

The Canonical Normalized Tight Frame of a Wavelet Frame Need Not Have Wavelet Structure: An Explicit Riesz Wavelet for a Question of Han and Frank

Alper Ferudun*

October 17, 2026

Abstract

D. Han asked in 2003 (Trans. Amer. Math. Soc. 355, Question 4) whether $S^{-1/2}\psi$ is always a wavelet frame when ψ is a wavelet frame with frame operator S ; M. Frank restated the question in 2004 (Oberwolfach Rep. 1): does the canonical normalized tight frame $\{S^{-1/2}D^jT_k\psi\}$ of a wavelet frame always have wavelet structure? Han proved that it does for semi-orthogonal wavelet frames, and Bownik, Jasper and Speegle (2011) asserted, without an example or a proof, that this fails for some frame wavelets. We give an explicit example with complete proofs. For an orthonormal dyadic wavelet ψ of $L^2(\mathbb{R})$ and $0 < \varepsilon < 1$, the function $\eta = \psi + \varepsilon D\psi$ generates a Riesz wavelet basis with optimal frame bounds $(1 - \varepsilon)^2$ and $(1 + \varepsilon)^2$; it is the classical example of a Riesz wavelet whose canonical dual frame is not a wavelet system (Daubechies; Chui; Chui and Shi). Its canonical normalized tight frame is the orthonormal basis $\{U\psi_{j,k}\}$, where U is the unitary polar factor of $I + \varepsilon B$ and $B\psi_{j,k} = \psi_{j+1,2k}$. We compute U exactly: B is a bilateral shift on the span of the $\psi_{j,0}$, where U is multiplication by $(1 + \varepsilon z)/|1 + \varepsilon z|$, and a unilateral shift on the other invariant subspaces, where the matrix of U is $(u_{n-m} - u_{n+m+2})_{n,m \geq 0}$ with explicit binomial series u_i , all nonzero and of known sign. Consequently $\|S^{-1/2}T_1\eta - T_1S^{-1/2}\eta\| \geq \varepsilon(\frac{1}{2} - \frac{3}{8}\varepsilon^2)$, so $S^{-1/2}$ is not in Han's local commutant, and the set $\{S^{-1/2}D^jT_k\eta\}$ is not of the form $\{D^{rj}T_{ck}\phi_l\}$ for any integer $r \geq 1$, nonzero dyadic rational c and family (ϕ_l) . For the Shannon wavelet and $1/\sqrt{2} \leq \varepsilon < 1$, $S^{-1/2}\eta$ does not even generate a Bessel system. The second part of Han's question, his question whether S^{-1} in the local commutant implies $S^{-1/2}$ in it, redundant wavelet frames and quasi-affine systems are not treated. This is an unrefereed note.

1 Introduction

1.1 The question

Let $Df(x) = \sqrt{2}f(2x)$ and $T_af(x) = f(x - a)$ ($a \in \mathbb{R}$) be the dyadic dilation and the translations on $L^2(\mathbb{R})$, and write $\psi_{j,k} := D^jT_k\psi = 2^{j/2}\psi(2^j \cdot - k)$ for $\psi \in L^2(\mathbb{R})$ and $j, k \in \mathbb{Z}$. Following Han [Han03, §§1 and 3], in the case of dilation 2 on $\mathbb{R}$, the function ψ is a *wavelet frame* if $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, *semi-orthogonal* if $\psi_{j,k} \perp \psi_{j',k'}$ whenever $j \neq j'$, and an *orthonormal wavelet* if $\{\psi_{j,k}\}$ is an orthonormal basis; it is a *Riesz wavelet* if $\{\psi_{j,k}\}$ is a Riesz basis. The frame operator $Sf = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k}$ is positive and invertible; $\{S^{-1}\psi_{j,k}\}$ is the canonical dual frame, and $\{S^{-1/2}\psi_{j,k}\}$ is a normalized tight (Parseval) frame, the *canonical normalized tight frame* of ψ . If $\{\psi_{j,k}\}$ is a Riesz basis, then $\{S^{-1/2}\psi_{j,k}\}$ is an orthonormal basis, the symmetric (Löwdin) orthogonalization of $\{\psi_{j,k}\}$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

[FPT02], [Fra04, p. 530]. The *local commutant* of the wavelet unitary system $\mathcal{U}_{D,T} = \{D^n T_\ell : n, \ell \in \mathbb{Z}\}$ at ψ is

$$\mathcal{C}_\psi(\mathcal{U}_{D,T}) = \{A \in B(L^2(\mathbb{R})) : AD^n T_\ell \psi = D^n T_\ell A\psi \text{ for all } n, \ell \in \mathbb{Z}\}.$$

In Section 3 of [Han03] Han observes that the canonical dual of a wavelet frame ψ is a wavelet system exactly when $S^{-1} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$, and he asks in Question 4 (p. 3338): “Is $S^{-1/2}\psi$ always a wavelet frame?”, and, if not, under which conditions it is. He notes that $S^{-1/2}\psi$ is a normalized tight wavelet frame if $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$, and that he does not know whether $S^{-1} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$ implies $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$ [Han03, pp. 3338–3339]. For semi-orthogonal wavelet frames he proves $S^{-1/2}D^n T_\ell \psi = D^n T_\ell S^{-1/2}\psi$ for all n, ℓ , in the proof of his Theorem 1.3 [Han03, p. 3339]. (Han works in $L^2(\mathbb{R}^d)$ with a real expansive dilation matrix and the lattice $\mathbb{Z}^d$.)

In his abstract for the Oberwolfach mini-workshop “Wavelets and Frames” [Fra04], M. Frank surveys approximations of frames by normalized tight ones. He writes $S = (\theta^* \theta)^{-1}$, where θ is the analysis operator, so that his $S^{1/2}(x_i)$ is our $S^{-1/2}x_i$. After recalling Han’s results he states that it is unknown whether the normalized tight frame $\{S^{1/2}(x_i)\}_i$ of a wavelet frame $\{x_i\}_i$ always has wavelet structure [Fra04, p. 531]. This is the first part of Han’s Question 4, asked a year earlier; Frank’s abstract cites [Han03] for the surrounding results. Just before, Frank mentions that wavelet frames without any wavelet-type dual frame had been found in discussions of Han with Daubechies, Wexler and Bownik; for published work on canonical and alternate duals of wavelet frames see [DH02, BW03, BL07]. In the same report, O. Christensen lists among the open problems of his lecture the question which conditions on the generator make the canonical dual a wavelet system [Chr04, p. 488].

We distinguish three readings of the first part of the question for a wavelet frame ψ .

- (I) *Indexed*: there is $\phi \in L^2(\mathbb{R})$ with $S^{-1/2}\psi_{j,k} = D^j T_k \phi$ for all $j, k \in \mathbb{Z}$. Taking $(j, k) = (0, 0)$ gives $\phi = S^{-1/2}\psi$, so (I) says $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$. This is Han’s gloss, and it is the hypothesis (4.1) of [BJS11, Theorem 4.1].
- (II) *As a set*: the set $\{S^{-1/2}\psi_{j,k} : j, k \in \mathbb{Z}\}$ equals $\{D^j T_k \phi_l : j, k \in \mathbb{Z}, l \in L\}$ for some family $(\phi_l)_{l \in L}$; this allows any re-indexing and any number of generators, as in [BL07] for the canonical dual.
- (III) *Literal*: the function $S^{-1/2}\psi$ is a wavelet frame, or a normalized tight wavelet frame.

Reading (I) implies (II) with one generator, and it implies (III) in the normalized tight form.

1.2 Results

Put $\beta_b := \binom{2b}{b} 4^{-b}$ ($b \geq 0$), so that $(1+w)^{-1/2} = \sum_{b \geq 0} (-1)^b \beta_b w^b$ for $|w| < 1$. For $0 < \varepsilon < 1$ and $k \geq 0$ let

$$\Gamma_k := \sum_{b \geq 0} \beta_b \beta_{b+k} \varepsilon^{2b+k}, \quad \Lambda_k := \Gamma_k - \varepsilon \Gamma_{k+1}, \quad \Delta_{k+1} := \varepsilon \Gamma_k - \Gamma_{k+1}, \quad (1)$$

and

$$u_i := (-1)^i \Lambda_{|i|} \quad (i \leq 0), \quad u_i := (-1)^{i+1} \Delta_i \quad (i \geq 1). \quad (2)$$

For a nonzero dyadic rational $t = 2^a b$ with $a \in \mathbb{Z}$ and b an odd integer we write $\nu(t) := a$; on the integers, ν is the 2-adic valuation.

Theorem 1.1. *Let ψ be an orthonormal wavelet of $L^2(\mathbb{R})$, let $0 < \varepsilon < 1$, put $\eta := \psi + \varepsilon D\psi$ and $\eta_{j,k} := D^j T_k \eta$, let S be the frame operator of $\{\eta_{j,k}\}$, and let $E := \{S^{-1/2} \eta_{j,k} : j, k \in \mathbb{Z}\}$.*

- (a) $\eta_{j,k} = \psi_{j,k} + \varepsilon \psi_{j+1,2k}$, and $\{\eta_{j,k}\}$ is a Riesz basis of $L^2(\mathbb{R})$ with optimal frame bounds $(1 - \varepsilon)^2$ and $(1 + \varepsilon)^2$. It is not semi-orthogonal: $\langle \eta_{0,0}, \eta_{1,0} \rangle = \varepsilon$.

- (b) Let B be the isometry with $B\psi_{j,k} = \psi_{j+1,2k}$, let $A := I + \varepsilon B$, and let $U := A(A^*A)^{-1/2}$ be the unitary factor in the polar decomposition of A . Then $S^{-1/2}\eta_{j,k} = U\psi_{j,k}$ for all j, k ; thus E is the orthonormal basis $\{U\psi_{j,k}\}$.
- (c) $\|S^{-1/2}T_1\eta - T_1S^{-1/2}\eta\| = (1 - u_0^2 - u_1^2)^{1/2} \geq \varepsilon(\frac{1}{2} - \frac{3}{8}\varepsilon^2) > \frac{\varepsilon}{8}$. Hence $S^{-1/2} \notin \mathcal{C}_\eta(\mathcal{U}_{D,T})$, and there is no $\phi \in L^2(\mathbb{R})$ with $S^{-1/2}\eta_{j,k} = D^jT_k\phi$ for all $j, k \in \mathbb{Z}$.
- (d) Let $r \geq 1$ be an integer, c a nonzero dyadic rational and $(\phi_l)_{l \in L}$ any family in $L^2(\mathbb{R})$. Then $E \neq \{D^{rj}T_{ck}\phi_l : j, k \in \mathbb{Z}, l \in L\}$. In particular ($r = c = 1$) the set E is not a wavelet system with any number of generators, however it is indexed.

Theorem 1.2. In the situation of Theorem 1.1, the numbers u_i are the Fourier coefficients of $g(z) = (1 + \varepsilon z)/|1 + \varepsilon z|$ on the unit circle, and for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z} \setminus \{0\}$

$$U\psi_{j,0} = \sum_{q \in \mathbb{Z}} u_q \psi_{j+q,0}, \quad U\psi_{j,k} = \sum_{q=-\nu(k)}^{\infty} (u_q - u_{q+2\nu(k)+2}) \psi_{j+q,2^q k}. \quad (3)$$

Every u_q is nonzero: u_q has the sign $(-1)^q$ for $q \leq 0$ and $(-1)^{q+1}$ for $q \geq 1$. For $q \geq -\nu(k)$ the coefficient $u_q - u_{q+2\nu(k)+2}$ is nonzero and has the sign of u_q . Moreover $u_{-1} = -\Lambda_1$ and $\Lambda_1 \geq \varepsilon(\frac{1}{2} - \frac{3}{8}\varepsilon^2)$.

Proposition 1.3. Let ψ be the Shannon wavelet, $\hat{\psi} = (2\pi)^{-1/2} \mathbf{1}_{[-2\pi, -\pi) \cup [\pi, 2\pi)}$, and let η and S be as in Theorem 1.1. Then

$$\sum_{j,k \in \mathbb{Z}} |\langle \psi, D^jT_k S^{-1/2}\eta \rangle|^2 = 1 + \sum_{i \geq 1} (2^i - 1) \Lambda_i^2 > 1.$$

Hence $\{D^jT_k S^{-1/2}\eta\}$ is not a normalized tight frame for any $\varepsilon \in (0, 1)$; for $1/\sqrt{2} \leq \varepsilon < 1$ it is not a Bessel sequence, and then $S^{-1/2}\eta$ is not a wavelet frame.

Corollary 1.4. The canonical normalized tight frame of a wavelet frame need not have wavelet structure. For the first part of Han's Question 4 [Han03, p. 3338] and for Frank's question [Fra04, p. 531] the answer is negative in reading (I), for every orthonormal wavelet ψ and every $\varepsilon \in (0, 1)$; in reading (II), with any number of generators; and, for the Shannon wavelet and $1/\sqrt{2} \leq \varepsilon < 1$, in reading (III).

1.3 What is known, what is new and what stays open

The example. The function $\eta = \psi + \varepsilon D\psi$ is the classical example of a Riesz wavelet whose canonical dual frame is not a wavelet system. Christensen [Chr14, §2.3] presents it in this form, for an orthonormal wavelet ψ and $\varepsilon \in (0, 1)$, and takes it from Daubechies [Dau90] and from the tutorial volume [Chu92]; Bownik and Lemvig count the proof by Daubechies [Dau90] and by Chui and Shi [CS93] that the canonical dual of a wavelet frame need not have wavelet structure among the first results on canonical and alternate duals, and note that their example is a non-biorthogonal Riesz wavelet [BL07, §1]. In our notation the canonical dual is written out in Remark 7.1. Theorem 1.1(a) is elementary and is proved for completeness.

Positive results. For tight wavelet frames S is a multiple of the identity. For semi-orthogonal wavelet frames $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$ by Han's proof of [Han03, Theorem 1.3]; our η is not semi-orthogonal. For group-like unitary systems, in particular for Gabor systems, the frame operator commutes with the system (proof of [Han03, Theorem 1.2]), so the canonical normalized tight frame keeps the structure; $\mathcal{U}_{D,T}$ is not group-like, since the group it generates contains $T_{1/2} = DT_1D^{-1}$, which is not a scalar multiple of any D^nT_ℓ . Riesz wavelets are wavelet frames in the sense of [Han03, Fra04]: Frank's

remark on symmetric orthogonalization concerns Riesz bases [Fra04, p. 530], and Han's Corollary 3.1 concerns Riesz wavelets ψ with $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$ [Han03, p. 3340].

A statement without proof. Bownik, Jasper and Speegle show that a frame wavelet admits an orthonormal dilation of the kind described in their Theorem 4.1 if and only if $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$, that is, if and only if its canonical Parseval frame is a wavelet system in reading (I) [BJS11, Theorem 4.1 and Remark 4.1]. They state that such dilation is possible “only for a proper subset of frame wavelets” [BJS11, §1; see also §4], which asserts that reading (I) can fail. They give no example or proof of this assertion, and they do not refer to [Han03, Fra04]. We found no other printed statement on the question (see “Scope and priority” in Section 8).

What is new, to our knowledge: the exact computation of the polar factor for this example (Theorem 1.2), the proof that it destroys wavelet structure in readings (I) and (II) (Theorem 1.1(c), (d)), the exact distance of Proposition 4.1, the description of the translation symmetries of E (Proposition 5.2), and Proposition 1.3 for reading (III). Together with [BJS11, Theorem 4.1], Theorem 1.1(c) gives an explicit frame wavelet without orthonormal dilation in their sense (Remark 7.2).

What stays open. The second part of Han's Question 4 (under which conditions $S^{-1/2}\psi$ is a wavelet frame); Han's question whether $S^{-1} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$ implies $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$, on which our example is silent because both conditions fail (Remark 7.1); redundant, that is non-Riesz, wavelet frames; other dilations and dimensions (one dyadic example suffices for the negative answer); the canonical normalized tight frame of quasi-affine systems [RS97]; and reading (III) for other orthonormal wavelets ψ , such as the Haar wavelet, or for $\varepsilon < 1/\sqrt{2}$.

2 The operator B and the frame

Throughout, ψ is an orthonormal wavelet, $0 < \varepsilon < 1$, $\eta = \psi + \varepsilon D\psi$ and $\eta_{j,k} = D^j T_k \eta$. Since $(T_a Df)(x) = \sqrt{2}f(2x - 2a) = (DT_{2a}f)(x)$, we have $T_a D = DT_{2a}$, hence $T_a D^{-1} = D^{-1}T_{a/2}$, and by induction

$$T_a D^j = D^j T_{2^j a} \quad (a \in \mathbb{R}, j \in \mathbb{Z}). \quad (4)$$

For nonzero dyadic rationals x, y with $x + y \neq 0$ the valuation ν of Section 1.2 satisfies $\nu(x + y) = \min(\nu(x), \nu(y))$ if $\nu(x) \neq \nu(y)$, and $\nu(x + y) > \nu(x)$ if $\nu(x) = \nu(y)$ (the sum of two odd numbers is even).

Lemma 2.1. (i) *There is a unique bounded operator B on $L^2(\mathbb{R})$ with $B\psi_{j,k} = \psi_{j+1,2k}$ for all j, k . It is an isometry, $B^*\psi_{j,k} = \psi_{j-1,k/2}$ for even k and $B^*\psi_{j,k} = 0$ for odd k , and $DB = BD$, $DB^* = B^*D$.*

(ii) $\eta_{j,k} = \psi_{j,k} + \varepsilon\psi_{j+1,2k} = A\psi_{j,k}$, where $A = I + \varepsilon B$.

(iii) *For $i \in \mathbb{Z}$ and odd $m \in \mathbb{Z}$ let $C_{i,m} := \{(i+n, 2^m n) : n \geq 0\}$, and let $O_0 := \{(j, 0) : j \in \mathbb{Z}\}$. Then $\mathbb{Z}^2$ is the disjoint union of O_0 and the chains $C_{i,m}$; a pair (j, k) with $k \neq 0$ is the element with $n = \nu(k)$ of the chain $C_{j-\nu(k), k2^{-\nu(k)}}$. Let $\mathcal{H}_0$ and $\mathcal{H}_{i,m}$ be the closed linear spans of the $\psi_{j,k}$ with (j, k) in O_0 and in $C_{i,m}$. Then $L^2(\mathbb{R}) = \mathcal{H}_0 \oplus \bigoplus_{i,m} \mathcal{H}_{i,m}$, and each summand is invariant under B and B^* . On $\mathcal{H}_0$, B is the bilateral shift $\psi_{j,0} \mapsto \psi_{j+1,0}$. On $\mathcal{H}_{i,m}$, with the orthonormal basis $e_n := \psi_{i+n, 2^m n}$ ($n \geq 0$), B is the unilateral shift $e_n \mapsto e_{n+1}$, and $B^*e_n = e_{n-1}$ for $n \geq 1$, $B^*e_0 = 0$.*

Proof. (i) The map $(j, k) \mapsto (j+1, 2k)$ is injective on $\mathbb{Z}^2$, so B maps the orthonormal basis $\{\psi_{j,k}\}$ onto an orthonormal family and extends to an isometry. The formula for B^* follows from $\langle B^*\psi_{j,k}, \psi_{j',k'} \rangle = \langle \psi_{j,k}, \psi_{j'+1,2k'} \rangle$. Next, $DB\psi_{j,k} = \psi_{j+2,2k} = BD\psi_{j,k}$, so $DB = BD$; then $B = D^{-1}BD$, and taking adjoints ($D^* = D^{-1}$) gives $B^* = D^{-1}B^*D$. (ii) By (4), $D^j T_k D\psi = D^{j+1} T_{2k} \psi = \psi_{j+1,2k}$. (iii) For

$k \neq 0$ write $k = 2^{\nu(k)}m$ with m odd; then $(j, k) = (i+n, 2^n m')$ with m' odd holds if and only if $n = \nu(k)$, $m' = m$ and $i = j - \nu(k)$, while no pair $(j, 0)$ lies on a chain. The operator B maps $(i+n, 2^n m)$ to $(i+n+1, 2^{n+1}m)$ and $(j, 0)$ to $(j+1, 0)$; by (i), B^* maps $(i+n, 2^n m)$ to $(i+n-1, 2^{n-1}m)$ if $n \geq 1$, it kills the first element (i, m) of each chain because m is odd, and it maps $(j, 0)$ to $(j-1, 0)$. $\square$

Proof of Theorem 1.1(a). Since $\|\varepsilon B\| = \varepsilon < 1$, A is invertible, so $\{\eta_{j,k}\} = \{A\psi_{j,k}\}$ is a Riesz basis. Its frame operator is $Sf = \sum_{j,k} \langle f, A\psi_{j,k} \rangle A\psi_{j,k} = A \sum_{j,k} \langle A^*f, \psi_{j,k} \rangle \psi_{j,k} = AA^*f$. The optimal frame bounds are the least and the largest point of the spectrum of $S = AA^* = A(A^*A)A^{-1}$, which is also the spectrum of $A^*A = (1 + \varepsilon^2)I + \varepsilon(B + B^*)$ (we used $B^*B = I$). As $\|B + B^*\| \leq 2$, this spectrum lies in $[(1 - \varepsilon)^2, (1 + \varepsilon)^2]$. On $\mathcal{H}_0$, which reduces B , the unitary map $\psi_{j,0} \mapsto z^j$ onto $L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ carries B to multiplication by z and $B + B^*$ to multiplication by $2 \cos \theta$, whose spectrum is $[-2, 2]$; so the spectrum of A^*A is the whole interval $[(1 - \varepsilon)^2, (1 + \varepsilon)^2]$, and the optimal bounds are $(1 - \varepsilon)^2$ and $(1 + \varepsilon)^2$. Finally $\langle \eta_{0,0}, \eta_{1,0} \rangle = \langle \psi_{0,0} + \varepsilon \psi_{1,0}, \psi_{1,0} + \varepsilon \psi_{2,0} \rangle = \varepsilon$. $\square$

Proof of Theorem 1.1(b). Since A is invertible, so is $|A| = (A^*A)^{1/2}$, and $U = A|A|^{-1}$ is invertible with $U^*U = |A|^{-1}A^*A|A|^{-1} = I$; hence U is unitary. From $AA^* = U|A|^2U^* = U(A^*A)U^*$ we get $(AA^*)^{-1/2} = U(A^*A)^{-1/2}U^*$, and so $S^{-1/2}\eta_{j,k} = (AA^*)^{-1/2}A\psi_{j,k} = U|A|^{-1}U^*U|A|\psi_{j,k} = U\psi_{j,k}$. $\square$

Lemma 2.2. U commutes with D and with the orthogonal projections onto $\mathcal{H}_0$ and the $\mathcal{H}_{i,m}$. Hence U maps each of these summands onto itself, $U\psi_{j,k}$ lies in the summand that contains $\psi_{j,k}$, every element of E lies in exactly one summand, and $D^r E = E$ for every $r \in \mathbb{Z}$.

Proof. Let P be one of the projections. By Lemma 2.1, P and D commute with B and B^* , hence with every norm limit of polynomials in B and B^* . The operator $(A^*A)^{-1/2}$ is such a limit, because $t^{-1/2}$ is a uniform limit of polynomials on the spectrum of A^*A , and therefore so are U and U^* . So U and U^* commute with P and D , and U maps $P(L^2(\mathbb{R}))$ onto itself. The element $U\psi_{j,k} \neq 0$ lies in the summand of $\psi_{j,k}$, and in no other since the summands are mutually orthogonal. Finally $D^r U\psi_{j,k} = U D^r \psi_{j,k} = U \psi_{j+r,k}$. $\square$

3 The polar factor

By Lemmas 2.1 and 2.2 it suffices to compute U on $\mathcal{H}_0$, where B is a bilateral shift, and on one chain space, where B is a unilateral shift. Let

$$h(\theta) := |1 + \varepsilon e^{i\theta}|^{-1} = (1 + \varepsilon^2 + 2\varepsilon \cos \theta)^{-1/2}, \quad \gamma_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta) e^{-ik\theta} d\theta \quad (k \in \mathbb{Z}).$$

Lemma 3.1. $\gamma_k = (-1)^k \Gamma_{|k|}$ for $k \in \mathbb{Z}$, with $0 < \Gamma_k \leq \varepsilon^k / (1 - \varepsilon^2)$. The Fourier coefficients of $g(e^{i\theta}) = (1 + \varepsilon e^{i\theta})h(\theta)$ are $\gamma_i + \varepsilon \gamma_{i-1} = u_i$ ($i \in \mathbb{Z}$), and $\sum_i u_i^2 = 1$.

Proof. Since $\binom{-1/2}{b} = (-1)^b \frac{1 \cdot 3 \cdots (2b-1)}{2^b b!} = (-1)^b \beta_b$, the principal branch satisfies $(1 + w)^{-1/2} = \sum_b (-1)^b \beta_b w^b$ for $|w| < 1$. For $z = e^{i\theta}$ the series for $(1 + \varepsilon \bar{z})^{-1/2}$ is the complex conjugate of that for $(1 + \varepsilon z)^{-1/2}$ (the coefficients are real), so

$$h(\theta) = |(1 + \varepsilon z)^{-1/2}|^2 = (1 + \varepsilon z)^{-1/2} (1 + \varepsilon \bar{z})^{-1/2} = \sum_{a,b \geq 0} (-1)^{a+b} \beta_a \beta_b \varepsilon^{a+b} z^{a-b}.$$

The double series converges absolutely, since $\sum_{a,b} \beta_a \beta_b \varepsilon^{a+b} = (1 - \varepsilon)^{-1}$ (because $\sum_b \beta_b x^b = (1 - x)^{-1/2}$). Collecting the terms with $a - b = k \geq 0$, that is $a = b + k$ and $(-1)^{a+b} = (-1)^k$, gives $\gamma_k = (-1)^k \Gamma_k$, and $\gamma_{-k} = \gamma_k$ by symmetry. The bound on Γ_k follows from $\beta_b \leq 1$. The coefficients of

$g = (1 + \varepsilon z)h$ are $\gamma_i + \varepsilon\gamma_{i-1}$. For $i \geq 1$ this is $(-1)^i(\Gamma_i - \varepsilon\Gamma_{i-1}) = (-1)^{i+1}\Delta_i$, and for $i \leq 0$, where $|i-1| = |i| + 1$, it is $(-1)^i(\Gamma_{|i|} - \varepsilon\Gamma_{|i|+1}) = (-1)^i\Lambda_{|i|}$; both agree with (2). Finally $\sum_i u_i^2 = \frac{1}{2\pi} \int |g|^2 = 1$. $\square$

Lemma 3.2. *Let $0 < \varepsilon < 1$ and $\delta_b := \beta_b - \beta_{b+1}$.*

- (i) $\beta_{b+1} = \beta_b \frac{2b+1}{2b+2}$, $\delta_b = \frac{\beta_b}{2b+2} > 0$ and $\delta_{b+1} = \delta_b \frac{2b+1}{2b+4} < \delta_b$ for $b \geq 0$.
- (ii) $0 < \Gamma_{k+1} < \varepsilon\Gamma_k$ for $k \geq 0$.
- (iii) $\Lambda_k = \sum_b \beta_b \varepsilon^{2b+k} (\beta_{b+k} - \varepsilon^2 \beta_{b+k+1}) > 0$ for $k \geq 0$, and $\Lambda_1 \geq \varepsilon(\beta_1 - \varepsilon^2 \beta_2) = \varepsilon(\frac{1}{2} - \frac{3}{8}\varepsilon^2) > \frac{\varepsilon}{8}$.
- (iv) $\Delta_k = \sum_b \beta_b \varepsilon^{2b+k} \delta_{b+k-1} > 0$ and $\Delta_{k+1} < \varepsilon\Delta_k$ for $k \geq 1$.

Consequently $|u_q| = \Lambda_{|q|}$ for $q \leq 0$ and $|u_q| = \Delta_q$ for $q \geq 1$; every u_q is nonzero with the sign given in Theorem 1.2, and $|u_q| > |u_{q'}|$ whenever $1 \leq q < q'$.

Proof. (i) is a computation with $\binom{2b+2}{b+1} = \binom{2b}{b} \frac{(2b+2)(2b+1)}{(b+1)^2}$. (ii) Termwise, $\beta_b \beta_{b+k+1} \varepsilon^{2b+k+1} < \varepsilon \cdot \beta_b \beta_{b+k} \varepsilon^{2b+k}$ by (i). (iii) Subtract the series for $\varepsilon\Gamma_{k+1}$ from that for Γ_k term by term; every term is positive because $\beta_{b+k+1} < \beta_{b+k}$ and $\varepsilon < 1$; keep only the term $b = 0$, and use $\beta_0 = 1$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{3}{8}$ and $\varepsilon < 1$. (iv) In the same way $\Delta_k = \varepsilon\Gamma_{k-1} - \Gamma_k = \sum_b \beta_b \varepsilon^{2b+k} \delta_{b+k-1}$, and $\Delta_{k+1} = \sum_b \beta_b \varepsilon^{2b+k+1} \delta_{b+k} < \varepsilon\Delta_k$ by (i). The consequences follow from (2). $\square$

Lemma 3.3. *Let $(e_n)_{n \geq 0}$ be the basis of a chain space $\mathcal{H}_{i,m}$ as in Lemma 2.1(iii). For $n, \mu \geq 0$,*

$$\langle (A^*A)^{-1/2} e_\mu, e_n \rangle = \gamma_{n-\mu} - \gamma_{n+\mu+2}, \quad \langle U e_\mu, e_n \rangle = u_{n-\mu} - u_{n+\mu+2}.$$

Proof. The functions $s_n(\theta) := \sin((n+1)\theta)$, $n \geq 0$, form an orthonormal basis of $L^2([0, \pi], \frac{2}{\pi} d\theta)$ (orthonormality is $\frac{1}{\pi} \int_0^\pi (\cos((n-p)\theta) - \cos((n+p+2)\theta)) d\theta = \delta_{np}$; completeness follows from that of the Fourier sine series), so $e_n \mapsto s_n$ defines a unitary map Φ of $\mathcal{H}_{i,m}$ onto this space. Since $s_{n+1} + s_{n-1} = 2 \cos \theta s_n$ for $n \geq 0$, with $s_{-1} := 0$, and $(B + B^*)e_n = e_{n+1} + e_{n-1}$ with $e_{-1} := 0$ (Lemma 2.1(iii)), the operator $\Phi(B + B^*)\Phi^{-1}$ is multiplication by $2 \cos \theta$. On $\mathcal{H}_{i,m}$ we have $A^*A = (1 + \varepsilon^2)I + \varepsilon(B + B^*)$, so $\Phi(A^*A)^{-1/2}\Phi^{-1}$ is multiplication by $h(\theta) = (1 + \varepsilon^2 + 2\varepsilon \cos \theta)^{-1/2}$. Hence, as h is even,

$$\begin{aligned} \langle (A^*A)^{-1/2} e_\mu, e_n \rangle &= \frac{2}{\pi} \int_0^\pi h s_\mu s_n d\theta = \frac{1}{\pi} \int_0^\pi h(\theta) (\cos((n-\mu)\theta) - \cos((n+\mu+2)\theta)) d\theta \\ &= \gamma_{n-\mu} - \gamma_{n+\mu+2} =: c_{n,\mu}. \end{aligned}$$

The same formula gives $c_{-1,\mu} = \gamma_{-1-\mu} - \gamma_{\mu+1} = 0$. Since $U = (I + \varepsilon B)(A^*A)^{-1/2}$ and $B^*e_n = e_{n-1}$ (with $e_{-1} = 0$),

$$\langle U e_\mu, e_n \rangle = c_{n,\mu} + \varepsilon c_{n-1,\mu} = (\gamma_{n-\mu} + \varepsilon \gamma_{n-\mu-1}) - (\gamma_{n+\mu+2} + \varepsilon \gamma_{n+\mu+1}) = u_{n-\mu} - u_{n+\mu+2}$$

by Lemma 3.1, for every $n \geq 0$. $\square$

Proof of Theorem 1.2. On $\mathcal{H}_0$ the unitary map $\psi_{j,0} \mapsto z^j$ onto $L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ carries B , B^* , A^*A and $(A^*A)^{-1/2}$ to the multiplications by z , $\bar{z}$, $|1 + \varepsilon z|^2$ and h , hence U to multiplication by $g = (1 + \varepsilon z)h = \sum_q u_q z^q$ (Lemma 3.1). So $U\psi_{j,0}$ corresponds to $z^j g$, which gives the first formula of (3). For $k \neq 0$, by Lemma 2.1(iii) $\psi_{j,k}$ is the basis vector e_μ , $\mu = \nu(k)$, of the chain space with $i = j - \mu$ and $m = k2^{-\mu}$, and $e_n = \psi_{j-\mu+n, 2^{n-\mu}k}$. By Lemma 2.2 and Lemma 3.3, $U\psi_{j,k} = \sum_{n \geq 0} (u_{n-\mu} - u_{n+\mu+2})e_n$; with $q = n - \mu$ this is the second formula. The signs of the u_q and the bound on $\Lambda_1 = -u_{-1}$ are in Lemma 3.2. Let $q \geq -\mu$ and $q' := q + 2\mu + 2 \geq 2$. If $q \leq 0$, then u_q has the sign $(-1)^q$ and $u_{q'}$ the opposite sign $(-1)^{q'+1}$, so $u_q - u_{q'}$ is nonzero with the sign of u_q . If $q \geq 1$, then u_q and $u_{q'}$ have the same sign $(-1)^{q+1}$ and $|u_q| > |u_{q'}|$ by Lemma 3.2; again $u_q - u_{q'}$ is nonzero with the sign of u_q . $\square$

4 The indexed reading

Proposition 4.1. *The number $d(\varepsilon) := \|S^{-1/2}T_1\eta - T_1S^{-1/2}\eta\| = \|U\psi_{0,1} - T_1U\psi\|$ satisfies*

$$d(\varepsilon)^2 = \sum_{i \geq 1} \Lambda_i^2 + \sum_{i \geq 2} \Delta_i^2 = 1 - u_0^2 - u_1^2.$$

In particular $d(\varepsilon) \geq \Lambda_1$, and $d(\varepsilon) = \frac{\varepsilon}{2} + O(\varepsilon^3)$ as $\varepsilon \rightarrow 0$.

Proof. By Theorem 1.1(b), $S^{-1/2}T_1\eta = S^{-1/2}\eta_{0,1} = U\psi_{0,1}$ and $T_1S^{-1/2}\eta = T_1U\psi$. Put $w_i := T_1\psi_{i,0}$ ($i \in \mathbb{Z}$); this is an orthonormal family because T_1 is unitary. For $i \geq 0$, (4) gives $w_i = D^i T_{2^i}\psi = \psi_{i,2^i}$, which is the basis vector e_i of the chain space $\mathcal{H}_{0,1}$ (the chain $C_{0,1} = \{(n, 2^n) : n \geq 0\}$). So the w_i with $i < 0$ are orthogonal to $\mathcal{H}_{0,1}$. By Theorem 1.2, $T_1U\psi = \sum_{i \in \mathbb{Z}} u_i w_i$ and $U\psi_{0,1} = Ue_0 = \sum_{n \geq 0} (u_n - u_{n+2})e_n$, hence

$$T_1U\psi - U\psi_{0,1} = \sum_{i < 0} u_i w_i + \sum_{n \geq 0} u_{n+2} e_n,$$

an orthogonal expansion. Therefore $d(\varepsilon)^2 = \sum_{i < 0} u_i^2 + \sum_{i \geq 2} u_i^2 = 1 - u_0^2 - u_1^2$ by Lemma 3.1, and $|u_{-i}| = \Lambda_i$, $|u_i| = \Delta_i$. For the asymptotics, $\Lambda_1 = \frac{\varepsilon}{2} + O(\varepsilon^3)$ by Lemma 3.2(iii), while $\Lambda_i \leq \Gamma_i \leq \varepsilon^i/(1 - \varepsilon^2)$ and $\Delta_i \leq \varepsilon \Gamma_{i-1} \leq \varepsilon^i/(1 - \varepsilon^2)$ give $\sum_{i \geq 2} (\Lambda_i^2 + \Delta_i^2) = O(\varepsilon^4)$. $\square$

ε	$d(\varepsilon)$	$\Lambda_1 = -u_{-1}$	$\varepsilon(\frac{1}{2} - \frac{3}{8}\varepsilon^2)$
0.05	0.0249960898	0.0249765503	0.0249531250
0.1	0.0499686224	0.0498121077	0.0496250000
0.25	0.1244989071	0.1220310842	0.1191406250
0.5	0.2456370636	0.2251858510	0.2031250000
0.75	0.3574732050	0.2831874439	0.2167968750
0.9	0.4132693683	0.2734533055	0.1766250000
0.99	0.4345989626	0.2259976232	0.1311378750

Table 1: The distance $d(\varepsilon) = \|S^{-1/2}T_1\eta - T_1S^{-1/2}\eta\|$ of Proposition 4.1 and the lower bounds of Theorem 1.1(c), rounded to ten decimals (computed by the program of Section 8).

Proof of Theorem 1.1(c). By Proposition 4.1 and Lemma 3.2(iii), $\|S^{-1/2}T_1\eta - T_1S^{-1/2}\eta\| = (1 - u_0^2 - u_1^2)^{1/2} \geq \Lambda_1 \geq \varepsilon(\frac{1}{2} - \frac{3}{8}\varepsilon^2)$, and $\varepsilon(\frac{1}{2} - \frac{3}{8}\varepsilon^2) > \frac{\varepsilon}{8}$ because $\varepsilon < 1$. So $S^{-1/2}T_1\eta \neq T_1S^{-1/2}\eta$, that is, $S^{-1/2} \notin \mathcal{C}_\eta(\mathcal{U}_{D,T})$. If $S^{-1/2}\eta_{j,k} = D^j T_k \phi$ held for all j, k , then $(j, k) = (0, 0)$ would give $\phi = S^{-1/2}\eta$ and hence $S^{-1/2} \in \mathcal{C}_\eta(\mathcal{U}_{D,T})$. $\square$

5 Sets of translates

Lemma 5.1. *For every $j \in \mathbb{Z}$ and every nonzero dyadic rational t , $T_t U \psi_{j,0} \notin E$.*

Proof. First let $j = 0$, and write $t = 2^a b$ with $a = \nu(t)$ and b odd. The vectors $v_i := T_t \psi_{i,0} = D^i T_{2^i t} \psi$ ($i \in \mathbb{Z}$) are orthonormal. If $i \geq -a$, then $2^i t = 2^{i+a} b \in \mathbb{Z}$ and $v_i = \psi_{i,2^{i+a}b}$ is the element with $n = i + a$ of the chain $C_{-a,b}$; as i runs through $i \geq -a$, these are all the basis vectors of $\mathcal{H}_{-a,b}$. Hence $v_i \perp \mathcal{H}_{-a,b}$ for $i < -a$. By Theorem 1.2, $T_t U \psi = \sum_i u_i v_i$, so the orthogonal projection of $T_t U \psi$ onto $\mathcal{H}_{-a,b}$ is $\sum_{i \geq -a} u_i v_i \neq 0$ (as $u_{-a} \neq 0$), and its component orthogonal to $\mathcal{H}_{-a,b}$ is $\sum_{i < -a} u_i v_i \neq 0$ (as

$u_{-a-1} \neq 0$). Every element of E lies in a single summand of the decomposition of Lemma 2.1(iii) (Lemma 2.2), which is either $\mathcal{H}_{-a,b}$ or orthogonal to it; in both cases one of the two components would vanish. So $T_t U\psi \notin E$. For general j , (4) and Lemma 2.2 give $T_t U\psi_{j,0} = T_t D^j U\psi = D^j T_{2^j t} U\psi$. Here $2^j t$ is a nonzero dyadic rational, so $T_{2^j t} U\psi \notin E$, and D^j maps the complement of E onto itself because $D^j E = E$. $\square$

Proof of Theorem 1.1(d). Suppose that $E = \{D^{rj} T_{ck} \phi_l : j, k \in \mathbb{Z}, l \in L\}$. Since $U\psi \in E$, there are j, k, l with $U\psi = D^{rj} T_{ck} \phi_l$, so $\phi_l = T_{-ck} D^{-rj} U\psi = T_{-ck} U\psi_{-rj,0}$ by Lemma 2.2. Both $\phi_l = D^0 T_0 \phi_l$ and $T_c \phi_l$ belong to E . If $k \neq 0$, then $-ck$ is a nonzero dyadic rational and $\phi_l \notin E$ by Lemma 5.1; if $k = 0$, then $\phi_l = U\psi_{-rj,0}$ and $T_c \phi_l \notin E$ by Lemma 5.1. Both cases are impossible. $\square$

The family (ϕ_l) in Theorem 1.1(d) is arbitrary (each ϕ_l lies in E). Many translates of elements of E lie again in E , which is why the set reading needs an argument of its own:

Proposition 5.2. *Let $e = U\psi_{j_0, k_0} \in E$ and $k \in \mathbb{Z} \setminus \{0\}$. Then $T_k e \in E$ if and only if $k_0 \neq 0$ and $j_0 + \nu(k) > \nu(k_0)$; in that case $T_k e = U\psi_{j_0, k_0 + 2^{j_0} k}$.*

For instance $T_2 U\psi_{0,1} = U\psi_{0,3}$, while $T_1 U\psi_{0,1} \notin E$ and no nonzero integer translate of any $U\psi_{j,0}$ lies in E .

Proof. Put $t := 2^{j_0} k$. By (4) and Lemma 2.2, $T_k e = T_k D^{j_0} U\psi_{0,k_0} = D^{j_0} T_t U\psi_{0,k_0}$, and since $D^{j_0} E = E$, $T_k e \in E$ if and only if $T_t U\psi_{0,k_0} \in E$. If $k_0 = 0$ this fails by Lemma 5.1. Let $k_0 \neq 0$, $\mu := \nu(k_0)$ and $m := k_0 2^{-\mu}$. Then ψ_{0,k_0} is the basis vector e_μ of the chain space $\mathcal{H}_{-\mu,m}$, whose basis is $e_n = \psi_{n-\mu, 2^{n-\mu} k_0}$, and by Theorem 1.2 $U\psi_{0,k_0} = \sum_{n \geq 0} a_n e_n$ with $a_n := u_{n-\mu} - u_{n+\mu+2} \neq 0$. By (4), $T_t e_n = D^{n-\mu} T_{2^{n-\mu} s} \psi =: v_n$ with $s := t + k_0$, and the v_n are orthonormal.

If $s = 0$, then $v_n = \psi_{n-\mu,0}$, and $T_t U\psi_{0,k_0} = \sum_n a_n \psi_{n-\mu,0}$ is a nonzero vector of $\mathcal{H}_0$ whose coefficients on the $\psi_{i,0}$ with $i < -\mu$ vanish. The elements of E in $\mathcal{H}_0$ are the $U\psi_{j,0}$, all of whose coefficients u_{i-j} are nonzero, and the other elements of E are orthogonal to $\mathcal{H}_0$; so $T_t U\psi_{0,k_0} \notin E$.

If $s \neq 0$, let $a := \nu(s)$ and $b := s 2^{-a}$. Then $2^{n-\mu} s = 2^{n-\mu+a} b$ is an integer if and only if $n \geq \mu - a$, and then $v_n = \psi_{n-\mu, 2^{n-\mu+a} b}$ is the basis vector e'_p , $p = n - \mu + a$, of the chain space $\mathcal{H}_{-a,b}$.

$a < \mu$: the v_n with $n \geq \mu - a$ are all the e'_p , and v_0 (and every v_n with $n < \mu - a$) is orthogonal to $\mathcal{H}_{-a,b}$. The components of $T_t U\psi_{0,k_0}$ in $\mathcal{H}_{-a,b}$ and in its orthogonal complement contain $a_{\mu-a} e'_0$ and $a_0 v_0$, so both are nonzero, and $T_t U\psi_{0,k_0} \notin E$ as in the proof of Lemma 5.1.

$a = \mu$: $v_n = e'_n$ for all n , and $T_t U\psi_{0,k_0} = \sum_n a_n e'_n$. By Lemma 3.3, which is the same on every chain, this is $U e'_\mu = U\psi_{0, 2^\mu b} = U\psi_{0,s} \in E$.

$a > \mu$: every $v_n = e'_{n+\mu-a}$ lies in $\mathcal{H}_{-a,b}$, so $T_t U\psi_{0,k_0}$ is a nonzero vector of $\mathcal{H}_{-a,b}$ with coefficient 0 on e'_0 . The elements of E in $\mathcal{H}_{-a,b}$ are the $U e'_p$, whose coefficient on e'_0 is $u_{-p} - u_{p+2} \neq 0$, and the other elements of E are orthogonal to $\mathcal{H}_{-a,b}$; so $T_t U\psi_{0,k_0} \notin E$.

Thus, for $k_0 \neq 0$, $T_k e \in E$ if and only if $s \neq 0$ and $\nu(s) = \nu(k_0)$, and then $T_k e = D^{j_0} U\psi_{0,s} = U\psi_{j_0,s}$. Since $s = k_0 + 2^{j_0} k$ and $\nu(2^{j_0} k) = j_0 + \nu(k)$, the rules for ν in Section 2 show that $s \neq 0$ and $\nu(s) = \nu(k_0)$ hold if and only if $j_0 + \nu(k) > \nu(k_0)$. $\square$

Remark 5.3 (No uniform margin). Unlike part (c), the proof of Theorem 1.1(d) gives no lower bound for the distance from E of translates that do not lie in E , and there is none. For $J \geq 0$ we have $T_1 U\psi_{J,0} \notin E$ by Lemma 5.1, but

$$\text{dist}(T_1 U\psi_{J,0}, E) \leq \|T_1 U\psi_{J,0} - U\psi_{J,2^J}\| = \left(\sum_{i < -J} u_i^2 + \sum_{i \geq J+2} u_i^2 \right)^{1/2} \leq \sqrt{2} \varepsilon^{J+1} (1 - \varepsilon^2)^{-3/2}.$$

Indeed, $T_1 U \psi_{J,0} = D^J T_{2^J} U \psi$, and with $t = 2^J$ in the proof of Lemma 5.1 we get $T_{2^J} U \psi = \sum_i u_i v_i$, where $v_i = e'_{i+J}$ for $i \geq -J$ are the basis vectors of $\mathcal{H}_{-J,1}$ and $v_i \perp \mathcal{H}_{-J,1}$ for $i < -J$; moreover $U \psi_{0,2^J} = U e'_J = \sum_{p \geq 0} (u_{p-J} - u_{p+J+2}) e'_p$. Apply the unitary D^J to the difference $\sum_{i < -J} u_i v_i + \sum_{p \geq 0} u_{p+J+2} e'_p$, and use $|u_{-i}| = \Lambda_i \leq \varepsilon^i / (1 - \varepsilon^2)$ and $|u_i| = \Delta_i \leq \varepsilon^i / (1 - \varepsilon^2)$. For $J = 0$ this is the distance of Proposition 4.1. For $\varepsilon = 0.1$ and $J = 3$ the middle term is about $2.7 \cdot 10^{-5}$. A numerical test of Proposition 5.2 therefore needs thresholds adapted to the rounding errors.

Remark 5.4. By Lemma 5.1, E is not invariant under T_1 ; so it is neither of the form $\{T_k D^j \phi_l\}$ nor a quasi-affine system [RS97]. The canonical normalized tight frame of the quasi-affine system of η is a different object, not treated here.

6 The literal reading: the Shannon wavelet

We use the unitary Fourier transform $\hat{f}(\xi) = (2\pi)^{-1/2} \int f(x) e^{-ix\xi} dx$, so that $\widehat{T_a f}(\xi) = e^{-ia\xi} \hat{f}(\xi)$ and $\widehat{D^j f}(\xi) = 2^{-j/2} \hat{f}(2^{-j}\xi)$. Let $K := [-2\pi, -\pi) \cup [\pi, 2\pi)$ and $\hat{\psi} = (2\pi)^{-1/2} \mathbf{1}_K$. The sets $2^j K$ ($j \in \mathbb{Z}$) are pairwise disjoint, K is congruent to $[-\pi, \pi)$ modulo 2π , and $\{\psi_{j,k}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ (the Shannon wavelet). The Fourier transform of $D^j T_a \psi$ is supported in $2^j K$ for every $a \in \mathbb{R}$.

Proof of Proposition 1.3. Put $\phi := S^{-1/2} \eta = U \psi = \sum_i u_i \psi_{i,0}$ (Theorem 1.1(b), Theorem 1.2). By (4), $D^j T_k \phi = \sum_i u_i D^{i+j} T_{2^i k} \psi$, an L^2 -convergent series of functions with disjoint Fourier supports $2^{i+j} K$. Only the term with $i = -j$ has a Fourier transform that meets K , so

$$\langle \psi, D^j T_k \phi \rangle = u_{-j} \langle \psi, T_{2^{-j}k} \psi \rangle, \quad \langle \psi, T_s \psi \rangle = \frac{1}{2\pi} \int_K e^{is\xi} d\xi.$$

For $j \leq 0$ the number $2^{-j}k$ is an integer, and the integer translates of ψ are orthonormal, so $\sum_k |\langle \psi, D^j T_k \phi \rangle|^2 = u_{-j}^2$. For $j \geq 1$, K lies in an interval I of length $2^{j+1}\pi \geq 4\pi$, and the functions $|I|^{-1/2} e^{-i2^{-j}k\xi}$ ($k \in \mathbb{Z}$) form an orthonormal basis of $L^2(I)$; by Parseval's identity

$$\sum_k |\langle \psi, T_{2^{-j}k} \psi \rangle|^2 = \frac{|I|}{4\pi^2} \sum_k \left| \langle \mathbf{1}_K, |I|^{-1/2} e^{-i2^{-j}k\cdot} \rangle_{L^2(I)} \right|^2 = \frac{|I||K|}{4\pi^2} = 2^j.$$

Summing over j and using $\sum_i u_i^2 = 1$ (Lemma 3.1) and $|u_{-i}| = \Lambda_i$,

$$\sum_{j,k} |\langle \psi, D^j T_k \phi \rangle|^2 = \sum_{i \geq 0} u_i^2 + \sum_{i \geq 1} 2^i u_{-i}^2 = 1 + \sum_{i \geq 1} (2^i - 1) \Lambda_i^2,$$

which exceeds $1 = \|\psi\|^2$ because $\Lambda_1 > 0$. A normalized tight frame would give $\|\psi\|^2$ here. For the last assertion, Lemma 3.2(iii) (the term $b = 0$) and $\beta_{i+1} < \beta_i$ give $\Lambda_i \geq \varepsilon^i (\beta_i - \varepsilon^2 \beta_{i+1}) \geq (1 - \varepsilon^2) \varepsilon^i \beta_i$. Moreover $\beta_i \geq 1/(2\sqrt{i})$ for $i \geq 1$, because $\beta_1 = \frac{1}{2}$ and $\beta_{i+1}^2 (i+1) / (\beta_i^2 i) = (2i+1)^2 / (4i(i+1)) > 1$. With $2^i - 1 \geq 2^{i-1}$ we get

$$(2^i - 1) \Lambda_i^2 \geq \frac{(1 - \varepsilon^2)^2}{8} \cdot \frac{(2\varepsilon^2)^i}{i},$$

whose sum over $i \geq 1$ diverges when $2\varepsilon^2 \geq 1$. Then the series in the proposition is infinite, so $\{D^j T_k \phi\}$ is not a Bessel sequence and in particular not a frame. $\square$

For $\varepsilon < 1/\sqrt{2}$ the sum is finite ($1.0858 \dots$ at $\varepsilon = \frac{1}{2}$), and we do not know whether $\{D^j T_k S^{-1/2} \eta\}$ is then a frame; for the Haar wavelet reading (III) is not decided.

7 Remarks

Remark 7.1 (The canonical dual). Since $S = AA^*$, the canonical dual is $S^{-1}\eta_{j,k} = (A^*)^{-1}\psi_{j,k}$ with $(A^*)^{-1} = \sum_{n \geq 0} (-\varepsilon)^n B^{*n}$, that is,

$$S^{-1}\eta_{j,k} = \sum_{n=0}^{\nu(k)} (-\varepsilon)^n \psi_{j-n, k2^{-n}} \quad (k \neq 0), \quad S^{-1}\eta_{j,0} = \sum_{n \geq 0} (-\varepsilon)^n \psi_{j-n,0}.$$

So $S^{-1}T_1\eta = \psi_{0,1}$, while $T_1S^{-1}\eta = \psi_{0,1} + \sum_{n \geq 1} (-\varepsilon)^n T_1\psi_{-n,0}$, and the vectors $T_1\psi_{-n,0}$ ($n \geq 1$) are orthonormal and orthogonal to $\psi_{0,1}$ (proof of Proposition 4.1). Hence $\|S^{-1}T_1\eta - T_1S^{-1}\eta\| = \varepsilon(1 - \varepsilon^2)^{-1/2}$ and $S^{-1} \notin \mathcal{C}_\eta(\mathcal{U}_{D,T})$: this is the classical fact recalled in Section 1.3, in our notation. As both S^{-1} and $S^{-1/2}$ are outside $\mathcal{C}_\eta(\mathcal{U}_{D,T})$, the example says nothing about Han's question whether the first condition implies the second. The proofs of Theorem 1.1(c), (d) do not use the canonical dual, and we know no general implication between the two properties.

Remark 7.2 (Orthonormal dilations). Bownik, Jasper and Speegle [BJS11, Theorem 4.1] consider a unitary representation $\{D', T'\}$ of the Baumslag–Solitar group, $D'T'D'^{-1} = T'^2$, on a Hilbert space H , and a frame wavelet ψ for it, with optimal frame bounds a^2 and b^2 . They show that the following holds if and only if $S^{-1/2} \in \mathcal{C}_\psi(\{D'^j T'^k\})$: there are a representation $\{D'', T''\}$ on a Hilbert space H' , an orthonormal wavelet $\psi_1 \oplus \psi_2$ for $\{D' \oplus D'', T' \oplus T''\}$, and a positive operator P on H with $\{a, b\} \subseteq \sigma(P) \subseteq [a, b]$ and $PD'^j T'^k \psi_1 = D'^j T'^k \psi$ for all j, k . On $L^2(\mathbb{R})$ the pair $D' = D^{-1}$, $T' = T_1$ is such a representation by (4), and $\{D'^j T'^k\} = \mathcal{U}_{D,T}$ as a set. By Theorem 1.1(c), η is therefore a frame wavelet of this representation without an orthonormal dilation in the sense of [BJS11, Theorem 4.1], for every $\varepsilon \in (0, 1)$; this is an explicit instance of the statement of [BJS11] quoted in Section 1.3.

Remark 7.3 (Scope). The question asks whether the canonical normalized tight frame *always* has wavelet structure, so a single dyadic example in $L^2(\mathbb{R})$ answers it. We make no claim about other dilation matrices or dimensions, nor about redundant wavelet frames, for which the canonical normalized tight frame is not an orthonormal basis and the method of this note does not apply directly.

8 Computations and reproducibility

No proof in this note depends on a computation. The reproducibility package has four parts; its `README.md` gives the command lines, the running times and what each program checks. *The program of this note*, `check_note.py` (Python 3, standard library only, a few seconds), checks the exact and finite parts: the identities for β_b ; Theorem 1.1(b) and Theorem 1.2 as identities of formal power series in ε with exact rational coefficients up to ε^{12} , with U computed from its definition $A(A^*A)^{-1/2}$ on the bilateral part, on chains and on ten index pairs (j, k) , together with $(AA^*)^{-1/2}A = A(A^*A)^{-1/2}$ and the formulas of Remark 7.1; the signs in Lemma 3.2 and Theorem 1.2 for fifteen values of ε in $[0.01, 0.99]$ (18,696 comparisons in 100-digit decimal arithmetic with bounded truncation error); Table 1, with $d(\varepsilon)$ computed both by Proposition 4.1 and by the direct sum in its proof, and compared with values obtained earlier by a fast Fourier transform; the index bookkeeping of Lemma 2.1(iii), Lemma 5.1 and Proposition 5.2 (84,240 cases of the case analysis of the proof of Proposition 5.2 against the closed criterion); the numbers of Remark 5.3; and the inequality $\beta_i \geq 1/(2\sqrt{i})$, the series and the identity $\sum_k |\langle \psi, T_{k/2^j} \psi \rangle|^2 = 2^j$ ($j \leq 3$) used for Proposition 1.3. *The programs with which the results were first obtained* (`original/`) compare the series for the u_i with a fast Fourier transform of g and the chain coefficients with finite sections of A^*A , and check identities for the Haar wavelet in exact arithmetic. The programs and recorded outputs of the *three independent verification runs* are in `verification_run_A/`, `verification_run_B/` and `independent_run_2/` (the third run; the folder name refers to the second verification stage).

Verification. The results were first obtained with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; they examined the first written version of the results, and each wrote its own programs. The first run re-derived every identity, inequality and case distinction; it compared the closed forms with quadrature of the defining integrals (50 digits) and with polar factors of finite sections of A (sizes up to 3000), tested the signs in 58,236 cases, checked the bookkeeping of Proposition 5.2 in 25,872 cases, assembled the functions $U\psi_{j,k}$ for the Haar wavelet and $\varepsilon = \frac{1}{2}$ on a dyadic grid, found the proof of Theorem 1.1(d) through Lemma 5.1 with the extension to the lattices $c\mathbb{Z}$ and the dilations D^r , and proved Proposition 1.3. The second run re-derived all proofs, computed U from its definition as formal power series in ε with exact rational coefficients up to ε^{24} (three deliberately planted errors were detected), evaluated the definitions numerically (33,066 sign checks), checked for the Haar wavelet that the frame operator is AA^* by exact integration, decided the membership in E of 1,013 integer and dyadic translates of elements of E (Haar wavelet, four values of ε) numerically, without the case analysis, in agreement with Lemma 5.1 and Proposition 5.2, found the literal reading for the Shannon wavelet independently, and searched the literature. Both runs re-ran the programs of the first stage and obtained identical outputs, confirmed Theorem 1.1 and found no gap. All corrections they required were made: the credit of the question to Han’s Question 4 [Han03, p. 3338], with Frank’s abstract as a restatement; the statement of what the note settles and what it does not (the second part of Question 4, Han’s question on S^{-1} and $S^{-1/2}$); the literal reading (Proposition 1.3); the page of Christensen’s open problem (p. 488); the description of Frank’s abstract (Frank does not specify the dilation, his Theorem 5 states less than Han’s proof gives, his sentence on duals concerns frames without any wavelet-type dual, and his remark on symmetric orthogonalization cites only Löwdin); “optimal” rather than “attained” for the frame bounds; bibliographic details; and Remark 5.3.

When this note was written, the following was added, which the two runs did not see: the form $u_{n-\mu} - u_{n+\mu+2}$ of the chain coefficients (Lemma 3.3; the first version gave them as $c_{n,\mu} + \varepsilon c_{n-1,\mu}$ with a separate sign analysis), the identity $d(\varepsilon)^2 = 1 - u_0^2 - u_1^2$ of Proposition 4.1, the bound in Remark 5.3, and Remark 7.2; the first three are also checked by the program of this note. The reference [BJS11] was found in the search made for this note. A third independent verification run, also AI-assisted, then examined the written note, and these parts first. It read the sources again ([Fra04, Chr04] in the open report, [Han03, Han04], both versions of [BJS11], [Chr14, BL07]), re-derived every proof, and wrote its own programs before opening those of the package: the u_i from the factorisation $g = (1 + \varepsilon z)^{1/2}(1 + \varepsilon \bar{z})^{-1/2}$, and the Γ_k as hypergeometric values and by quadrature; the chain matrix $M = (u_{n-m} - u_{n+m+2})$ through the identities of a polar decomposition ($M^T M = M M^T = I$ and $M^T A$ symmetric, exactly up to ε^{18}) and against polar factors of finite sections of size 4000; 34,481 sign tests; Table 1 to ten decimals, and $d(\varepsilon)$ assembled from numerical polar factors; the identity and the bound of Remark 5.3; Lemma 5.1, Proposition 5.2 and Theorem 1.1(d) for the Shannon wavelet from inner products of functions, without the bookkeeping of their proofs (1,005 cases, no disagreement); the sums in Proposition 1.3 through values of the Hurwitz zeta function; and the relation used in Remark 7.2. It re-ran every program of the package (identical outputs), searched the literature again, confirmed all results and found no gap. The corrections it required were made: the wording on [BJS11] (an assertion without example or proof) and on [BL07], the credit of the example in the abstract, the source of the definition of a Riesz wavelet, a reason why $\mathcal{U}_{D,T}$ is not group-like, and the record of the searches.

Scope and priority. The note answers the first part of Question 4 of Han [Han03, p. 3338], restated by Frank [Fra04, p. 531], negatively: in readings (I) and (II) for every orthonormal wavelet and every $\varepsilon \in (0, 1)$, and in reading (III) for the Shannon wavelet and $1/\sqrt{2} \leq \varepsilon < 1$. It does not treat the second part of Question 4, Han’s question whether $S^{-1} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$ implies $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$, redundant

wavelet frames, other dilations or dimensions, or quasi-affine systems. The question is Han’s, restated by Frank; the positive answer for semi-orthogonal wavelet frames is Han’s [Han03, Theorem 1.3]; the example $\eta = \psi + \varepsilon D\psi$ is the classical example for canonical duals [Dau90, Chu92, CS93], as described in [Chr14, §2.3] and [BL07, §1]; and the failure of $S^{-1/2} \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$ for some frame wavelets is asserted, without an example or a proof, in [BJS11]. What this note adds is the exact computation of the polar factor for this example and the proofs that it destroys wavelet structure.

What was read: [Han03] completely and [Han04] completely, in copies of the publishers’ versions ([Han04] concerns multi-frames and super-frames generated by projective unitary representations of groups; wavelet frames occur only in its introduction, and it contains no statement on Question 4 of [Han03]); pp. 529–532 and p. 488 of the report [Fra04, Chr04] in its open version; [BJS11] in a copy of the journal version on the first author’s website and in the arXiv version; [BL07, BW03] in authors’ copies; [Chr14] in its arXiv version 1; [FPT02] in its arXiv version; [BR19], two surveys of D. Larson (arXiv:math/0604615 and math/0604619) and arXiv:1901.08024 (listed with the question in the problem corpus), searched for the question. Not read: [HL09] (not accessible to us; its abstract announces an asymptotically tight approximation result for wavelet frames), [HL11] (only its abstract), [Han01] (the full text is behind an automated-access check; its abstract in a university repository announces the results of [Han03] for Gabor frames and a result for semi-orthogonal wavelet frames, so we cannot say whether Question 4 already appears there), [HL00], the books [Dau92, Chu92, Chr16], and the papers [Dau90, CS93], whose content is cited as described in [Chr14, BL07]. The searches were made on 17 October 2026: queries to arXiv, Crossref and zbMATH Open; the lists of works citing [Han03] in zbMATH Open (19) and Semantic Scholar (28), citing [Fra04] (Semantic Scholar, 3), citing [BJS11] (Semantic Scholar and zbMATH Open, 5 each; OpenAlex refused the request) and citing [DH02, BW03, BL07] (Semantic Scholar), with titles and available abstracts; the later work on symmetric approximation of frames [AC18, Chi19], which concerns general Hilbert spaces; and ten web searches. Apart from the statement of [BJS11], no printed answer to the question, and no printed example or proof of a negative answer, was found in these sources. A search that finds nothing is not a proof of novelty, and we make no priority claim.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AC18] Jorge Antezana and Eduardo Chiumiento. Approximation by partial isometries and symmetric approximation of finite frames. *Journal of Fourier Analysis and Applications*, 24(4):1098–1118, 2018. doi:10.1007/s00041-017-9547-5.
- [BJS11] Marcin Bownik, John Jasper, and Darrin Speegle. Orthonormal dilations of non-tight frames. *Proceedings of the American Mathematical Society*, 139(9):3247–3256, 2011. arXiv:1004.3552, doi:10.1090/S0002-9939-2011-10887-6.
- [BL07] Marcin Bownik and Jakob Lemvig. The canonical and alternate duals of a wavelet frame. *Applied and Computational Harmonic Analysis*, 23(2):263–272, 2007. doi:10.1016/j.acha.2007.04.004.
- [BR19] Marcin Bownik and Ziemowit Rzeszutnik. Open problems in wavelet theory, 2019. arXiv:1905.08230.

- [BW03] Marcin Bownik and Eric Weber. Affine frames, GMRA's, and the canonical dual. *Studia Mathematica*, 159(3):453–479, 2003. doi:10.4064/sm159-3-8.
- [Chi19] Eduardo Chiumiento. Global symmetric approximation of frames. *Journal of Fourier Analysis and Applications*, 25(4):1395–1423, 2019. doi:10.1007/s00041-018-9632-4.
- [Chr04] Ole Christensen. Trends in frame theory. *Oberwolfach Reports*, 1(1):488, 2004. Abstract in: Mini-Workshop: Wavelets and Frames, Report No. 10/2004, pp. 479–544. doi:10.4171/OWR/2004/10.
- [Chr14] Ole Christensen. Six (seven) problems in frame theory. In Ahmed I. Zayed and Gerhard Schmeisser, editors, *New Perspectives on Approximation and Sampling Theory*, Applied and Numerical Harmonic Analysis, pages 337–358. Birkhäuser/Springer, Cham, 2014. arXiv version v1 (2013) titled “Six problems in frame theory”. arXiv:1308.5065, doi:10.1007/978-3-319-08801-3_14.
- [Chr16] Ole Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Cham, second edition, 2016. doi:10.1007/978-3-319-25613-9.
- [Chu92] Charles K. Chui, editor. *Wavelets: A Tutorial in Theory and Applications*, volume 2 of *Wavelet Analysis and its Applications*. Academic Press, Boston, MA, 1992.
- [CS93] Charles K. Chui and Xianliang Shi. Inequalities of Littlewood–Paley type for frames and wavelets. *SIAM Journal on Mathematical Analysis*, 24(1):263–277, 1993. doi:10.1137/0524017.
- [Dau90] Ingrid Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Transactions on Information Theory*, 36(5):961–1005, 1990. doi:10.1109/18.57199.
- [Dau92] Ingrid Daubechies. *Ten Lectures on Wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992. doi:10.1137/1.9781611970104.
- [DH02] Ingrid Daubechies and Bin Han. The canonical dual frame of a wavelet frame. *Applied and Computational Harmonic Analysis*, 12(3):269–285, 2002. doi:10.1006/acha.2002.0381.
- [FPT02] Michael Frank, Vern I. Paulsen, and Terry R. Tiballi. Symmetric approximation of frames and bases in Hilbert spaces. *Transactions of the American Mathematical Society*, 354(2):777–793, 2002. arXiv:math/9812052, doi:10.1090/S0002-9947-01-02838-0.
- [Fra04] Michael Frank. Approximation of frames by normalized tight ones. *Oberwolfach Reports*, 1(1):529–532, 2004. Abstract in: Mini-Workshop: Wavelets and Frames, organized by H. G. Feichtinger, P. E. T. Jorgensen, D. R. Larson and G. Ólafsson, Report No. 10/2004, pp. 479–544. doi:10.4171/OWR/2004/10.
- [Han01] Deguang Han. Tight frame approximations for Gabor and wavelet frames. In Andrew F. Laine, Michael A. Unser, and Akram Aldroubi, editors, *Wavelets: Applications in Signal and Image Processing IX*, volume 4478 of *Proceedings of SPIE*, pages 135–141, 2001. doi:10.1117/12.449695.
- [Han03] Deguang Han. Approximations for Gabor and wavelet frames. *Transactions of the American Mathematical Society*, 355(8):3329–3342, 2003. doi:10.1090/S0002-9947-03-03047-2.

- [Han04] Deguang Han. Tight frame approximation for multi-frames and super-frames. *Journal of Approximation Theory*, 129(1):78–93, 2004. doi:10.1016/j.jat.2004.04.012.
- [HL00] Deguang Han and David R. Larson. Frames, bases and group representations. *Memoirs of the American Mathematical Society*, 147(697), 2000. doi:10.1090/memo/0697.
- [HL09] Deguang Han and David R. Larson. On the orthogonality of frames and the density and connectivity of wavelet frames. *Acta Applicandae Mathematicae*, 107(1–3):211–222, 2009. doi:10.1007/s10440-008-9420-2.
- [HL11] Deguang Han and David R. Larson. Unitary systems and Bessel generator multipliers. In *Wavelets and Multiscale Analysis*, Applied and Numerical Harmonic Analysis, pages 131–150. Birkhäuser, Boston, MA, 2011. doi:10.1007/978-0-8176-8095-4_7.
- [RS97] Amos Ron and Zuowei Shen. Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator. *Journal of Functional Analysis*, 148(2):408–447, 1997. doi:10.1006/jfan.1996.3079.