

An Odd-Divisor Obstruction for Cyclically Invariant Grassmannian Initial Ideals

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Abstract

Let $I_{k,n}$ be the Plücker ideal in its natural coordinates over a field of characteristic zero. If $\gcd(k, n)$ has an odd divisor greater than one, no monomial initial ideal of $I_{k,n}$ is invariant under cyclic rotation of the labels. More generally, write $n = mq$ and $k = rq$, where $q > 1$ is odd and $1 \leq r < m$. For a permutation consisting of m disjoint q -cycles, the trace on the quadratic relations is $-\binom{m}{r, 2}$. A stable monomial initial space would have nonnegative trace, giving the obstruction. The same calculation excludes an invariant monomial basis even in degree two, without requiring that the basis come from a term order. The proof uses classical standard monomial theory and a two-column character identity. It extends the obstruction observed by Santos, Stump and Welker for $\text{Gr}(3, 6)$; no squarefreeness, quadratic-generation or Gorenstein hypothesis is imposed on the putative initial ideal.

1 Statement and context

Let K be a field of characteristic zero and put

$$S = K[p_A : A \subseteq [n], |A| = k], \quad R = S/I_{k,n}, \quad 1 \leq k < n.$$

Here p_A represents the maximal minor with column set A of a generic $k \times n$ matrix, and $I_{k,n}$ is the kernel of this minor map. All variables have degree one. We use S_d, R_d , and $(I_{k,n})_d$ for the homogeneous degree- d spaces.

A permutation $\sigma \in S_n$ acts on the variables with the usual exterior-algebra signs:

$$\sigma(p_{\{a_1 < \dots < a_k\}}) = p_{\sigma(a_1), \dots, \sigma(a_k)}.$$

The right-hand side is sorted increasingly, with the sign of the sorting permutation. This action preserves $I_{k,n}$. For a monomial ideal, stability depends only on the underlying permutation of its monomials, so it is unaffected by these signs. A *stable monomial basis* of R_d means a set of degree- d monomials in the variables p_A , stable as a set up to these signs, whose images form a K -basis of R_d . It does not mean an arbitrary polynomial basis.

Theorem 1. *Suppose $n = mq$, $k = rq$, $q > 1$ is odd, and $1 \leq r < m$. Let $\sigma \in S_n$ have cycle type q^m , and set $s = \binom{m}{r}$. Then*

$$\text{tr}(\sigma \mid S_2) = \frac{s(s+1)}{2}, \quad \text{tr}(\sigma \mid R_2) = s^2, \quad \text{tr}(\sigma \mid (I_{k,n})_2) = -\binom{s}{2}. \quad (1)$$

There is no σ -stable monomial basis of R_2 . Consequently no monomial initial ideal of $I_{k,n}$, for any term order in the given coordinates, is σ -stable.

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Corollary 2. *If $\gcd(k, n)$ has an odd divisor $q > 1$, no monomial initial ideal of $I_{k,n}$ is invariant under $c = (1\ 2\ \cdots\ n)$. There is also no c -stable monomial basis of R_2 .*

Proof. Write $n = mq$ and $k = rq$. Stability under c implies stability under c^m , which has cycle type q^m , and apply Theorem 1. $\square$

The result concerns an additional symmetry requirement. Conca's Question 6 [1, p. 1716] asks whether a Gorenstein classical ideal has a Gorenstein initial ideal in the given coordinates with the same number of minimal generators. For Grassmannians, the regular unimodular flag Gorenstein noncrossing triangulations of Petersen, Pylyavskyy and Speyer [2], and Santos, Stump and Welker [3], provide such initial ideals. Indeed, both ideals are generated by quadrics, and equality of Hilbert functions gives equality of their degree-two dimensions and hence of their numbers of minimal generators. Our result does not contradict these constructions or settle Conca's broader question.

Santos, Stump and Welker [3, Remark 1.10] already exclude cyclic symmetry for a flag complex on the Plücker labels of $\text{Gr}(3, 6)$ with the required h -vector, by counting edge orbits. Theorem 1 treats all the stated odd-divisor parameters and all monomial initial ideals, including those that are not assumed squarefree or generated by quadrics. Its trace argument also applies to monomial bases independently of a Gröbner construction.

2 A character obstruction for stable monomials

We first isolate the elementary linear algebra.

Lemma 3. *Let V have a finite distinguished basis B , and suppose an operator g of odd finite order acts on B by a signed permutation. Let $W \subseteq V$ be g -stable. If a subset $C \subseteq B$ is stable up to signs and its images form a basis of V/W , then*

$$\text{tr}(g \mid W) = \#\{b \in B \setminus C : g(b) = b\} \geq 0.$$

The equality is an equality of integers embedded in K .

Proof. Every basis vector fixed up to sign by g has sign $+1$: a sign -1 would persist after taking the odd order of g . Nontrivial permutation cycles contribute zero to the trace. Thus the trace on V counts its fixed basis vectors. The images of C form a signed permutation basis of the quotient, so its trace counts the fixed members of C . Subtract the two traces using the invariant exact sequence $0 \rightarrow W \rightarrow V \rightarrow V/W \rightarrow 0$. $\square$

Corollary 4. *Let a finite odd-order signed permutation of polynomial variables preserve a homogeneous ideal I . If some homogeneous piece I_d has negative integer trace, then no invariant monomial initial ideal of I exists.*

Proof. For a term order, the degree- d monomials outside $\text{in}(I)$ map to a basis of $(S/I)_d$. If the initial ideal is invariant, these monomials are stable up to signs. Apply Lemma 3 to $V = S_d$ and $W = I_d$. $\square$

The argument does not require the chosen term order to be preserved by the group, and does not assume that a flat degeneration is equivariant. Invariance of the resulting set of standard monomials suffices.

3 The degree-two character

For variables $x = (x_1, \dots, x_n)$, let $e_j(x)$ be the elementary symmetric polynomial, with $e_0 = 1$. Classical standard monomial theory gives a basis of R_2 consisting of $p_A p_B$ with

$$A = (a_1 < \dots < a_k), \quad B = (b_1 < \dots < b_k), \quad a_i \leq b_i \quad (1 \leq i \leq k).$$

See, for example, [2, 3]. Thus, for a diagonal operator with eigenvalues $x_1, \dots, x_n$, the character of R_2 is the sum of $x_A x_B$ over these componentwise ordered pairs, where $x_A = \prod_{a \in A} x_a$.

Lemma 5. *The degree-two character of the Plücker algebra is*

$$\chi_{R_2}(x) = e_k(x)^2 - e_{k-1}(x)e_{k+1}(x). \quad (2)$$

Proof. This is the two-column Schur identity; we recall a direct weight-preserving proof. The first term counts all ordered pairs of k -subsets. For a pair (A, B) , set

$$h(t) = |A \cap [t]| - |B \cap [t]|.$$

The componentwise inequalities $a_i \leq b_i$ hold exactly when $h(t) \geq 0$ for all t . For a bad pair, take the first t with $h(t) = -1$, and exchange the memberships of A and B within $[t]$. The resulting ordered pair (C, D) has sizes $k+1, k-1$. The multiset union, and therefore the monomial weight, is unchanged.

Conversely, for any pair of sizes $k+1, k-1$, its prefix-count difference ends at 2, so there is a first prefix with difference $+1$. Exchanging memberships on that prefix recovers a bad pair of k -subsets, and reverses the construction. The bad-pair weight sum is therefore $e_{k+1}e_{k-1}$, proving (2). $\square$

The identity is a character identity for the natural general-linear action, so it may be evaluated at the eigenvalues of any diagonalizable operator. In characteristic zero a finite order operator is diagonalizable after extending the field. Traces and the existence of a specified monomial basis are unchanged by this field extension.

Proof of Theorem 1. For a permutation σ of cycle type q^m , with q odd,

$$\det(1 + t\sigma) = (1 + t^q)^m. \quad (3)$$

Indeed, a single q -cycle has characteristic polynomial $z^q - 1$, and hence contributes $1 - (-t)^q = 1 + t^q$. Writing $e_j(\sigma)$ for the corresponding elementary symmetric function of its eigenvalues, (3) gives

$$e_k(\sigma) = \binom{m}{r} = s, \quad e_{k-1}(\sigma) = e_{k+1}(\sigma) = 0.$$

Lemma 5 now gives $\text{tr}(\sigma | R_2) = s^2$.

The degree-one variable space is the exterior permutation representation $E = \bigwedge^k K^n$, and $S_2 = \text{Sym}^2 E$. For any operator g on E ,

$$\text{tr}(g | \text{Sym}^2 E) = \frac{\text{tr}(g | E)^2 + \text{tr}(g^2 | E)}{2}.$$

This follows directly by adding the eigenvalue products with weakly increasing indices. Because q is odd, σ^2 has the same cycle type as σ , so both exterior-power traces equal s . This proves the first equality in (1); subtracting the quotient trace gives the third.

Finally $1 \leq r < m$ implies $s \geq 2$, so the trace on $(I_{k,n})_2$ is strictly negative. The degree-two monomials of S form a signed permutation basis for the odd-order action. Lemma 3 excludes a stable monomial basis of R_2 , and Corollary 4 excludes every invariant monomial initial ideal. $\square$

There is a direct fixed-monomial interpretation of the same calculation. The fixed Plücker variables correspond to unions of r of the m cycles, so there are s of them. An odd-order permutation cannot interchange two distinct variables. Consequently the fixed quadratic monomials are exactly the s squares and the $\binom{s}{2}$ products of distinct fixed variables. Their total $s(s+1)/2$ is smaller than the trace s^2 required of R_2 .

4 Examples, checks, and limitations

For $\text{Gr}(3, 6)$, take $\sigma = (135)(246)$. The fixed variables are p_{135}, p_{246} , giving three fixed quadratic monomials. The quotient trace is four, and the relation-space trace is -1 . For $\text{Gr}(3, 9)$, three 3-cycles give $s = 3$, quotient trace nine, ambient trace six, and relation trace -3 . For $\text{Gr}(5, 10)$, two 5-cycles again give relation trace -1 . Composite odd q are allowed; primality is not used.

(k, n)	cycle length q	$\text{tr}(S_2)$	$\text{tr}(R_2)$	$\text{tr}(I_2)$
$(3, 6)$	3	3	4	-1
$(3, 9)$	3	6	9	-3
$(6, 9)$	3	6	9	-3
$(5, 10)$	5	3	4	-1
$(4, 8)$	4	5	4	1
$(2, 6)$	2	12	9	3

The first four rows satisfy the theorem; the last two are even-order controls and make no existence claim.

The accompanying script independently constructs the signed Plücker quadratic relations and reproduces all six traces by exact rational row reduction. It also checks the quotient character by eigenvalue-weight enumeration on five examples, including $\text{Gr}(6, 12)$, and the reflection bijection for all $1 \leq k < n \leq 8$. These finite diagnostics support the written proof; they are not a formal proof-assistant verification.

The conclusion is sufficient rather than a classification: no assertion is made when $\gcd(k, n)$ is a power of two, including one, or in positive characteristic. The theorem concerns actual products of the given Plücker coordinates. It does not exclude invariant polynomial bases. Likewise, tableau promotion is not literal rotation of each label in a monomial. The promotion bijections for positroid standard monomials in [4, Theorem C] concern that distinct operation and are consistent with the obstruction here.

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References

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