

Algebraic Shifting Can Destroy Every Polynomial HVC Bound

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Abstract

Kalai asked whether algebraic shifting preserves homological VC dimension, defined by a uniform polynomial bound on total Betti numbers in a family closed under induced subcomplexes and links. We give a negative answer without a bound on simplicial dimension. A strongly hereditary family of minimum polynomial HVC exponent exactly 1 has, after either exterior or symmetric algebraic shifting over $\mathbb{Q}$, a strongly hereditary closure with no finite polynomial HVC exponent. The obstruction comes from cross-polytope boundaries. If Δ_r is either shift of the boundary of the r -dimensional cross-polytope, then the link of the first s vertices has $2r - s$ vertices and top reduced Betti number $\binom{r}{s}$. Central links therefore have exponentially growing homology. The proof combines standard shifting properties with the classical facet interpretation of the h -vector.

1 The family formulation and the result

All homology is over $\mathbb{Q}$. For a finite simplicial complex K , put $n(K) = f_0(K)$ and $b(K) = \sum_{i \geq 0} \dim H_i(K; \mathbb{Q})$. A family is *strongly hereditary* if it is closed, up to relabeling, under induced subcomplexes and links of faces. Following Kalai [1, p. 1730, Definitions 1–2], a polynomial HVC bound of exponent d means that one constant $A > 0$ satisfies

$$b(K) < A n(K)^d \tag{1}$$

throughout the family. We impose this inequality for $n(K) \geq 1$. Equivalently, replace $n(K)$ by $\max(1, n(K))$; the complex $\{\emptyset\}$ has $b = 0$. This removes the immaterial strict inequality $0 < 0$ at zero vertices. We distinguish the least possible polynomial exponent from the simplicial dimensions of the complexes.

Kalai's Conjecture 3 asks whether algebraic shifting preserves this bound [1]. The family interpretation, including the need to close the shifted family under links and induced subcomplexes, is explicit in Hell [2, Definition 5.29, Conjecture 5.31 and Observation 5.32]. Write $\text{cl}_{\text{sh}}(\mathcal{F})$ for this strongly hereditary closure. Let D_m be the complex of m isolated vertices, and let Σ_t be the simplex on t vertices, with $\Sigma_0 = \{\emptyset\}$. Define

$$J_{r,t} = (D_2)^{*r} * \Sigma_t, \quad \mathcal{C} = \{J_{r,t} : r, t \geq 0\} \cup \{D_m : m \geq 0\},$$

including all relabelings. In particular, $K_r := J_{r,0}$ is the boundary of the r -dimensional cross-polytope.

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Theorem 1. *The family $\mathcal{C}$ is strongly hereditary and has minimum polynomial HVC exponent exactly 1, with constant $A = 3$. For either standard exterior or symmetric algebraic shifting Δ over $\mathbb{Q}$, the family*

$$\text{cl}_{\text{sh}}(\{\Delta(K) : K \in \mathcal{C}\})$$

has no finite polynomial HVC exponent. More precisely, if $r \geq 2$, $0 \leq s \leq r - 1$, and $\Delta_r = \Delta(K_r)$ on the ordered vertex set $[2r]$, then

$$n(\text{lk}_{\Delta_r}([s])) = 2r - s, \quad \tilde{\beta}_{r-s-1}(\text{lk}_{\Delta_r}([s])) = \binom{r}{s}. \quad (2)$$

Thus the unrestricted preservation assertion fails even if the constant and the exponent are both allowed to change after shifting. The simplicial dimensions in this example are unbounded. The theorem makes no claim about a version with a fixed dimension bound, or about the adjacent fractional Helly and nerve conjectures in [1].

2 The input family

Proposition 2. *The family $\mathcal{C}$ has the input properties in Theorem 1.*

Proof. On restriction to a vertex subset, each two-point factor of $J_{r,t}$ leaves two, one, or zero vertices. The one-point factors merge with the remaining simplex factor, so the induced complex is another $J_{r',t'}$. A face of a two-point factor contains at most one vertex. Taking its link removes that entire factor if a vertex was chosen, and otherwise retains it. In the simplex factor, taking a link deletes the chosen vertices. Links are therefore also of the form $J_{r',t'}$. Induced subcomplexes of D_m are discrete, and a nonempty face has link $\{\emptyset\}$. This proves strong heredity.

If $t > 0$, then $J_{r,t}$ is contractible. If $t = 0$ and $r \geq 1$, it is the sphere S^{r-1} , whose total ordinary Betti number is 2, also when $r = 1$. The zero-vertex complex has $b = 0$. Moreover $b(D_m) = m$. Hence, for every $K \in \mathcal{C}$ on $n \geq 1$ vertices,

$$b(K) \leq \max(2, n) < 3n.$$

For any real $e < 1$, the ratios $b(D_m)/m^e = m^{1-e}$ are unbounded. Thus no smaller polynomial exponent works. $\square$

3 Initial links of pure shifted complexes

A complex on $[N]$ is *shifted* if replacing a vertex of a face by a smaller missing vertex produces a face. It is *pure* of dimension $r - 1$ when every facet has size r . Use the face and h -polynomials

$$F_K(x) = \sum_{A \in K} x^{|A|}, \quad h_K(t) = (1-t)^r F_K\left(\frac{t}{1-t}\right) = \sum_{i=0}^r h_i(K)t^i.$$

We recall two elementary properties with proofs. The facet formula below is classical, due to Kalai; see Murai [3, Lemma 2.1] or [4, Lemma 1.4].

Lemma 3. *Let K be pure and shifted of dimension $r - 1$. For $0 \leq s \leq r$, the number of facets containing $[s]$ and omitting $s + 1$ is $h_{r-s}(K)$.*

Proof. For a facet F , let $s(F)$ be the largest s for which $[s] \subseteq F$, and put $R(F) = F \setminus [s(F)]$. The intervals of faces

$$[R(F), F] = \{A : R(F) \subseteq A \subseteq F\}$$

partition K . To verify this, assign each face A to the lexicographically first facet F containing it. If A omitted a vertex $j \in R(F)$, then $j > s(F) + 1$, and replacing j by the missing vertex $s(F) + 1$ would give an earlier facet containing A . Thus $R(F) \subseteq A$.

Conversely, suppose $R(F) \subseteq A \subseteq F$. Any distinct facet G containing A must retain all of $R(F)$. Consequently $F \setminus G \subseteq [s(F)]$, whereas every element of $G \setminus F$ exceeds $s(F)$. Since the facets have equal size, both differences are nonempty, and their smallest element belongs to F . Therefore F precedes G . This proves the partition. It gives

$$F_K(x) = \sum_{F \text{ facet}} x^{r-s(F)}(1+x)^{s(F)}, \quad h_K(t) = \sum_{F \text{ facet}} t^{r-s(F)}.$$

The coefficient of t^{r-s} is the stated count. $\square$

Lemma 4. *If L is a pure shifted q -complex, $q \geq 0$, with least vertex a , then $\tilde{\beta}_q(L)$ equals the number of facets of L that do not contain a .*

Proof. For every facet T avoiding a , consider the formal chain $z_T = \partial([a] * [T])$. Every term of this boundary lies in L : the term T is a facet and the other terms replace one vertex of T by a . Thus z_T is a q -cycle. These cycles are independent, because T occurs with coefficient 1 in its own cycle and in none of the others.

Subtract suitable multiples of the z_T 's from any top cycle. The remainder is supported only on facets containing a . For such a facet $\{a\} \cup U$, its coefficient, up to sign, is the coefficient of U in the boundary of the remainder: no other facet containing a has U as its opposite face. Therefore all coefficients vanish. There are no $(q+1)$ -chains in L , so this cycle basis is a homology basis. For $q = 0$, use the augmented boundary; the cycles are $[v] - [a]$, and the same argument applies. $\square$

Proposition 5. *For a pure shifted $(r-1)$ -complex K on $[N]$, and $0 \leq s \leq r-1$,*

$$n(\text{lk}_K([s])) = N - s, \quad \tilde{\beta}_{r-s-1}(\text{lk}_K([s])) = h_{r-s}(K).$$

Proof. The link is pure of dimension $r-s-1$ and shifted on the remaining ordered vertices. Every $v \notin [s]$ occurs in it: choose a facet containing v and shift other vertices into each missing position of $[s]$. There are enough other positions since $r-1 \geq s$, and all replacements decrease the labels. This preserves v . Hence the link has exactly $N-s$ vertices and least vertex $s+1$. By Lemma 4, its top reduced Betti number counts facets avoiding $s+1$. These correspond to the facets of K containing $[s]$ and omitting $s+1$, counted by Lemma 3. $\square$

4 Cross-polytopes and exponential growth

Proof of Theorem 1. Proposition 2 proves the assertions about $\mathcal{C}$. Standard exterior and symmetric algebraic shifting over $\mathbb{Q}$ preserve the f -vector and take a Cohen–Macaulay complex to a pure shifted complex of the same dimension; see Murai [3, Section 1 and Lemma 1.2], also [4, Lemma 1.3]. These are the only algebraic-shifting properties needed here.

The cross-polytope boundary K_r is a simplicial sphere, hence is Cohen–Macaulay over $\mathbb{Q}$. Its face polynomial and h -polynomial are

$$F_{K_r}(x) = (1+2x)^r, \quad h_{K_r}(t) = (1-t)^r \left(1 + \frac{2t}{1-t}\right)^r = (1+t)^r.$$

Thus Δ_r is pure and shifted on $2r$ vertices with $h_i(\Delta_r) = \binom{r}{i}$. Proposition 5 proves (2).

Set $s_r = \lfloor r/2 \rfloor$ and $L_r = \text{lk}_{\Delta_r}([s_r])$. Then $m_r = n(L_r) = 2r - s_r \leq 2r$. The central binomial coefficient is at least the average of all $r + 1$ binomial coefficients, so

$$b(L_r) \geq \tilde{\beta}_{r-s_r-1}(L_r) = \binom{r}{\lfloor r/2 \rfloor} \geq \frac{2^r}{r+1}. \quad (3)$$

For every fixed $e \geq 0$, the quantity $2^r / ((r+1)(2r)^e)$ tends to infinity; its successive ratio tends to 2. Consequently, for any $B > 0$, $b(L_r) > B m_r^e$ for sufficiently large r . Every strongly hereditary family containing all Δ_r contains these links. Hence no finite polynomial HVC exponent is possible. $\square$

Remark 6. The individual complexes retain their ordinary homology under algebraic shifting. The obstruction lies in links required by strong heredity. It does not conflict with preservation of a fixed d -Leray property: the dimensions of the input spheres here tend to infinity.

5 Verification and scope

The proof above is uniform in r ; finite calculations are not used to infer the asymptotic conclusion. The accompanying standard-library Python checker performs 4,317 exact checks using integer arithmetic and rational boundary-matrix ranks. It enumerates all 133 pure shifted complexes on at most six vertices with facet size at most four, checks their face partitions and 387 initial links, and checks the input join decompositions. Two negative controls detect the missing shiftedness hypothesis and a wrong h -index. Normal and optimized Python runs produce identical results. No numerical computation of a generic initial ideal is required.

The result answers the unrestricted source assertion in [1, p. 1730, Conjecture 3], recorded as OWR-734-007 in the frozen UnsolvedMath v1.6.0 corpus. The contribution is the cross-polytope family obstruction to that assertion; the shifting properties and h -vector identities used in its proof are established results. A bounded search of primary sources found no earlier occurrence of this counterexample. This statement is not an absolute-priority claim.

This manuscript was developed with AI assistance in mathematical exploration, source retrieval, exact computation and exposition. The argument was self-audited; the preprint has not undergone independent peer review or proof-assistant verification.

References

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