

Hive Polytopes with Arbitrary Integral Right-Hand Sides Have Integral Points Exactly up to Rank 5: A Counterexample to a Conjecture of De Loera and McAllister

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Abstract

Knutson and Tao proved that a nonempty hive polytope has an integral vertex. De Loera and McAllister conjectured that this extends to arbitrary integral right-hand sides: if B and R are the matrices of the boundary equations and of the rhombus inequalities of hive patterns of rank r , every nonempty polytope $\{h : Bh = b, Rh \leq c\}$ with integral b and c should have an integral vertex, and the matrix $\begin{pmatrix} B & 0 \\ R & I \end{pmatrix}$ should have a unimodular triangulation (Oberwolfach Report 39/2004; Experiment. Math. 15 (2006), Conjecture 4.5). We show that for every $r \geq 6$ there is an integral c for which $\{h : Bh = 0, Rh \leq c\}$ is a single point with four coordinates equal to $1/2$. Hence for $r \geq 6$ the conjecture fails in both printed forms: the cone generated by the columns of the matrix has no unimodular cover, and neither has their convex hull. For $r = 6$ the example is derived in a few lines from fifteen of the 45 rhombus inequalities. For $r \leq 5$ every nonempty polytope of this form contains an integral point. This is contained in what the sources report as verified by computation; we prove it independently, by a reduction to finitely many cases and an exact computation. So the question whether such polytopes always contain an integral point, record OWR-741-001 of the UnsolvedMath collection, has the answer yes for $r \leq 5$ and no for $r \geq 6$. The sources state that the conjecture was verified for $r \leq 6$. In the indexing of their definitions the example has $r = 6$; a hive pattern printed there with the label $r = 5$ has five rows, which is rank 4 in that indexing. We cannot tell from the texts which indexing the verification used. The journal version of the paper of De Loera and McAllister was not accessible to us and was not read; we used the arXiv version. No counterexample was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 Hive polytopes and the conjecture

Let $r \geq 1$ and

$$H_r = \{(i, j, k) \in \mathbb{Z}_{\geq 0}^3 : i + j + k = r\}.$$

A point of H_r is a *boundary point* if one of its coordinates is 0 and an *interior point* otherwise; I_r denotes the set of interior points. So H_r has $(r+1)(r+2)/2$ points, of which $3r$ are boundary points and $(r-1)(r-2)/2$ are interior points. Two points are *adjacent* if their difference is a permutation of $(1, -1, 0)$. A *rhombus* ρ of H_r consists of two adjacent points s, s' , its *short diagonal*, and of the two

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common neighbours t, t' of s and s' , its *tips*; all four points must lie in H_r . The *excess* of a function $h: H_r \rightarrow \mathbb{R}$ on ρ is

$$g_\rho(h) = h(s) + h(s') - h(t) - h(t').$$

There are $3r(r-1)/2$ rhombi. A *hive pattern* of rank r is a function h on H_r with $g_\rho(h) \geq 0$ for all rhombi ρ ; these are the *rhombus inequalities*. In [LM06, Definition 2.2] they are written as three families of inequalities indexed by the points $(i, j, k) \in H_r$ with $i, j \geq 1$,

$$\begin{aligned} h_{i,j-1,k+1} + h_{i-1,j+1,k} &\leq h_{i,j,k} + h_{i-1,j,k+1}, \\ h_{i,j,k} + h_{i-1,j-1,k+2} &\leq h_{i,j-1,k+1} + h_{i-1,j,k+1}, \\ h_{i+1,j-1,k} + h_{i-1,j,k+1} &\leq h_{i,j,k} + h_{i,j-1,k+1}; \end{aligned}$$

these are the inequalities $g_\rho(h) \geq 0$, each rhombus occurring once (the program `hive_system.py` of the package generates the three families as printed and checks this for $r \leq 10$). The values of a hive pattern are required to be non-negative in [LM06].

For partitions λ, μ, ν the *hive polytope* $H_{\lambda\mu}^\nu$ is the set of hive patterns whose boundary values are prescribed by λ, μ, ν [LM06, Definition 2.3]; the number of its integral points is the Littlewood–Richardson coefficient $c_{\lambda\mu}^\nu$ [LM06, Lemma 2.4]. In matrix form [McA04, p. 2073], [LM06, equation (2)],

$$H_{\lambda\mu}^\nu = \{h \in \mathbb{R}^{(r+1)(r+2)/2} : Bh = b(\lambda, \mu, \nu), Rh \leq 0\}, \quad (1)$$

where B and R are integral matrices: Bh is the vector of the $3r$ boundary values of h , and Rh is the vector of the numbers $-g_\rho(h)$. According to [McA04, LM06], Knutson and Tao [KT99] proved the saturation theorem by showing that every nonempty hive polytope has an integral vertex. The abstract [McA04] then says that the authors conjecture that this integrality result extends to all polytopes

$$\{h \in \mathbb{R}^{(r+1)(r+2)/2} : Bh = b, Rh \leq c\}, \quad (2)$$

where b and c are arbitrary integral vectors for which the polytope is not empty, and that for this it suffices to prove that the columns of the matrix

$$M_r = \begin{pmatrix} B & 0 \\ R & I \end{pmatrix} \quad (3)$$

admit a unimodular triangulation [McA04, Conjecture 1]. In [LM06, Section 4.1] the matrix M_r is called the *homogenized hive matrix*; for an integral vector β the polytope $\{\tilde{h} : M_r \tilde{h} = \beta, \tilde{h} \geq 0\}$ is called a *generalized hive polytope* or *g-hive polytope* [LM06, Definition 4.2], and the conjecture reads as follows.

Conjecture (De Loera and McAllister [LM06, Conjecture 4.5]). For each r , the homogenized hive matrix has a unimodular triangulation. Consequently, every g-hive polytope has an integral vertex.

In [McA04, Conjecture 1] the statement on triangulations is made for the convex hull of the points whose coordinates are the columns of M_r . In [LM06, Section 4.1] a unimodular set is a set of columns which form a square matrix of determinant ± 1 , and the cones generated by such sets are to cover the cone generated by all columns. Both forms are treated below (Corollary 1.3).

Record OWR-741-001 of the UnsolvedMath collection [ula26] is this conjecture; it is listed there as open. Its original statement is the conjecture of [McA04] that the integrality result extends to every nonempty polytope (2) with integral b and c ; in the current form of the record this has become the question whether every such polytope contains an integral point.

1.2 Results

It is convenient to write the right-hand side as a lower bound for the excess. For a function b on the boundary points and a function w on the rhombi of H_r put

$$P_r(b, w) = \{h \in \mathbb{R}^{H_r} : h = b \text{ on the boundary, } g_\rho(h) \geq w_\rho \text{ for all rhombi } \rho\},$$

$$P_r^+(b, w) = P_r(b, w) \cap \{h \geq 0\}.$$

So $P_r(b, w)$ is the polytope (2) with $c = -w$, and $P_r^+(b, w)$ is, up to the slack variables, the g-hive polytope with $\beta = (b, -w)$.

Theorem 1.1. *Let $r \geq 6$. There is an integral w such that $P_r(0, w)$ consists of a single point, and this point has the value $\frac{1}{2}$ at four interior points and the value 0 at all other points. In particular $P_r(0, w) = P_r^+(0, w)$ is not empty and contains no integral point.*

Theorem 1.2. *Let $r \leq 5$, and let b and w be integral. If $P_r(b, w)$ is not empty, it contains an integral point. If $P_r^+(b, w)$ is not empty, it contains an integral point.*

Corollary 1.3. *Let $r \geq 6$, and let $m = 3r + 3r(r - 1)/2$ be the number of rows of the homogenized hive matrix M_r .*

- (a) *There is an integral vector β in the cone generated by the columns of M_r which lies in no cone generated by m columns that form a matrix of determinant ± 1 . So the columns of M_r have no unimodular cover and no unimodular triangulation in the sense of [LM06, Section 4.1], and there is a g-hive polytope which is not empty and has no integral point.*
- (b) *The convex hull of the columns of M_r is not the union of simplices $\text{conv}(a_0, \dots, a_m)$ such that $a_0, \dots, a_m$ are columns and $a_1 - a_0, \dots, a_m - a_0$ form a basis of $\mathbb{Z}^m$. So it has no unimodular triangulation by simplices whose vertices are among the columns, which is the form of the conjecture in [McA04, Conjecture 1].*

In particular the conjecture of Section 1.1 does not hold for any $r \geq 6$.

The example of rank 6 is given in Section 3 with all its data; that it is a single point follows from fifteen of its 45 inequalities (Proposition 3.1). Part (b) of Corollary 1.3 needs examples in which the sum of all coordinates, slack variables included, is an integer; they are given in Propositions 3.2 and 4.2. Theorem 1.2 is proved by a reduction to finitely many cases (Lemmas 2.1 and 2.2) and an exact computation (Section 5). The hive polytopes themselves are not affected: they are the case $w = 0$ with special boundary values, and they have integral vertices by the theorem of Knutson and Tao.

1.3 Conventions that could change the answer

The rank. Throughout, r is the number in $i + j + k = r$: a pattern of rank r has $r + 1$ rows and $(r + 1)(r + 2)/2$ entries, of which $3r$ lie on the boundary. This is the indexing of the record, of [McA04], where the polytope lies in $\mathbb{R}^{(r+1)(r+2)/2}$ and $b(\lambda, \mu, \nu)$ is a vector with $3r$ entries, and of [LM06, Definition 2.2]. The example of Section 3 has $r = 6$: it has 28 entries, 18 boundary entries and 45 rhombi. See Remark 6.1 for the range in which the sources state that the conjecture was verified.

Sign condition. The polytopes (2) of [McA04] and of the record carry no condition $h \geq 0$; the g-hive polytopes of [LM06] do. Theorem 1.1 holds in both readings, because the single point is non-negative, and Theorem 1.2 is proved in both.

Integral point or integral vertex. The current form of the record asks for an integral point. Its original statement and the sources [McA04, LM06] speak of the integrality result of Knutson and Tao, that is, of an integral vertex. For Theorem 1.1 and Corollary 1.3 this makes no difference: the conjecture fails for $r \geq 6$ in both forms. Theorem 1.2 gives an integral point, not an integral vertex; see Remark 6.2.

The matrices. We take B to be the restriction to the $3r$ boundary points and R to have one row for each rhombus, with entries $0, \pm 1$. In [LM06, Section 4.1] it is said that there is some choice in how the equations and inequalities are encoded and that all encodings are equivalent for the purposes there. The examples of Theorem 1.1 have $b = 0$, which is integral in every encoding of the boundary equations, and they use the rhombus inequalities as printed; further rows or multiples of rows would only add right-hand sides. Theorem 1.2 refers to the rows as printed, with entries $0, \pm 1$.

1.4 What was known

The conjecture is due to De Loera and McAllister [McA04, LM06]; it is also Conjecture 3.5 of McAllister’s thesis [McA06]. The three texts say that it was verified for $r \leq 6$: “up to $r = 6$ using placing triangulations” [McA04, p. 2073], and as a theorem, computed with the software TOPCOM [Ram02], in [LM06, Theorem 4.6] and [McA06, Theorem 3.6]. The survey of Haase, Paffenholz, Piechnik and Santos mentions the conjecture as “validated up to A_6 ” [HPPS21, Section 1.2.1] and asks, in its list of open questions, whether the homogenized hive matrices always have unimodular triangulations [HPPS21, end of Section 1].

A unimodular cover in the sense of [LM06, Section 4.1] gives an integral vertex of every nonempty g -hive polytope [LM06, Corollary 4.4]. So Theorem 1.2 is contained in [LM06, Theorem 4.6], in each of the readings of its range discussed in Remark 6.1, and with an integral vertex in place of an integral point (for the polytopes $P_r(b, w)$ one adds a constant to h , which changes no excess). The proof of Theorem 1.2 given here does not use that theorem.

The programme of a meeting of the Seminar on Nonlinear Algebra in Berlin on 26 June 2015 lists a talk by C. Haase with the title “A unimodular triangulation for the hives matrix”. Its abstract recalls the conjecture of De Loera and McAllister for the homogenized hives matrix and announces an argument that the proof of Knutson and Tao implies the existence of a regular unimodular triangulation [Haa15]. By Corollary 1.3 there is no such triangulation for $r \geq 6$. We found no written account of the talk, and the version of [HPPS21] of December 2017 lists the question as open. We record these facts only.

We found no counterexample to the conjecture and no statement that it fails; see “Scope and priority” at the end of Section 7 for the searches. New, to our knowledge and relative to the literature we found, are Theorem 1.1 and Corollary 1.3, that is, the failure of the conjecture from rank 6 on, and with them the exact range of the ranks in which integral points always exist.

2 Reduction to finitely many cases

A function on I_r is extended by 0 to the boundary points; in this way $\mathbb{R}^{I_r}$ is the space of functions on H_r with boundary values 0, and g_ρ is a linear form with integer coefficients on it.

Lemma 2.1. *For a fixed r the following two statements are equivalent.*

- (i) *For all integral b and w , the polytope $P_r(b, w)$ is empty or contains an integral point.*
- (ii) *For every $x \in \mathbb{R}^{I_r}$ there is $z \in \mathbb{Z}^{I_r}$ with $g_\rho(z) \geq \lfloor g_\rho(x) \rfloor$ for all rhombi ρ .*

Likewise the following two statements are equivalent.

(i⁺) For all integral b and w , the polytope $P_r^+(b, w)$ is empty or contains an integral point.

(ii⁺) For every $x \in \mathbb{R}^{I_r}$ there is $z \in \mathbb{Z}^{I_r}$ with $g_\rho(z) \geq \lfloor g_\rho(x) \rfloor$ for all rhombi ρ and $z_p \geq \lfloor x_p \rfloor$ for all $p \in I_r$.

Proof. (i) $\Rightarrow$ (ii): x lies in $P_r(0, \lfloor g(x) \rfloor)$, so this polytope contains an integral point z .

(ii) $\Rightarrow$ (i): Let $h_0 \in P_r(b, w)$. Let β be the function which is b on the boundary and 0 on I_r , and $x = h_0 - \beta$. The numbers $g_\rho(\beta)$ are integers. With z from (ii), the function $h = \beta + z$ is integral, equals b on the boundary, and

$$g_\rho(h) = g_\rho(\beta) + g_\rho(z) \geq g_\rho(\beta) + \lfloor g_\rho(x) \rfloor = \lfloor g_\rho(h_0) \rfloor \geq w_\rho,$$

because w_ρ is an integer and $w_\rho \leq g_\rho(h_0)$.

(ii⁺) $\Rightarrow$ (i⁺): The same argument with $h_0 \geq 0$: then $b \geq 0$, and $x = h_0$ on I_r , so $z_p \geq \lfloor x_p \rfloor \geq 0$ and $h \geq 0$.

(i⁺) $\Rightarrow$ (ii⁺): Let $y = \lfloor x \rfloor$, coordinate by coordinate, and $x' = x - y \in [0, 1)^{I_r}$. Then $x' \in P_r^+(0, \lfloor g(x') \rfloor)$, so this polytope contains an integral point $z' \geq 0$. Put $z = z' + y$. Then $g_\rho(z) \geq \lfloor g_\rho(x') \rfloor + g_\rho(y) = \lfloor g_\rho(x) \rfloor$ and $z_p \geq y_p = \lfloor x_p \rfloor$. $\square$

Lemma 2.2. Let $\ell_1, \dots, \ell_m$ be linear forms on $\mathbb{R}^n$ with integer coefficients which span the dual space of $\mathbb{R}^n$. Call $x \in \mathbb{R}^n$ bad if there is no $z \in \mathbb{Z}^n$ with $\ell_k(z) \geq \lfloor \ell_k(x) \rfloor$ for $k = 1, \dots, m$.

(a) If x is bad, then $x + t$ is bad for every $t \in \mathbb{Z}^n$.

(b) If there is a bad point, then there is a bad point v such that the forms ℓ_k with $\ell_k(v) \in \mathbb{Z}$ span the dual space.

(c) A point v has the property that the forms ℓ_k with $\ell_k(v) \in \mathbb{Z}$ span the dual space if and only if $Lv \in \mathbb{Z}^n$ for some $n \times n$ matrix L of rank n whose rows are among the ℓ_k . For a fixed L these points form $|\det L|$ classes modulo $\mathbb{Z}^n$, namely the subgroup of $\mathbb{R}^n/\mathbb{Z}^n$ generated by the columns of L^{-1} .

Proof. (a) $\ell_k(x + t) = \ell_k(x) + \ell_k(t)$ and $\ell_k(t) \in \mathbb{Z}$; so z works for x if and only if $z + t$ works for $x + t$.

(b) Let x be bad, $w_k = \lfloor \ell_k(x) \rfloor$, and $Q = \{y \in \mathbb{R}^n : \ell_k(y) \geq w_k \text{ for all } k\}$. Then $x \in Q$. Every $y \in Q$ is bad: $\lfloor \ell_k(y) \rfloor \geq w_k$, so a vector z which works for y would work for x . The polyhedron Q is not empty and contains no line, because the forms span the dual space. Hence it has a vertex v . The forms of the inequalities which hold with equality at a vertex span the dual space, and for these forms $\ell_k(v) = w_k \in \mathbb{Z}$.

(c) The first statement is a reformulation. For the second, the set of these points is $L^{-1}\mathbb{Z}^n$, a lattice which contains $\mathbb{Z}^n$ with index $|\det L|$ and is generated by the columns of L^{-1} . $\square$

By Lemma 2.2, statement (ii) of Lemma 2.1 holds for a given r as soon as it holds for one representative x of each of the finitely many classes described in (c), where the forms are the excess forms g_ρ on $\mathbb{R}^{I_r}$; for (ii⁺) the coordinate forms are added. This is used in Section 5.

3 The example of rank 6

Let $r = 6$. We name the values of a function h on H_6 at the ten interior points:

$$\begin{aligned} x &= h(1, 1, 4), & y &= h(1, 2, 3), & z &= h(1, 3, 2), & d &= h(1, 4, 1), & a &= h(2, 1, 3), \\ e &= h(2, 2, 2), & c &= h(2, 3, 1), & b &= h(3, 1, 2), & f &= h(3, 2, 1), & g &= h(4, 1, 1). \end{aligned}$$

Let φ be the function on H_6 with

$$a = b = c = d = \frac{1}{2}, \quad x = y = z = e = f = g = 0, \quad \varphi = 0 \text{ on the boundary}$$

(Figure 1), and let $w_\rho = \lfloor g_\rho(\varphi) \rfloor$ for the 45 rhombi ρ of H_6 . Then $w_\rho = 1$ for two rhombi, $w_\rho = -1$ for ten rhombi, and $w_\rho = 0$ for the other 33; the twelve rhombi with $w_\rho \neq 0$ are listed in Table 1. In the notation of (2), $b = 0$ and $c = -w \in \{-1, 0, 1\}$ ⁴⁵.

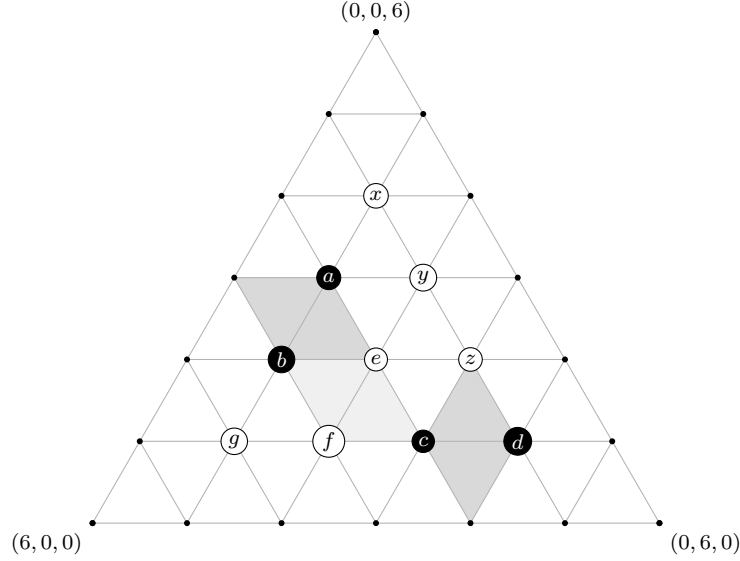


Figure 1: The pattern of rank 6; the point (i, j, k) is drawn in row $i + j$, counted from the top, and j increases to the right. The function φ has the value $\frac{1}{2}$ at the four black points a, b, c, d and the value 0 at all other points. Shaded dark: the rhombi with short diagonals $\{c, d\}$ and $\{a, b\}$, where $w_\rho = 1$. Shaded light: the rhombus with short diagonal $\{e, f\}$ and tips b, c , where $w_\rho = -1$.

short diagonal	tips	inequality $g_\rho(h) \geq w_\rho$	$g_\rho(\varphi)$
$(2, 3, 1), (1, 4, 1)$	$(2, 4, 0), (1, 3, 2)$	$c + d - z \geq 1$	1
$(3, 1, 2), (2, 1, 3)$	$(3, 0, 3), (2, 2, 2)$	$a + b - e \geq 1$	1
$(1, 2, 3), (1, 1, 4)$	$(2, 1, 3), (0, 2, 4)$	$x + y - a \geq -1$	-1/2
$(1, 3, 2), (0, 4, 2)$	$(1, 4, 1), (0, 3, 3)$	$z - d \geq -1$	-1/2
$(1, 5, 0), (0, 5, 1)$	$(1, 4, 1), (0, 6, 0)$	$-d \geq -1$	-1/2
$(2, 0, 4), (1, 1, 4)$	$(2, 1, 3), (1, 0, 5)$	$x - a \geq -1$	-1/2
$(2, 2, 2), (1, 2, 3)$	$(2, 1, 3), (1, 3, 2)$	$e + y - a - z \geq -1$	-1/2
$(2, 2, 2), (1, 3, 2)$	$(2, 3, 1), (1, 2, 3)$	$e + z - c - y \geq -1$	-1/2
$(3, 2, 1), (2, 2, 2)$	$(3, 1, 2), (2, 3, 1)$	$e + f - b - c \geq -1$	-1
$(3, 3, 0), (3, 2, 1)$	$(4, 2, 0), (2, 3, 1)$	$f - c \geq -1$	-1/2
$(4, 1, 1), (4, 0, 2)$	$(5, 0, 1), (3, 1, 2)$	$g - b \geq -1$	-1/2
$(4, 1, 1), (3, 2, 1)$	$(4, 2, 0), (3, 1, 2)$	$f + g - b \geq -1$	-1/2

Table 1: The twelve rhombi of H_6 with $w_\rho \neq 0$, with the boundary values 0 inserted. For the other 33 rhombi $w_\rho = 0$.

Proposition 3.1. $P_6(0, w) = \{\varphi\}$. In particular $P_6(0, w)$ is not empty and contains no integral point.

Proof. By the definition of w , $\varphi \in P_6(0, w)$. Among the 45 inequalities $g_\rho(h) \geq w_\rho$ are the following fifteen; the first two columns give the short diagonal and the tips of the rhombus, and the boundary values 0 are inserted.

- | | | | |
|------|----------------------|----------------------|-------------------------|
| (1) | (1, 1, 4), (0, 1, 5) | (1, 0, 5), (0, 2, 4) | $x \geq 0$ |
| (2) | (1, 2, 3), (0, 2, 4) | (1, 1, 4), (0, 3, 3) | $y - x \geq 0$ |
| (3) | (1, 3, 2), (0, 3, 3) | (1, 2, 3), (0, 4, 2) | $z - y \geq 0$ |
| (4) | (2, 4, 0), (1, 4, 1) | (2, 3, 1), (1, 5, 0) | $d - c \geq 0$ |
| (5) | (2, 3, 1), (1, 4, 1) | (2, 4, 0), (1, 3, 2) | $d + c - z \geq 1$ |
| (6) | (1, 0, 5), (0, 1, 5) | (1, 1, 4), (0, 0, 6) | $-x \geq 0$ |
| (7) | (1, 1, 4), (0, 2, 4) | (1, 2, 3), (0, 1, 5) | $x - y \geq 0$ |
| (8) | (1, 2, 3), (0, 3, 3) | (1, 3, 2), (0, 2, 4) | $y - z \geq 0$ |
| (9) | (2, 3, 1), (1, 3, 2) | (2, 2, 2), (1, 4, 1) | $z + c - d - e \geq 0$ |
| (10) | (3, 1, 2), (2, 1, 3) | (3, 0, 3), (2, 2, 2) | $a + b - e \geq 1$ |
| (11) | (3, 2, 1), (2, 2, 2) | (3, 1, 2), (2, 3, 1) | $e + f - c - b \geq -1$ |
| (12) | (2, 4, 0), (2, 3, 1) | (3, 3, 0), (1, 4, 1) | $c - d \geq 0$ |
| (13) | (3, 1, 2), (3, 0, 3) | (4, 0, 2), (2, 1, 3) | $b - a \geq 0$ |
| (14) | (4, 2, 0), (4, 1, 1) | (5, 1, 0), (3, 2, 1) | $g - f \geq 0$ |
| (15) | (5, 1, 0), (5, 0, 1) | (6, 0, 0), (4, 1, 1) | $-g \geq 0$ |

Let $h \in P_6(0, w)$. By (1) and (6), $x = 0$; by (2) and (7), $y = 0$; by (3) and (8), $z = 0$. By (4) and (12), $c = d$, and then (5) gives $2d = c + d \geq 1$. By (9), $e \leq z + c - d = 0$. By (14) and (15), $f \leq g \leq 0$. By (10) and (13), $2b \geq a + b \geq 1 + e$. By (11), $b + c \leq 1 + e + f \leq 1 + e$. Therefore

$$\frac{1}{2} + \frac{1}{2}(1 + e) \leq c + b \leq 1 + e,$$

that is, $e \geq 0$. So $e = 0$, $b \geq \frac{1}{2}$, $c \geq \frac{1}{2}$ and $b + c \leq 1$, which gives $b = c = \frac{1}{2}$, and $d = c = \frac{1}{2}$. Now (11) gives $f \geq b + c - 1 - e = 0$, so $f = g = 0$; (10) gives $a \geq 1 + e - b = \frac{1}{2}$, and (13) gives $a \leq b = \frac{1}{2}$. Hence $h = \varphi$. $\square$

As an independent check, the package contains for each of the ten interior points p non-negative rational multipliers of the 45 inequalities whose combination is the inequality $h(p) \geq \varphi(p)$, and multipliers whose combination is $-h(p) \geq -\varphi(p)$; the program `example_rank6.py` verifies these twenty combinations in exact arithmetic.

The total. If $h \in P_r^+(0, w)$, the corresponding point of the g-hive polytope is (h, s) with the slack variables $s_\rho = g_\rho(h) - w_\rho$. We call the sum of all its coordinates,

$$\tau(h, w) = \sum_{p \in H_r} h(p) + \sum_{\rho} (g_\rho(h) - w_\rho),$$

the *total* of h . For φ the sum of the values is 2, and $g_\rho(\varphi) - w_\rho$ is $\frac{1}{2}$ for the 21 rhombi on which the excess of φ is not an integer and 0 for the others; so $\tau(\varphi, w) = 25/2$. For part (b) of Corollary 1.3 we need points whose total is an integer.

Proposition 3.2. *Let ψ be the function on H_6 with $a = b = c = d = g = \frac{1}{2}$, $x = y = z = e = f = 0$ and boundary values 0, and let $w_\rho^\psi = \lfloor g_\rho(\psi) \rfloor$. Then $P_6(0, w^\psi) = \{\psi\}$, and $\tau(\psi, w^\psi) = 13$.*

Proof. $\psi \in P_6(0, w^\psi)$ by definition. The inequalities (1) to (13) of the proof of Proposition 3.1 are also among the 45 inequalities $g_\rho(h) \geq w_\rho^\psi$, with the same right-hand sides, and so are

$$\begin{aligned}
(16) \quad & (4, 0, 2), (3, 1, 2) \quad (4, 1, 1), (3, 0, 3) \quad b - g \geq 0 \\
(17) \quad & (4, 1, 1), (4, 0, 2) \quad (5, 0, 1), (3, 1, 2) \quad g - b \geq 0 \\
(18) \quad & (2, 2, 2), (2, 1, 3) \quad (3, 1, 2), (1, 2, 3) \quad e + a - b - y \geq 0 \\
(19) \quad & (3, 1, 2), (2, 2, 2) \quad (3, 2, 1), (2, 1, 3) \quad b + e - f - a \geq 0 \\
(20) \quad & (3, 2, 1), (3, 1, 2) \quad (4, 1, 1), (2, 2, 2) \quad f + b - g - e \geq 0
\end{aligned}$$

Let $h \in P_6(0, w^\psi)$. As before, $x = y = z = 0$, $c = d \geq \frac{1}{2}$, $e \leq 0$ and $b \geq a$. By (18), $e \geq b - a \geq 0$; so $e = 0$ and $a = b$. By (10), $b \geq \frac{1}{2}$. By (19), $f \leq 0$. By (16) and (17), $g = b$. By (20), $f \geq g - b + e = 0$; so $f = 0$. By (11), $b + c \leq 1$. Hence $b = c = \frac{1}{2}$, and $a = g = b$, $d = c$: $h = \psi$. The sum of the values of ψ is $5/2$, and the excess of ψ is not an integer on 21 rhombi; so $\tau(\psi, w^\psi) = 5/2 + 21/2 = 13$. $\square$

The program `example_psi_and_totals.py` of the package checks in exact arithmetic that the eighteen inequalities are among the 45, and the totals.

4 All ranks from 6 on

Embed H_r in H_{r+1} by $\iota(i, j, k) = (i, j, k+1)$. The image is the set of points with $k \geq 1$. Its complement is the side $S = \{k = 0\}$ of H_{r+1} , which consists of boundary points. The map ι sends the sides $i = 0$ and $j = 0$ of H_r into the boundary of H_{r+1} and the side $k = 0$ of H_r onto the row $T = \{k = 1\}$ of H_{r+1} ; the two ends $(0, r, 1)$ and $(r, 0, 1)$ of T are boundary points. The points of a rhombus lie on two rows $k, k+1$ with two points on each, or on three rows $k, k+1, k+2$ with one, two and one point, the two points in the middle forming the short diagonal. Hence the rhombi of H_{r+1} without a point on S are the images of the rhombi of H_r , and a rhombus with a point on S is of one of two kinds:

- two points on S : its short diagonal joins a point of S and a point of T , and its tips are one point of S and one point of T ;
- one point on S : this point is a tip, the short diagonal lies on T , and the other tip lies on the row $k = 2$.

(The short diagonal cannot lie on S , since one tip would have $k = -1$.) The rhombi of the second kind correspond to the r pairs of adjacent points of T , and their tips on the row $k = 2$ are the r points of this row, each once. For an integral w on the rhombi of H_r define w' on the rhombi of H_{r+1} by

$$w'_\rho = \begin{cases} w_{\iota^{-1}\rho} & \text{if } \rho \text{ has no point on } S, \\ 0 & \text{if } \rho \text{ has two points on } S, \\ -1 & \text{if } \rho \text{ has one point on } S. \end{cases}$$

Lemma 4.1. *Let w be integral on the rhombi of H_r .*

- Every $h' \in P_{r+1}(0, w')$ vanishes on T , and $h' \circ \iota \in P_r(0, w)$.
- If $h \in P_r(0, w)$ and $h \leq 1$ on the row $k = 1$ of H_r , then the function h' with $h' \circ \iota = h$ and $h' = 0$ on S lies in $P_{r+1}(0, w')$, and

$$\tau(h', w') = \tau(h, w) + r - \sum_{p \in H_r, k(p)=1} h(p).$$

- (c) If $P_r(0, w) = \{\chi\}$ and $\chi \leq 1$ on the row $k = 1$ of H_r , then $P_{r+1}(0, w') = \{\chi'\}$, where $\chi' \circ \iota = \chi$ and $\chi' = 0$ on S .

Proof. (a) Let $p = (i, r - i, 1)$ and $p' = (i + 1, r - i - 1, 1)$ be adjacent points of T , $0 \leq i \leq r - 1$. Their common neighbour on S is $q = (i + 1, r - i, 0)$. The rhombus with short diagonal $\{p, q\}$ has the tips p' and $(i, r - i + 1, 0) \in S$; the rhombus with short diagonal $\{p', q\}$ has the tips p and $(i + 2, r - i - 1, 0) \in S$. Both have two points on S , where $h' = 0$, and $w' = 0$ for both. So $h'(p) - h'(p') \geq 0$ and $h'(p') - h'(p) \geq 0$. Hence h' is constant on T , and it is 0 at the ends of T ; so $h' = 0$ on T . Therefore $h = h' \circ \iota$ vanishes on the whole boundary of H_r , and $g_\rho(h) = g_{\iota\rho}(h') \geq w'_{\iota\rho} = w_\rho$ for every rhombus ρ of H_r .

(b) The function h' vanishes on the boundary of H_{r+1} and on T . For a rhombus without a point on S the inequality is the one of h . For a rhombus with two points on S all four values are 0. For a rhombus with one point on S the excess is $-h'(t)$, where t is the tip on the row $k = 2$; this is $-h(\iota^{-1}t)$ with $\iota^{-1}t$ on the row $k = 1$ of H_r , and it is at least -1 by assumption. For the total: the values of h' are those of h and zeros; the slack variables of the rhombi without a point on S are those of h ; they are 0 for the rhombi with two points on S and $1 - h(\iota^{-1}t)$ for the rhombi with one point on S , and $\iota^{-1}t$ runs once through the r points of the row $k = 1$ of H_r .

(c) By (b), $\chi' \in P_{r+1}(0, w')$. By (a), every point of $P_{r+1}(0, w')$ vanishes on $S \cup T$, and its composition with ι is χ . $\square$

Proof of Theorem 1.1. For $r = 6$ this is Proposition 3.1, with the point $\varphi_6 = \varphi$. If $P_r(0, w) = \{\varphi_r\}$, where φ_r has the value $\frac{1}{2}$ at four interior points and 0 elsewhere, then Lemma 4.1(c) gives an integral w' with $P_{r+1}(0, w') = \{\varphi_{r+1}\}$, where $\varphi_{r+1} \circ \iota = \varphi_r$ and $\varphi_{r+1} = 0$ on S . $\square$

Proposition 4.2. *For every $r \geq 6$ there is an integral w such that $P_r(0, w)$ consists of a single point, which is non-negative and not integral and whose total is an integer.*

Proof. For $r = 6$ this is Proposition 3.2. For $r \geq 7$ we start from the image of φ under the exchange of the coordinates i and k : let $\varphi^*(i, j, k) = \varphi(k, j, i)$ and $w_\rho^* = \lfloor g_\rho(\varphi^*) \rfloor$. The exchange maps rhombi to rhombi; so $P_6(0, w^*) = \{\varphi^*\}$ and $\tau(\varphi^*, w^*) = 25/2$ by Proposition 3.1. The function φ^* has the value $\frac{1}{2}$ at $(1, 4, 1)$, $(1, 3, 2)$, $(3, 1, 2)$, $(2, 1, 3)$; on the row $k = 1$ of H_6 its only non-zero value is the value $\frac{1}{2}$ at $(1, 4, 1)$. By Lemma 4.1(c) and (b), $P_7(0, w')$ is a single point χ_7 with total $25/2 + 6 - \frac{1}{2} = 18$. The row $k = 1$ of H_7 is the set T of this step, where $\chi_7 = 0$. Applying Lemma 4.1 again and again gives, for every $r \geq 7$, a single point χ_r which vanishes on the row $k = 1$ of H_r and whose total is $\tau_r = \tau_{r-1} + (r - 1)$ for $r \geq 8$, an integer. The points χ_r have the values 0 and $\frac{1}{2}$. $\square$

The program `extension.py` of the package carries out the construction of Lemma 4.1 for the ranks 7 to 12, starting from φ , and checks in exact arithmetic that the extended point satisfies all inequalities, that the two inequalities used in part (a) are present for every pair of adjacent points of T , and that the inequalities of the rhombi without a point on S are the old ones; the program `example_psi_and_totals.py` does the same starting from φ^* and finds the totals 18, 25, 33, 42, 52, 63 for the ranks 7 to 12. These runs are tests of the construction, not part of the proof.

Lemma 4.3. *Let $r \geq 6$, let $a_1, \dots, a_N$ be the columns of M_r and m the number of its rows.*

- (a) *The columns generate the lattice $\mathbb{Z}^m$, and the zero vector is a combination $\sum_i n_i a_i$ with integers n_i and $\sum_i n_i = 1$. In particular the columns do not lie in an affine hyperplane.*
- (b) *Suppose that the convex hull of the columns is the union of simplices $\text{conv}(a_{i_0}, \dots, a_{i_m})$ such that $a_{i_1} - a_{i_0}, \dots, a_{i_m} - a_{i_0}$ form a basis of $\mathbb{Z}^m$. Let $t \geq 1$ be an integer and $\tilde{h} \geq 0$ a real vector with $\sum_i \tilde{h}_i = t$ such that $\beta = M_r \tilde{h}$ is integral. Then $\beta = M_r n$ for a vector n of non-negative integers with $\sum_i n_i = t$.*

Proof. (a) The column of the slack variable of a rhombus ρ is the unit vector e_ρ . The column of a boundary point p is the unit vector e_p plus a combination of the e_ρ with integer coefficients, and the column of an interior point p is $\sum_\rho R_{\rho p} e_\rho$. So the columns generate $\mathbb{Z}^m$. Let p be a point of H_r all of whose coordinates are at least 2; such a point exists because $r \geq 6$. Its six neighbours lie in H_r , so p lies on the short diagonal of six rhombi; and p is a tip of six rhombi, the other tips being the points which differ from p by a permutation of $\pm(1, 1, -2)$, which lie in H_r . As $R_{\rho p}$ is 1 if p is a tip of ρ and -1 if p lies on its short diagonal, $\sum_\rho R_{\rho p} = 0$. Hence $a_p - \sum_\rho R_{\rho p} e_\rho = 0$ is a combination of columns with integer coefficients whose sum is 1. Finally, suppose that the columns lie in an affine hyperplane $\{u = \gamma\}$ with a linear form $u \neq 0$. If $\gamma = 0$, they do not span $\mathbb{R}^m$; if $\gamma \neq 0$, applying u to the combination just found gives $0 = \gamma$.

(b) The point β/t lies in the convex hull of the columns, hence in one of the simplices: $\beta = \sum_{l=0}^m \mu_l a_{i_l}$ with $\mu_l \geq 0$ and $\sum_l \mu_l = t$. Then $\beta - t a_{i_0} = \sum_{l \geq 1} \mu_l (a_{i_l} - a_{i_0})$ is an integral vector, so $\mu_1, \dots, \mu_m$ are integers, and so is $\mu_0 = t - \sum_{l \geq 1} \mu_l$. $\square$

By Lemma 4.3(a) the lattice generated by the columns, and the lattice of the integral combinations of the columns with coefficient sum 0, are both $\mathbb{Z}^m$. So a simplex whose vertices are $m+1$ columns is unimodular, in the usual sense, if and only if its edge vectors at one vertex form a basis of $\mathbb{Z}^m$; this is the notion used in Corollary 1.3(b) and in Lemma 4.3(b).

Proof of Corollary 1.3. Let $r \geq 6$. For an integral w put $\beta_w = (0, -w)$, with the entry 0 for every boundary point and the entry $-w_\rho$ for every rhombus ρ . The solutions $\tilde{h} = (h, s) \geq 0$ of $M_r \tilde{h} = \beta_w$ are the pairs with $h \in P_r^+(0, w)$ and $s_\rho = g_\rho(h) - w_\rho$, and the sum of the coordinates of $\tilde{h}$ is the total $\tau(h, w)$.

(a) Take w as in Theorem 1.1. Then $\{\tilde{h} \geq 0 : M_r \tilde{h} = \beta_w\}$ is a single point, which is not integral, and β_w lies in the cone generated by the columns of M_r . Suppose that β_w lies in the cone generated by m columns which form a matrix A' with $\det A' = \pm 1$. Then $A'^{-1} \beta_w$ is a non-negative integral vector, and filling it up with zeros gives an integral solution $\tilde{h} \geq 0$ of $M_r \tilde{h} = \beta_w$, a contradiction. A unimodular triangulation in the sense of [LM06, Section 4.1] is in particular a unimodular cover, that is, a family of such matrices A' whose cones cover the cone of all columns [LM06, Lemma 4.3 and Corollary 4.4].

(b) Take w as in Proposition 4.2, let $\tilde{h}$ be the unique solution of $M_r \tilde{h} = \beta_w$, $\tilde{h} \geq 0$, and t its total, a positive integer. If the convex hull of the columns were a union of simplices as in the statement, Lemma 4.3(b) would give an integral solution $n \geq 0$ of $M_r n = \beta_w$, which is impossible, since the only solution $\tilde{h}$ is not integral. $\square$

The restriction in Corollary 1.3(b) to simplices whose vertices are columns is no restriction for lattice triangulations: every integral point of the convex hull of the columns is a column. Indeed, let $\beta = M_r \tilde{h}$ be integral with $\tilde{h} = (h, s) \geq 0$ and coordinate sum 1. The boundary values of h are entries of β , hence integers in $[0, 1]$; if one of them is 1, then $\tilde{h}$ is a unit vector. Otherwise h vanishes on the boundary. If $h = 0$, then $s = \beta$ is integral with sum 1. If $h \neq 0$, let $p = (i, j, k)$ be a point with $h(p) > 0$ and i maximal, and let ρ be the rhombus with the tips p and $p + (2, -1, -1)$; the other three points of ρ have a larger first coordinate, so h vanishes there, and the entry $\beta_\rho = h(p) + s_\rho$ is an integer in $(0, 1]$. So it is 1, and all other coordinates of $\tilde{h}$ vanish. For a rhombus $\rho' \neq \rho$ with p on its short diagonal (there is one, because p is an interior point), $\beta_{\rho'} = -h(p)$ is then an integer, so $h(p) = 1$ and $\tilde{h}$ is a unit vector.

5 Ranks at most 5

Proof of Theorem 1.2. For $r \leq 2$ there is no interior point, and the polytope is empty or consists of the integral point b . Let $3 \leq r \leq 5$ and $n = |I_r|$, so $n = 1, 3, 6$. By Lemma 2.1 it suffices to prove (ii) and (ii⁺). The excess forms span the dual space of $\mathbb{R}^{I_r}$ (the computation below finds square submatrices of non-zero determinant), so Lemma 2.2 applies to the forms g_ρ , and to these forms together with the coordinate forms. By Lemma 2.2(a) and (b) it suffices to find, for one representative v of each class of Lemma 2.2(c), an integral vector z with $\ell_k(z) \geq \lfloor \ell_k(v) \rfloor$ for all k .

The program `small_ranks.py` of the package does this in exact rational arithmetic. It forms the distinct non-zero forms up to sign, runs through all sets of n of them (the singular ones included), computes the determinant, and for every non-singular matrix L with $|\det L| \geq 2$ the subgroup of $\mathbb{R}^n/\mathbb{Z}^n$ generated by the columns of L^{-1} (for $|\det L| = 1$ the only class is that of 0). For every class found it searches an integral z and verifies the inequalities for all forms. Table 2 gives the numbers. A vector z was found for every class. $\square$

A second proof of (ii⁺), and hence of (ii), for $r = 3, 4, 5$, which does not use Lemma 2.2, was given by one of the verification runs described in Section 7. By the reformulation $\ell(z) \geq \lfloor \ell(x) \rfloor \iff \ell(x - z) < 1$, statement (ii⁺) says that the integer translates of the open polytope $U = \{y \in \mathbb{R}^{I_r} : g_\rho(y) < 1 \text{ for all } \rho, y_p < 1 \text{ for all } p\}$ cover $\mathbb{R}^{I_r}$. The run found a partition of the cube $[0, 1)^{I_r}$ into half-open boxes with dyadic corners, 4 boxes for $r = 3$, 64 for $r = 4$ and 21 484 for $r = 5$, together with an integral vector z for each box such that the box lies in $z + U$; a separate program verifies in integer arithmetic that each box lies in its translate, that the boxes are pairwise disjoint and that their volumes add up to 1. The partition and both programs are in the package (`verification_run_B/`).

r	statement	n	inequalities	forms	sets of n forms	classes	denominators
3	(ii), (ii ⁺)	1	9, 10	1	1	1	1
4	(ii), (ii ⁺)	3	18, 21	9	84	4	1, 2
5	(ii)	6	30	18	18 564	394	1 to 7
5	(ii ⁺)	6	36	21	54 264	418	1 to 7

Table 2: The computation for Theorem 1.2. “Forms” is the number of distinct non-zero forms up to sign; the sets of n forms include the singular ones (7 827 of 18 564 and 29 181 of 54 264 for $r = 5$); “classes” is the number of classes of Lemma 2.2(c) modulo $\mathbb{Z}^n$, and the last column gives the denominators of their coordinates.

6 Remarks

Remark 6.1 (the range of the earlier verification). The sources say that the conjecture was verified for $r \leq 6$ (Section 1.4). The abstract [McA04], [LM06, Section 4.1] and [McA06, Section 3.1] print the dimension $(r + 1)(r + 2)/2$, which is the indexing of [LM06, Definition 2.2] and the one we use. In this indexing the example of Section 3 has $r = 6$, and the statement of [LM06, Theorem 4.6] for $r = 6$ and Corollary 1.3 cannot both hold.

The letter r is not used in one way throughout these texts. The example of a hive pattern printed in [LM06, p. 5] with the label “ $r = 5$ ” has five rows and fifteen entries, and its boundary consists of partitions with four parts; this is rank 4 in the indexing of Definition 2.2 there. The thesis prints the same pattern, reflected, with the label $r = 4$ [McA06, p. 8]. The experiments with Littlewood–Richardson coefficients are described for partitions with r parts in [LM06, Section 3.1], with the ranges $r \leq 5$, $r = 6$ and $r = 7$, and for partitions with $r + 1$ parts in [McA06, Section 2.2.1], with the ranges

$r \leq 4$, $r = 5$ and $r = 6$. The range $r \leq 6$ of the verification of the conjecture is the same in [McA04], [LM06] and [McA06]. None of the texts gives the size of a matrix that was triangulated.

If the range of the verification is counted in rows, it ends with rank 5 of Definition 2.2, which is the last rank in which integral points always exist (Theorems 1.1 and 1.2). We cannot tell from the texts which indexing the verification used, and other explanations, for instance another encoding of B and R or an error in one of the computations, cannot be excluded from the texts.

Remark 6.2 (integral vertices). The conjecture asks for an integral vertex. Theorem 1.2 gives an integral point for $r \leq 5$; we did not examine the triangulations of [LM06, Theorem 4.6]. For each of the 394 classes of rank 5 the polytope $P_5(0, \lfloor g(v) \rfloor)$ has an integral vertex; this was found by a separate exact computation and is a test, since it concerns only these right-hand sides.

Remark 6.3 (rank 6). The following computation is not used in any proof. For $r = 6$ there are 30 distinct forms up to sign and $\binom{30}{10} = 30\,045\,015$ sets of ten forms. The program `rank6_vertices.py` of the package finds 173 839 classes as in Lemma 2.2(c), with denominators up to 24. For 173 677 of them an integral z was found and verified. For the other 162 classes v the program gives rational bounds $\lambda \leq h(p) \leq \mu$ for one interior point p , valid on the polytope $P_6(0, \lfloor g(v) \rfloor)$ and verified in exact arithmetic, such that the interval $[\lambda, \mu]$ contains no integer; so these classes are bad. The 162 classes found form 27 orbits of size 6 under the permutations of the coordinates (i, j, k) ; their denominators are 2 (18 classes), 3 (30), 4 (30), 5 (24), 6 (6), 7 (6), 8 (6), 9 (24) and 18 (18). The half-integral ones form three orbits, with 4, 4 and 5 values $\frac{1}{2}$; among them are the orbits of φ and of ψ . For these the polytope is a point; for some of the others it has positive dimension. The enumeration of the classes uses floating-point determinants, which are rounded and, when the absolute value is at least 2, confirmed by an exact identity for the adjugate.

Remark 6.4 (what remains open). We do not know a description of the integral vectors c for which the polytope (2) of rank $r \geq 6$ is not empty and has no integral point, nor whether the points of such polytopes have bounded denominators as r grows.

7 Computations and reproducibility

The proofs of Lemmas 2.1, 2.2, 4.1 and 4.3, of Propositions 3.1, 3.2 and 4.2, of Theorem 1.1 and of Corollary 1.3 use no computation; the only numbers taken from a count are the two totals 25/2 and 13, which are sums of 28 values and of 45 slack variables. Theorem 1.2 rests on an exact computation (the program `small_ranks.py`). The programs for these statements use only integers and rational numbers and need no software beyond Python. The reproducibility package contains the programs, the command lines and the recorded outputs:

<code>hive_system.py</code>	the points and the rhombus inequalities, from the three families as printed in [LM06, Definition 2.2]; checks that these are the unit rhombi, each once
<code>example_rank6.py</code>	checks for Proposition 3.1: φ satisfies the 45 inequalities; the fifteen inequalities are among them; twenty rational combinations which give $P_6(0, w) = \{\varphi\}$ independently of the proof; a search for integral points in a box
<code>example_psi_and_totals.py</code>	checks for Propositions 3.2 and 4.2: the eighteen inequalities, the totals, the extensions of φ^*
<code>extension.py</code>	the construction of Lemma 4.1 for ranks 7 to 12; a search for integral points in rank 7
<code>small_ranks.py</code>	Theorem 1.2, Table 2
<code>integral_vertices_rank5.py</code>	the test of Remark 6.2
<code>rank6_vertices.py</code>	Remark 6.3 (needs NumPy and SciPy)

Verification. The example of rank 6 was found by a search over the half-integral points, and the statements and proofs were then written with the programs of the table above. Two independent verification runs, both AI-assisted, followed; they did not see each other, and each wrote its own programs from [LM06, Definition 2.2].

The first run compared the formalisation with [McA04, LM06]; checked for each of the fifteen inequalities of Proposition 3.1 that it is one of the printed inequalities, with its right-hand side, and checked the derivation; derived the proofs of Lemmas 2.1, 2.2 and 4.1 and of Corollary 1.3(a) again; read and ran `small_ranks.py`; and reproduced Table 2 with a separate implementation, in which the classes were also enumerated directly from their definition. It found no mathematical error. It observed that the fifteen inequalities already give the single point, and that the form of the conjecture in [McA04, Conjecture 1] needs examples whose total is an integer; Propositions 3.2 and 4.2, Lemma 4.3 and Corollary 1.3(b) were written after its report, with the examples it proposed.

The second run built B , R and M_r with its own code and showed, by an exact simplex method with certificates verified against M_6 , that the g-hive polytope of Section 3 is a single point with 25 non-integral coordinates; it did the same for the extended examples of ranks 7 and 8; it proved Theorem 1.2 a second time, by the partition into boxes described in Section 5; it tested all classes with denominators up to 20 in rank 5, and random right-hand sides in ranks 4 and 5, and found no nonempty polytope without integral point; it reproduced the recorded outputs of the programs of this note; and it searched the literature. It found no mathematical error. It found the passages of [McA06] recorded in Remark 6.1, and it also arrived at the examples with integral total. All corrections required by the two runs were made.

A third independent verification run, also AI-assisted, was made on the final text. It read all statements and proofs again and compared the sentences about the sources with the texts, and the record with the dataset. With programs of its own, in exact arithmetic, it checked Table 1, the inequalities (1) to (20) with their right-hand sides, the totals $25/2$ and 13, the point φ^* and the totals of its extensions; it found non-negative rational combinations of the fifteen inequalities which give $h = \varphi$, of the eighteen which give $h = \psi$, and of the inequalities of the extended examples of ranks 7, 8 and 9 which give the single point; it reproduced Table 2 with an implementation of its own and checked the partition into boxes of Section 5 with its own forms; and it verified the recorded bounds for the 162 classes of Remark 6.3. It found no mathematical error. The corrections it required, which concern the bibliography and the package, were made. The paragraph after the proof of Corollary 1.3, on the integral points of the convex hull of the columns, was added at this revision; its argument is due to this run. Its programs and outputs are in the package (`independent_run_2/`).

Scope and priority. The note answers the question of the record for every rank: integral points always exist for $r \leq 5$ (Theorem 1.2) and not for $r \geq 6$ (Theorem 1.1); and it shows that the conjecture of De Loera and McAllister does not hold for $r \geq 6$, in the form of [LM06, Conjecture 4.5] and in the form of [McA04, Conjecture 1] (Corollary 1.3). The question, the conjecture, the matrix M_r and the notion of a g-hive polytope are due to De Loera and McAllister. Theorem 1.2 concerns integral points, not integral vertices or triangulations; it is contained in [LM06, Theorem 4.6] (Section 1.4), of which our proof is independent. New, to our knowledge and relative to the literature we found, is the failure of the conjecture for every $r \geq 6$. What the sources say about the range $r \leq 6$ is recorded in Remark 6.1 without a judgement. We claim no priority.

What was read: the abstract [McA04] (pp. 2072–2074) in the publisher’s file; in the arXiv version of [LM06] the abstract, Section 1, Section 2 up to Lemma 2.4, the passages of Section 3.1 on the ranks, and Section 4.1; in [McA06] Definition 1.2 with the example which follows it, the passages of Section 2.2.1 on the ranks, and Section 3.1 up to Theorem 3.6; in version 3 of the arXiv version of [HPPS21] the passage of Section 1.2.1 on hive polytopes and the question on hive matrices at the

end of Section 1; the abstract [Haa15] on the web page of the meeting. Not read: the journal version of [LM06], which was not accessible to us; the published version of [HPPS21]; [KT99] and [Ram02], which are cited as in [McA04, LM06].

The searches were made on 12 October 2026: queries to the arXiv, zbMATH Open, Crossref, OpenAlex and Semantic Scholar, among them the lists of the works which cite [LM06] (75, 46 and 32 entries in Semantic Scholar, OpenAlex and Crossref, examined by title), and nine web searches. The texts found which mention the conjecture are [McA06], [HPPS21] and [Haa15]. MathSciNet and Google Scholar were not searched, and the authors of the conjecture were not contacted. No counterexample to the conjecture and no statement that it fails was found in the literature accessible to us; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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