then $|\varphi_d(X_d)| \leq c_q$, by the definition of c_q ; and the kernel $X_d \cap K_d$ has order at most $|K_d \cap P_m| = q - 1$ (Corollary 3.2(a)). As X permutes the q^2 affine points, $|X| = |d^X| |X_d| \leq q^2(q-1)c_q = M$, a contradiction.

(ii) *X fixes a point z of m .* The kernel of φ_z on X has order at most $|K_z| = q^2(q-1)$, so $|\varphi_z(X)| > M/(q^2(q-1)) = c_q$, and $\varphi_z(X_z) = \varphi_z(X)$ lies in no centraliser of a non-identity element of $\text{PSL}(V/z)$: X is not good at z . Let $d \in m$, $d \neq z$. Then X_d fixes d, z and $m = dz$, so $\varphi_d(X_d)$ lies in the stabiliser of m in $\text{PGL}(V/d)$. If X were good at d , then $|\varphi_d(X_d)| \leq q$ by Lemma 2.2; the kernel $X_d \cap K_d$ has order at most $|K_d \cap P_z| = q(q-1)$ (Corollary 3.2(b)); and $|d^X| \leq q$, since X permutes the q points of m other than z . So $|X| \leq q^3(q-1) < q^2(q-1)(q+1) \leq M$, a contradiction. Affine points are covered by (i).

(iii) *X fixes no point of m , that is, $\bar{H}$ fixes no point of $\text{PG}(1, q)$.* Let D be a point of m . First $N \neq 0$, since otherwise $|X| = |H| \leq |\text{GL}(2, q)| = q(q-1)^2(q+1) < M$. Next $N \not\subseteq D$: otherwise, for

$A \in H$ moving D , $N = NA \subseteq D \cap DA = 0$. Choose $n \in N$ outside D . Then $t_n \in X_D$, and $\varphi_D(t_n)$ is a non-identity translation of $[D]$, a non-identity p -element (Lemma 5.1(a)). If X were good at D , then, by Lemma 2.2, $\varphi_D(X_D)$ would be a p -group, so $\beta_A = 1$ for every A in $H_D = \{A \in H : DA = D\}$, the set of linear parts of the elements of X_D . A scalar $\lambda I \in H$ lies in H_D and has $\beta_{\lambda I} = \lambda$; hence $H \cap Z = 1$, and $|\bar{H}| = |H| = |X|/|N| > M/q^2 = (q-1)c_q \geq q^2 - 1$. By Lemmas 2.4 and 2.5(a), $H \supseteq \text{SL}(2, q)$. But the matrix with $e \mapsto \mu^{-1}e$, $e' \mapsto \mu e'$, for some $\mu \in \mathbb{F}_q^\times$, $\mu \neq 1$, lies in $\text{SL}(2, q) \cap H_D$ and has $\beta = \mu \neq 1$: a contradiction. Affine points are covered by (i). $\square$

6 Subgroups fixing a point

By duality (Lemma 3.3), subgroups fixing a point but no line through it correspond to subgroups fixing a line but no point on it, and goodness at points corresponds to goodness on lines. We therefore prove the following statement about lines.

Proposition 6.1. *Let $X \leq P_m$ with $|X| > M$, and suppose that X fixes no point of m . Then X is good on no line.*

Proof. We use the notation $N, H, \bar{H}$ of Section 5.

(i) *The line m .* We have $X_m = X$, and X induces $\bar{H}$ on the points of m , with kernel of order at most $q^2(q-1)$. So $|\bar{H}| > M/(q^2(q-1)) = c_q$, and $\bar{H}$ centralises no non-identity element of the group $\text{PSL}(2, q)$ of m .

(ii) *Step 1:* $N = W$. Suppose that $N \neq W$. Then $|N| \leq q^2/p$ and $|H| \leq (q-1)|\bar{H}|$, so

$$|\bar{H}| > \frac{M}{(q-1)q^2/p} = pc_q \geq 2(q+1):$$

for even q , $pc_q = 2(q+1)$, and for odd q , $pc_q \geq 3 \cdot 2(q-1) \geq 2(q+1)$. As $\bar{H}$ fixes no point of $\text{PG}(1, q)$ and $|\bar{H}| > 2(q+1)$, Lemma 2.3 leaves the types (D3) and (D4).

- $\bar{H} \supseteq \text{PSL}(2, q)$. Then $H \supseteq \text{SL}(2, q)$ (Lemma 2.5(a)), which is transitive on $W \setminus \{0\}$; so $N = 0$ or $N = W$. If $N = 0$, then $|X| = |H| \leq |\text{GL}(2, q)| < M$. So $N = W$, against the assumption.
- $\bar{H} \cong \text{PSL}(2, q_0)$ or $\text{PGL}(2, q_0)$ with $q_0 = p^e < q$, $e \mid f$, $q_0 \geq 4$; or $\bar{H} \cong A_5$ with $p = 2$ ($A_5 \cong \text{PSL}(2, 4)$, $q_0 = 4$) or $p = 5$ ($A_5 \cong \text{PSL}(2, 5)$, $q_0 = 5$). In each case the derived subgroup of $\bar{H}$ is isomorphic to $\text{PSL}(2, q_0)$ with $q_0 \geq 4$ a power of p , $q_0 \leq q$, and $|\bar{H}| \leq q_0(q_0^2 - 1)$. By Lemma 2.5(b), $|N| \leq q^2/q_0^2$, so

$$|X| = |N||H| \leq \frac{q^2}{q_0^2}(q-1)q_0(q_0^2 - 1) < q^2(q-1)q_0 \leq q^2(q-1)(q+1) \leq M,$$

a contradiction.

- $\bar{H} \cong \text{PSL}(2, q_0)$ or $\text{PGL}(2, q_0)$ with $q_0 \in \{2, 3\}$. Then $|\bar{H}| \leq 24$. For $p = 2$, $|\bar{H}| \leq 6 < 2(q+1)$. For $p = 3$, $q \geq 9$ and $pc_q \geq 3 \cdot 2(q-1) \geq 48$. Both contradict $|\bar{H}| > pc_q$.
- $\bar{H} \cong A_4$ or S_4 . If $p = 2$, the normal subgroup of order 4 of $\bar{H}$ is a 2-group, so $\bar{H}$ fixes a point (Lemma 2.3, supplement): excluded. If p is odd, then $q \geq 5$ and $pc_q \geq 3 \cdot 2(q-1) \geq 24 \geq |\bar{H}|$: a contradiction.
- $\bar{H} \cong A_5$ with $p \notin \{2, 5\}$. If $p \geq 7$, then $pc_q \geq 7 \cdot 2(q-1) \geq 84 > 60$. If $p = 3$, then $60 > 3c_q$ forces $c_q < 20$, that is, $q = 9$, as $c_{27} = 56$ and $c_q \geq q+1 \geq 82$ for $q \geq 81$. For $q = 9$: $M = 81 \cdot 8 \cdot 16 = 10368$, $|N| \leq 27$, and $|H| = 60|H \cap Z|$ with $|H \cap Z|$ dividing 8; so $|X| > M$ forces $|H \cap Z| = 8$, that is, $Z \leq H$. Then N is closed under multiplication by $\mathbb{F}_9^\times$, an $\mathbb{F}_9$ -subspace of W ; it is not W , and not a 1-space, since $\bar{H}$ fixes no point. So $N = 0$ and $|X| = |H| = 480 < M$, a contradiction.

Hence $N = W$.

(iii) *Step 2: the affine lines.* As $N = W$, the group X contains all translations and $X = \{g(A, b) : A \in H, b \in W\}$. Let $l = a + D$ be an affine line, D a point of m , and $H_D = \{A \in H : DA = D\}$. By Lemma 5.1(b), the group induced by X_l on l contains the q translations $s \mapsto s + c$, induced by the t_n with $n \in D$, and, for every $A \in H_D$, the map $s \mapsto \lambda_A s$, induced by $g(A, a - aA)$. This group fixes the point D of l . If X were good on l , it would be a p -group by Lemma 2.2, so $\lambda_A = 1$ for every $A \in H_D$. Scalars $\lambda I \in H$ lie in H_D and have $\lambda_{\lambda I} = \lambda$; so $H \cap Z = 1$ and $|\bar{H}| = |H| = |X|/q^2 > (q-1)c_q \geq q^2 - 1$. By Lemmas 2.4 and 2.5(a), $H \supseteq \text{SL}(2, q)$, and the matrix with $e \mapsto \mu e$, $e' \mapsto \mu^{-1}e'$ ($\mu \neq 1$) lies in H_D and has $\lambda = \mu \neq 1$: a contradiction. The lines are m and the affine lines, so X is good on no line. $\square$

Corollary 6.2. *Let $S \leq P$ fix a point z , with $|S| > M$. Then S is good at no point.*

Proof. If S fixes a line through z , then Proposition 5.2 applies, after conjugation in P (which is transitive on the lines and preserves goodness at points). Otherwise, by Lemma 3.3(b), the group $X = S^\tau$ fixes the line $\pi(z)$ and no point of it, and $|X| = |S| > M$. After conjugation in P we may apply Proposition 6.1: X is good on no line. By Lemma 3.3(a), S is good at no point. $\square$

7 Proofs of Theorems A and B

Proof of Theorem A. Let $S \leq P$ with $|S| > M$. If $q \geq 5$, then by Proposition 4.2 the group S contains $\text{PSL}(3, q)$, and is good at no point by Lemma 4.4; or S fixes a line, and is good at no point by Proposition 5.2 (after conjugation in P); or S fixes a point, and is good at no point by Corollary 6.2.

Let $q = 4$. Then $M = 240$ and $\text{PGL}(V/d) = \text{PSL}(V/d) \cong A_5$. The subgroups of A_5 of order at most 5 are abelian, and those of order 6, 10, 12, 60 have trivial centraliser in A_5 [Fer26, proof of Lemma 3.2]; so S is good at d if and only if $|\varphi_d(S_d)| \leq 5$. Hence Theorem A for $q = 4$ is [Fer26, Proposition 3.3]: if $|S| > 240$, then $|\varphi_d(S_d)| \geq 6$ for every point d . That proposition is proved in [Fer26, §4] by a written argument from the lists of maximal subgroups of $\text{PGL}(3, 4)$ and $\text{PSL}(3, 4)$ in the tables of Connor and Leemans [CL15], and independently by exhaustive computation [Fer26, §6]. $\square$

Proof of Theorem B. We verify the hypotheses of Lemma 1.1(b). The group $T = \text{PSL}(2, q)$ is non-abelian and simple as $q \geq 4$, and $\text{Inn}(T) \leq \varphi(Q) \cong \text{PGL}(2, q)$ by Lemma 1.5(a). As P is 2-transitive on the points, $Q = P_y$ is a maximal subgroup, and $\text{core}_P(Q) = 1$ because P acts faithfully. Suppose that $\psi: P \rightarrow \text{Aut}(T)$ extends φ . If ψ is injective, then $\psi^{-1}(\text{Inn}(T))$ is a non-trivial normal subgroup of P , and it lies in Q because $\psi(Q) = \varphi(Q) \supseteq \text{Inn}(T)$; this contradicts $\text{core}_P(Q) = 1$. Otherwise $N = \ker \psi \neq 1$. The subgroup $\text{PSL}(3, q)$ of P is simple, and its centraliser in P is trivial (a matrix whose image centralises $\text{PSL}(3, q)$ commutes with $\text{SL}(3, q)$ up to a homomorphism into the scalars, which is trivial as $\text{SL}(3, q)$ is perfect; so it is scalar, as $\text{SL}(3, q)$ is absolutely irreducible). If $N \cap \text{PSL}(3, q) = 1$, then $[N, \text{PSL}(3, q)] = 1$ and $N = 1$; so $N \supseteq \text{PSL}(3, q)$, and $\psi(P)$ has order at most $|P : \text{PSL}(3, q)| \leq 3$, although it contains $\varphi(Q) \cong \text{PGL}(2, q)$. So φ does not extend, and by Lemma 1.1(b) the group G_q is primitive of type TW with socle $B \cong T^k$ and primitive component $H = T \rtimes \varphi(Q)$.

If a good subgroup S of P had order greater than M , it would be good at y (Lemma 1.5(b)), against Theorem A. By Lemma 1.3, $M_G = M$, and by Lemma 1.1(a) and (3)

$$\text{MinSubDeg}(G_q) = \frac{|P|}{M} = \frac{q^3(q^3 - 1)(q^2 - 1)}{q^2(q - 1)c_q} = \frac{q(q^3 - 1)(q + 1)}{c_q},$$

and this equals $k \cdot q(q^2 - 1)/c_q = k \cdot |Q|/M = k \cdot \text{MinSubDeg}(H)$. The values of c_q in Lemma 2.1 give the three cases for $\text{MinSubDeg}(H)$. For $q = 4$ we obtain $\text{MinSubDeg}(H) = 12$ and $\text{MinSubDeg}(G_4) = 252$, as in [Fer26]. $\square$

8 Computations

No proof in this note depends on a computation; the computations below are tests. The programs and their recorded outputs are in the reproducibility package, whose `README.md` gives command lines and running times. All computations are exact, in Python 3 with numpy.

The program of this note (`check_note.py`, about 30 seconds) is written from the definitions of Section 1.3 and uses none of the lemmas of Sections 2 to 6. For $q = 5$ and $q = 7$ it computes c_q and $M = \max_t |Q_t|$ from the definitions (the rows $q = 5, 7$ of Table 1); it lists all conjugacy classes of subgroups of $\mathrm{GL}(2, q)$ (48 and 84) by a generic method (every subgroup $U \neq 1$ is generated by a maximal subgroup of U and an element of prime power order); and for the stabilisers $E \rtimes \mathrm{GL}(2, q)$ of a point and of a line (E the elations with that centre, resp. axis) it checks, listing all invariant subgroups of E , that every subgroup X of order greater than M contains E , and that for *every* point d the group induced by X_d on V/d centralises no non-identity element of $\mathrm{PSL}(V/d)$. Result: 10 ($q = 5$) and 9 ($q = 7$) such subgroups in each stabiliser, none good at any point; as a control, with the bound $|X| \geq M$ it finds good subgroups of order M . Its lists coincide with those of the second verification run (`compare_check_note.py`).

Programs of the first version (`original/`). The equality was first found by computation for $q \leq 9$ and then proved. `parabolic_q.py` lists all subgroups of $\mathrm{GL}(2, q)$ with a generic lattice program and tests every point ($q = 7, 8, 9$); `check_parabolic_prime.py` lifts all subgroup classes of $\mathrm{PGL}(2, q)$ to $\mathrm{GL}(2, q)$ and tests one point and one line of every orbit of the subgroups of a line stabiliser ($q = 5, 7, 11, 13$; the run for $q = 17$ was stopped after one subgroup and does not cover $q = 17$); `lemma_checks.py` and `lemma17b_check.py` test Lemmas 2.1, 2.2 and 2.4 on all subgroup classes of $\mathrm{PGL}(2, q)$ for $4 \leq q \leq 25$, and Lemma 2.5(b) for $(q, q_0) = (16, 4), (25, 5)$.

First verification run (`verification_run_A/`). `parab_check.py` lists all subgroups of order greater than M of a line stabiliser, including non-split ones (through 1-cocycles), for nine values of q (Table 2), with a control at $|X| \geq M$ for $q \leq 11$; `values_check.py` recomputes Table 1 for $4 \leq q \leq 32$; `ineq_sweep.py` checks every numerical inequality of Sections 2 and 4 to 6 in exact integers for all 217169 prime powers $4 \leq q \leq 3 \cdot 10^6$, with no failure (the 291 members of (C5) or (C8) of order greater than M all have one of the two shapes of Lemma 4.1); `lemma_pgl2_check.py` tests Lemmas 2.2 to 2.4 for $q \leq 16$ and Lemma 2.5(b) for $(16, 4), (25, 5), (49, 7)$.

Second verification run (`verification_run_B/`). `vb_parab.py` enumerates, through first cohomology groups, all subgroups of order greater than M of the stabilisers of a point and of a line, for eleven values of q (Table 2); in every case $N = E$, and none is good. With the threshold $M/2$ ($q = 5, 7, 8, 9$) it finds good subgroups, and with the threshold 239 for $q = 4$ it finds twelve subgroups, six in each parabolic subgroup, which give the eleven parabolic rows of [Fer26, Table 2] (the stabiliser of a flag lies in both). `vb_maximal.py` constructs members of (C2), (C3), (C5), (C6), (C8) in $\mathrm{PGL}(3, q)$ for $q = 5, 7, 8, 9, 11, 13, 16, 25$ (orders by Schreier–Sims, all less than M ; for $q = 16, 25$ the two shapes of Lemma 4.1) and checks Lemma 4.4; `vb_c9_q11.py` constructs $\mathrm{PSL}(2, 7) < \mathrm{SL}(3, 11)$; `vb_lemmas.py` tests Lemmas 2.1, 2.2 and 2.4 and Lemma 2.5(a) on all subgroups of $\mathrm{GL}(2, q)$, $q \leq 19$. Its lists coincide with those of `original/`, whose programs it re-ran with identical outputs.

Third verification run (`independent_run_2/`), on the final text. `thmA_dfs.py` tests Theorem A itself, from the definitions and without the reduction or any list of maximal subgroups. For one non-identity t in every orbit of Q on T and every subgroup R of Q_t with $k|R| > M$ (all classes of subgroups of the soluble group Q_t , by cyclic extension), it searches exhaustively for the subgroups $S \leq P$ with $S \cap Q = R$ and $|S| > M$: the orbit y^S is a union of orbits of R ; if it contains a point d , the elements of S that map y to d form one of the $|Q : R|$ cosets Rx in the set of elements of P that map y to d , so the search adjoins one such element at a time and tests the Schreier generators of the stabiliser of y . For $q = 4, 5, 7, 8, 9$ it finds no such subgroup, and with the threshold $M - 1$ it finds the good subgroups of order M . `values.py` recomputes Table 1, and c_q , M and $\mathrm{MinSubDeg}(H)$ for

$4 \leq q \leq 49$; `plane_lemmas.py` tests Lemmas 2.1 and 2.2 for $q \leq 32$ and Lemmas 3.1, 3.3 and 5.1 and Corollary 3.2 for $q \leq 9$; `tight_cases.py` checks the two cases of Lemma 4.1 that the proof of Proposition 4.2 needs, without Input R2, for $q = 16, 25, 64, 121$; `ineq_sweep2.py` repeats the sweep, with the orders of [AB15, Lemmas 4.10, 4.11 and 4.13].

q	M	X	programs	q	M	X	programs
4	240	5	A, B, C, F	13	48672	34	O2, A, B
5	800	10	N, O2, A, B, C, F	16	65280	20	A, B
7	4704	9	N, O1, O2, A, B, C	17	147968	32	A, B
8	4032	5	O1, A, B, C	19	259920	28	B
9	10368	19	O1, A, B, C	23	558624	9	B
11	29040	10	O2, A, B	25	720000	84	B

Table 2: Tests of the parabolic part of Theorem A. Column X : the number of subgroups of order greater than M of one parabolic subgroup, up to conjugacy (for $q = 25$ possibly with repetitions); none of them is good at any point. N: `check_note.py`; O1: `parabolic_q.py`; O2: `check_parabolic_prime.py`; A, B, C: the first, the second and the third verification run; F: the complete enumerations of [Fer26, §6]. C tests Theorem A as a whole and does not list the subgroups of the parabolic subgroups. Where several programs list them, the numbers agree, and so do the orbit data wherever two programs print them.

These computations are tests. Theorem A is proved by the written argument of Sections 2 to 7. Its parabolic part (Propositions 5.2 and 6.1 and Corollary 6.2) was tested by at least two independently written programs for $q \in \{4, 5, 7, 8, 9, 11, 13, 16, 17\}$ and by one program for $q = 19, 23, 25$. Theorem A as a whole, and so also the reduction, was tested for $q \in \{4, 5, 7, 8, 9\}$ by the third verification run, and for $q = 4$ and $q = 5$ the complete enumerations of [Fer26, §6] cover all subgroups of P . For $q \geq 11$ the reduction (Proposition 4.2) is not tested by computation beyond its arithmetic: the sweeps check the inequalities, and the constructions check the orders of some members of the non-parabolic classes, not the completeness of Input R1.

Verification. The equality was first found by computation for $q \leq 9$ and then written up with complete proofs. Two independent verification runs, both AI-assisted, examined this first written version; neither saw the other. The first re-derived every step of the proofs and compared the setting and the criterion with [GLP⁺06, Fer26], Input R1 with [LY22] and [AB15], and Lemma 2.3 for odd q with [COTR06]; the second made the computations described above, compared Input R1 with [Gil07] and the ATLAS, and searched the literature. Neither found a mathematical error. They required these changes, all of which were made: Input R1 is cited as read in a transcription, with [AB15] as an independent source for the part that is used (Remark 4.3); [SvdW28, Ste60] and Dickson’s list for even q are cited as standard results and not re-proved; an alternative that cannot occur was removed from the proof of Proposition 4.2; the types in Lemma 2.3 are said to overlap; Lemma 2.1 is proved in full, since the source first cited for it covers only fixed points; a count for $q = 7$ in a summary of the first computations is corrected; and the account of the computations is updated. This note was written after the two runs. In writing it, the first case of the proof of Proposition 4.2 was shortened, Lemma 4.1 was restricted to the two cases that are used, the argument for non-extension in the proof of Theorem B was replaced by one that does not use $|\text{Aut}(\text{PSL}(2, q))|$, and `check_note.py` was written.

A third verification run, also AI-assisted, examined the final text, and these four changes first. It read again [PS05], Sections 1, 2 and 4 of [GLP⁺06], the cited parts of [Fer26], [LY22, Lemma 2.2], [Gil07, Theorem 19], the cited parts of [AB15] and [COTR06, Theorem 2], compared them with the

statements of this note, and re-derived every proof. With its own programs, which share no code with the earlier ones, it tested Theorem A as a whole, without the reduction, for $q = 4, 5, 7, 8, 9$, recomputed Table 1, tested Lemmas 2.1 and 2.2 for $q \leq 32$ and Section 3 and Lemma 5.1 for $q \leq 9$, and checked the two cases of Lemma 4.1 that are used for $q = 16, 25, 64, 121$ without Input R2 (see above). It ran the package again from the archive, with the recorded outputs, repeated the literature search and checked the DOIs. It found no mathematical error. It required that this paragraph and Section 8 describe its checks, that a count in the description of the second run be corrected (twelve subgroups give eleven rows of [Fer26, Table 2]), and that Section 2 of [GLP⁺06], which is cited in Section 1.1, and the abstract [PS05] be listed among what was read; these changes were made.

Scope and priority. The note answers the question of M. Giudici mentioned in Section 1.1: infinitely many primitive groups of type TW attain the upper bound in (1), namely the groups G_q , $q \geq 4$. The bounds (1), Question 1.6 and the method of [GLP⁺06, §4] (Construction 4.1 and Lemmas 4.2–4.5), on which the criterion of Lemma 1.3 rests, are due to Giudici, Li, Praeger, Seress and Trofimov [GLP⁺06]; Question 4 of [PS05] is the same question in the abstract of Praeger and Seress. The example $q = 4$, and the computed example $q = 5$, are those of [Fer26]. The subgroup lists that we use are those of Mitchell [Mit11] and Hartley [Har25], in the tabulation of Bray, Holt and Roney-Dougal [BHRD13] as transcribed in [LY22], and Dickson’s list [Dic01] of the subgroups of $\text{PGL}(2, q)$. New, to our knowledge and relative to the literature we found, are the infinite family of Theorem B and its proof. No infinite family attaining the bound was found in the literature accessible to us, in particular in the papers citing [GLP⁺06] that we could find; a search that finds nothing is not a proof of novelty, and we claim no priority. We do not classify the groups of type TW with equality, and we make no statement about other top groups; for $P = \text{PSL}(3, 4)$, for instance, the inequality is strict [Fer26, Theorem 1.2].

What was read: the abstract [PS05]; of [GLP⁺06], in the copy that M. Giudici sent us for [Fer26], Sections 1, 2 and 4 up to Lemma 4.5; [LY22, §2]; [Gil07, §3.1]; of [AB15] (arXiv version 2), Sections 1 and 4.2–4.4 (Table 6, Proposition 4.7 with Lemmas 4.9–4.16, Definition 4.27, Proposition 4.28, Table 7); [COTR06, §3]; and the pages of $L_3(5)$, $L_3(7)$, $L_3(8)$ of [W⁺26]. Not read: [BHRD13, Mit11, Har25, Dic01, Hup67, KL90, Asc84, SvdW28, Ste60]; the list of maximal subgroups is cited through [LY22, Gil07, AB15], and the other results as standard ones, in the form stated. The literature search was made on 14 October 2026 by the second verification run (arXiv, OpenAlex, zbMATH Open and Semantic Scholar, for minimal subdegrees and twisted wreath products; the works citing [GLP⁺06]; three web searches); the OpenAlex list of citing works was retrieved again when this note was written, and the third verification run repeated the search on the same day (arXiv, OpenAlex, Crossref and zbMATH Open; one web search) and found nothing further. The citing works found are [PS05, GLP⁺07, BGL⁺13, CGM19, BS22]; [CGM19] was searched in full text, the abstracts of [BGL⁺13, BS22] were read, and [GLP⁺07] was not accessible to us. None of them shows that the upper bound is attained.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AB15] S. Hassan Alavi and Timothy C. Burness. Large subgroups of simple groups. *Journal of Algebra*, 421:187–233, 2015. Read in the arXiv version 2, whose numbering we use. arXiv:1311.6733, doi:10.1016/j.jalgebra.2014.08.026.

- [Asc84] Michael Aschbacher. On the maximal subgroups of the finite classical groups. *Inventiones Mathematicae*, 76(3):469–514, 1984. Not read. doi:10.1007/BF01388470.
- [BGL⁺13] John Bamberg, Michael Giudici, Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl. The classification of almost simple $\frac{3}{2}$ -transitive groups. *Transactions of the American Mathematical Society*, 365(8):4257–4311, 2013. doi:10.1090/S0002-9947-2013-05758-3.
- [BHRD13] John N. Bray, Derek F. Holt, and Colva M. Roney-Dougal. *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups*, volume 407 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2013. Not consulted; cited as transcribed in [LY22]. doi:10.1017/CB09781139192576.
- [BS22] Timothy C. Burness and Aner Shalev. Permutation groups with restricted stabilizers. *Journal of Algebra*, 607:160–185, 2022. doi:10.1016/j.jalgebra.2021.08.012.
- [CGM19] Alexander Y. Chua, Michael Giudici, and Luke Morgan. Coprime subdegrees of twisted wreath permutation groups. *Proceedings of the Edinburgh Mathematical Society*, 62(4):1137–1162, 2019. arXiv:1801.02456, doi:10.1017/S0013091519000130.
- [CL15] Thomas Connor and Dimitri Leemans. An atlas of subgroup lattices of finite almost simple groups. *Ars Mathematica Contemporanea*, 8(2):259–266, 2015. arXiv:1306.4820, doi:10.26493/1855-3974.455.422.
- [COTR06] Peter J. Cameron, G. R. Omid, and B. Tayfeh-Rezaie. 3-designs from $\text{PGL}(2, q)$. *The Electronic Journal of Combinatorics*, 13(1):Research Paper R50, 2006. doi:10.37236/1076.
- [Dic01] Leonard Eugene Dickson. *Linear Groups, with an Exposition of the Galois Field Theory*. Teubner, Leipzig, 1901. Not read.
- [Fer26] Alper Ferudun. A negative answer to a question of Praeger and Seress on minimal subdegrees of primitive groups of twisted wreath type. Zenodo, unrefereed preprint, version 1.1, 2026. All versions: doi:10.5281/zenodo.23299864. doi:10.5281/zenodo.23342394.
- [Gil07] Nick Gill. $\text{PSL}(3, q)$ and line-transitive linear spaces. Preprint, 2007. Version 3; Theorem 19 read. arXiv:math/0603293.
- [GLP⁺06] Michael Giudici, Cai Heng Li, Cheryl E. Praeger, Ákos Seress, and Vladimir Trofimov. On minimal subdegrees of finite primitive permutation groups. In *Finite Geometries, Groups, and Computation*, pages 75–94. Walter de Gruyter, Berlin, 2006. Read in a copy sent by M. Giudici, whose numbering we use. doi:10.1515/9783110199741.75.
- [GLP⁺07] Michael Giudici, Cai Heng Li, Cheryl E. Praeger, Ákos Seress, and Vladimir I. Trofimov. On limit graphs of finite vertex-primitive graphs. *Journal of Combinatorial Theory, Series A*, 114(1):110–134, 2007. Not accessible to us. doi:10.1016/j.jcta.2006.03.005.
- [Har25] Robert W. Hartley. Determination of the ternary collineation groups whose coefficients lie in the $\text{GF}(2^n)$. *Annals of Mathematics (2)*, 27(2):140–158, 1925. Not read. doi:10.2307/1967970.

- [Hup67] Bertram Huppert. *Endliche Gruppen I*, volume 134 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 1967. Not read. doi:10.1007/978-3-642-64981-3.
- [KL90] Peter Kleidman and Martin Liebeck. *The Subgroup Structure of the Finite Classical Groups*, volume 129 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990. Not read. doi:10.1017/CB09780511629235.
- [LY22] Yang Liu and Yong Yang. Huppert’s analogue conjecture for $\mathrm{PSL}(3, q)$ and $\mathrm{PSU}(3, q)$. Preprint, 2022. Version 1; Lemma 2.2 read. arXiv:2209.02660.
- [Mit11] Howard H. Mitchell. Determination of the ordinary and modular ternary linear groups. *Transactions of the American Mathematical Society*, 12(2):207–242, 1911. Not read. doi:10.1090/S0002-9947-1911-1500887-3.
- [PS05] Cheryl E. Praeger and Ákos Seress. On minimal subdegrees of finite primitive permutation groups. *Oberwolfach Reports*, 2(1):690–693, 2005. Abstract in Report No. 12/2005, Groups and Geometries. doi:10.4171/OWR/2005/12.
- [Ste60] Robert Steinberg. Automorphisms of finite linear groups. *Canadian Journal of Mathematics*, 12:606–615, 1960. Not read. doi:10.4153/CJM-1960-054-6.
- [SvdW28] Otto Schreier and Bartel L. van der Waerden. Die Automorphismen der projektiven Gruppen. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 6:303–322, 1928. Not read. doi:10.1007/BF02940620.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record OWR-793-006.
- [W⁺26] Robert A. Wilson et al. ATLAS of Finite Group Representations, version 3. <https://brauer.maths.qmul.ac.uk/Atlas/v3/>, 2026. Pages of $L_3(5)$, $L_3(7)$, $L_3(8)$.