

Sharp Kernel Penalties for Canonical Parseval Approximation

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Abstract

For a closed-range synthesis operator F , let P_F be the projection onto $(\ker F)^\perp$. We study the metrics

$$d_{p,\kappa}(F, G) = (\|F - G\|^p + \kappa^p \|P_F - P_G\|^p)^{1/p}, \quad 1 \leq p < \infty.$$

The polar factor of every F is a nearest Parseval synthesis operator among partial isometries with arbitrary ranges if and only if the universal penalty satisfies $\kappa \geq 1$. Every minimizer then has the same kernel as F . We parametrize all minimizers and their ranges by a contraction, an orthogonal defect isometry and an explicit operator inequality. Over complex Hilbert spaces we obtain an exact spectral uniqueness criterion and determine when distinct minimizing subspaces occur. A rank-collapse example shows that no metric compatible with the operator-norm topology can have the same universal nearest-frame property. The case $p = 2, \kappa = 1$ is the existing kernel metric of Corach, Maestripieri and Mbekhta; the results concern its metric projections and the sharp penalty in this family.

1 The frame approximation problem

A frame sequence is a frame for its own closed linear span. A normalized tight frame, also called a Parseval frame, has frame bound one. Frank asks for distances under which the canonical normalized tight frame, or an appropriate multiple of it, is closest among normalized tight frames for arbitrary closed subspaces, and for the geometry of the minimizing subspaces [6, pp. 529–531]. Frank and Larson already formulate the broad approximation problem and discuss the limitations of natural distances [7, Section 5].

We give a constructive answer with multiplier one. The coefficient space and indexing are fixed, and the competitors may span any closed subspace of the ambient Hilbert space. The main result identifies a sharp universal kernel penalty. We then describe every minimizing frame and every minimizing range for these metrics. The problem of classifying all possible distances, or optimizing a variable tight-frame bound simultaneously, is not part of this statement.

There is extensive prior work on the underlying operator geometry. Corach, Maestripieri and Mbekhta introduced the metric d_N , which is exactly $d_{2,1}$ below, and studied closed-range operators with this stronger topology [5]. Chiumiento’s polar-factor approximation theorem under an index condition implies the norm lower bound needed here [3, Theorem 2.4 and Corollary 2.5]; we supply a direct proof of the projection-gap case. Finite unitarily invariant norm approximation and global Hilbert–Schmidt approximation have their own developed theories [1, 2]. Further operator-ideal geometry is studied in [4]. Our focus is the finite- p penalty threshold, the resulting unrestricted minimizers and their ranges.

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1.1 Operator conventions

Fix complex Hilbert spaces $\mathcal{E}_0$ and $\mathcal{H}$, with inner products linear in the first variable. Write $\text{CR}(\mathcal{E}_0, \mathcal{H})$ for the bounded operators with closed range. Given a fixed orthonormal basis (e_i) of $\mathcal{E}_0$, a nonzero $F \in \text{CR}(\mathcal{E}_0, \mathcal{H})$ synthesizes the frame sequence (Fe_i) for $\text{ran } F$. Conversely, the synthesis operator of a frame sequence has closed range. Parseval synthesis operators are precisely the partial isometries: V^*V is an orthogonal projection. The zero operator represents the degenerate zero-space case and will be treated separately.

Let $F \neq 0$, and write its canonical polar decomposition as $F = U|F|$. Set

$$P = U^*U = P_{(\ker F)^\perp}, \quad E = \text{ran } P, \quad K = \text{ran } F, \quad A = |F|_E.$$

Then $U|_E : E \rightarrow K$ is unitary and A is positive and boundedly invertible. The canonical Parseval synthesis operator is U . Define

$$M = \|F\|, \quad m = \inf_{\substack{x \in E \\ \|x\|=1}} \|Fx\| > 0, \quad r = \|F - U\|.$$

Spectral calculus gives

$$r = \|A - I_E\| = \max\{M - 1, 1 - m\}. \quad (1)$$

The spectral endpoints need not be eigenvalues.

For $G \in \text{CR}(\mathcal{E}_0, \mathcal{H})$, let $P_G = P_{(\ker G)^\perp}$. For $1 \leq p < \infty$ and $\kappa > 0$, put

$$d_{p,\kappa}(F, G) = (\|F - G\|^p + \kappa^p \|P_F - P_G\|^p)^{1/p}. \quad (2)$$

This is a finite-valued metric: definiteness follows from the first term, and the two norm triangle inequalities followed by Minkowski's inequality give the triangle inequality. Simultaneous unitary changes in the domain and codomain preserve the metric. Complementing the kernel projections identifies $d_{2,1}$ with d_N in [5, Section 3].

2 The sharp penalty theorem

Lemma 1. *Let V be a partial isometry and put $Q = V^*V$. If $\|P - Q\| < 1$, then $\|F - V\| \geq r$.*

Proof. Write $\delta = \|P - Q\| < 1$. For $x \in E$,

$$\|Qx\|^2 = \|x\|^2 - \|(I - Q)x\|^2 \geq (1 - \delta^2)\|x\|^2.$$

Thus $B = PQP|_E$ is positive and boundedly invertible. For any unit vector $x \in E$, set $y = QB^{-1}x$. Then $y \in \text{ran } Q$, $P_y = x$ and $\|y\| \geq 1$. Since $F = FP$, the unit vector $z = y/\|y\|$ satisfies

$$\|Vz\| = 1, \quad \|Fz\| = \frac{\|Fx\|}{\|y\|} \leq \|Fx\|.$$

The reverse triangle inequality yields $\|F - V\| \geq 1 - \|Fx\|$. Taking unit vectors with $\|Fx\| \rightarrow m$ gives $\|F - V\| \geq 1 - m$. Also $\|F - V\| \geq M - 1$, because a partial isometry has norm at most one. Equation (1) proves the result. $\square$

Theorem 2. For $1 \leq p < \infty$ and $\kappa \geq 1$, the polar factor U is a global nearest partial isometry to F in $d_{p,\kappa}$, and

$$\min_{V \text{ partial isometry}} d_{p,\kappa}(F, V) = r, \quad (3)$$

$$\mathcal{M}(F) = \{V : V^*V = P, \|F - V\| \leq r\}. \quad (4)$$

Here $\mathcal{M}(F)$ is the complete set of minimizers. In particular, equality of kernels is a consequence of minimization, with no same-kernel restriction imposed on competitors. For $F = 0$, the unique minimizer is $V = 0$.

Proof. The polar factor has $P_U = P$, so $d_{p,\kappa}(F, U) = r$. For a competitor V , put $Q = V^*V$ and $\delta = \|P - Q\| \leq 1$.

If $\delta < 1$, Lemma 1 gives $\|F - V\| \geq r$. For $\delta > 0$, the additional positive summand in the finite- p metric makes $d_{p,\kappa}(F, V) > r$.

If $\delta = 1$ and $r < 1$, then $d_{p,\kappa}(F, V) \geq \kappa \geq 1 > r$. If $\delta = 1$ and $r \geq 1$, the strict inequality $1 - m < 1$ and (1) imply $r = M - 1$. Hence $\|F - V\| \geq r$, and again the positive penalty gives strict inequality. Every minimizer therefore satisfies $Q = P$. For these operators the metric equals $\|F - V\|$, whose lower bound is r by the lemma. This proves (4). The zero case follows from definiteness of the metric. $\square$

Proposition 3. The threshold $\kappa = 1$ is sharp for every finite $p \geq 1$. For each $0 < \kappa < 1$, the universal nearest-polar-factor property fails already on $\mathbb{C}^2$.

Proof. Let P_1, P_2 be orthogonal rank-one projections with $P_1 + P_2 = I$, and take

$$\varepsilon = \frac{1 - \kappa}{4}, \quad F = \varepsilon P_1 + P_2, \quad U = I, \quad V = P_2.$$

Then

$$d_{p,\kappa}(F, U) = 1 - \varepsilon, \quad d_{p,\kappa}(F, V) = (\varepsilon^p + \kappa^p)^{1/p} \leq \varepsilon + \kappa < 1 - \varepsilon.$$

The competitor V is nonzero. If zero individual frame vectors are excluded by convention, right-multiply F, U, V by the same real orthogonal matrix with cosine $3/5$ and sine $4/5$. Both columns of the resulting rank-one competitor are nonzero, and unitary invariance preserves every distance. $\square$

3 All minimizing frames and subspaces

For a contraction $C : E \rightarrow E$, let $D_C = (I_E - C^*C)^{1/2}$, and choose an isometry

$$J : \overline{\text{ran } D_C} \longrightarrow K^\perp. \quad (5)$$

Such J exists exactly when $\dim \overline{\text{ran } D_C} \leq \dim K^\perp$, where dimension means Hilbert-space dimension. If $D_C = 0$, its domain is the zero space. Define

$$W_{C,J}x = UCx + JD_Cx \quad (x \in E), \quad W_{C,J}|_{E^\perp} = 0. \quad (6)$$

Theorem 4. For the parameters in Theorem 2, the full set of minimizers is given by (6), subject to

$$AC + C^*A \geq A^2 + (1 - r^2)I_E. \quad (7)$$

The complete family of closest-frame subspaces is

$$L_{C,J} = \{UCx + JD_Cx : x \in E\}, \quad (8)$$

with the same contraction, defect-dimension and operator-inequality conditions. Every such subspace is closed.

Proof. The two summands in (6) are orthogonal, so

$$W_{C,J}^* W_{C,J}|_E = C^* C + D_C^2 = I_E.$$

Consequently $W_{C,J}^* W_{C,J} = P$. Conversely, if $V^* V = P$, define

$$C = U^* V|_E, \quad D = (I - U U^*) V|_E : E \rightarrow K^\perp.$$

Then $C^* C + D^* D = I_E$. The polar decomposition of D has the form $D = J D_C$, with J as in (5), and $V = U C + J D_C$ on E .

Both F and $W_{C,J}$ vanish on $E^\perp$, and orthogonality gives

$$\begin{aligned} (F - W_{C,J})^* (F - W_{C,J})|_E &= (A - C)^* (A - C) + D_C^2 \\ &= A^2 - AC - C^* A + I_E. \end{aligned} \tag{9}$$

Thus (7) is equivalent to $\|F - W_{C,J}\| \leq r$. Apply Theorem 2. The range in (8) is closed because $W_{C,J}|_E$ is an isometry. $\square$

The description is exhaustive, although distinct parameter pairs may produce the same subspace. In infinite dimension, $D_C = 0$ only says that C is an isometry; it does not imply that C is onto.

4 Uniqueness and variation of the range

The positive operator

$$R = r^2 I_E - (A - I_E)^2 \tag{10}$$

measures which directions have strictly smaller error than the maximal error r .

Theorem 5. *Over complex Hilbert spaces, U is the unique minimizer if and only if $R = 0$. If K is a proper subspace of $\mathcal{H}$, all minimizers span K if and only if $R = 0$. When $R \neq 0$ and $K^\perp \neq 0$, minimizing subspaces different from K occur arbitrarily close to K in the projection-gap metric.*

Proof. Suppose first that $R = 0$. Spectral calculus implies $\sigma(A) \subset \{1 - r, 1 + r\}$, with nonpositive values excluded because A is positive and invertible. For an eigenvector x with eigenvalue $a > 0$ and any minimizer V , we have

$$\|a U x - V x\|^2 = (a - 1)^2 \|x\|^2 + 2a(\|x\|^2 - \operatorname{Re}\langle U x, V x \rangle) \leq r^2 \|x\|^2.$$

The first term is already $r^2 \|x\|^2$, and the second is nonnegative because U and V are isometries on E . It vanishes only if $V x = U x$. The at most two spectral subspaces span E , and both operators vanish on $E^\perp$; hence $V = U$.

Suppose now that $R \neq 0$. Choose $\eta > 0$ such that the spectral projection $B = \mathbf{1}_{[\eta, \infty)}(R)$ is nonzero. It commutes with A , and $R \geq \eta B$. For a small nonzero real t , take $C = e^{itB}$. This is unitary and $D_C = 0$, while

$$(A - C)^* (A - C) = (A - I_E)^2 + 2(1 - \cos t)AB \leq r^2 I_E$$

provided $2M(1 - \cos t) \leq \eta$. Thus $U C \neq U$ is a minimizer with range K . The phase construction uses the complex field. Phase families of nonunique approximations already appear in [7, Section 5]; here the spectral projection locates all the slack needed for the criterion.

Also assume $K^\perp \neq 0$. Choose unit vectors $x \in \text{ran } B$ and $z \in K^\perp$. Let P_x denote the projection onto $\text{span}\{x\}$. For small $0 < \theta < \pi/2$, put

$$C = I_E - (1 - \cos \theta)P_x, \quad Dy = \sin \theta \langle y, x \rangle z, \quad V|_E = UC + D, \quad V|_{E^\perp} = 0.$$

Then $C^*C + D^*D = I_E$, so $V^*V = P$. With $\tau = 1 - \cos \theta$, equation (9) becomes

$$(F - V)^*(F - V)|_E = (A - I_E)^2 + \tau(AP_x + P_xA). \quad (11)$$

The self-adjoint operator $AP_x + P_xA$ is supported on $\text{ran } B$, since $P_x \leq B$ and $AB = BA$, and its norm is at most $2M$. Consequently $AP_x + P_xA \leq 2MB$; no eigenvector assumption on x is needed. Choose $2M\tau \leq \eta$ to obtain $(F - V)^*(F - V)|_E \leq r^2I_E$. The resulting minimizing range is

$$L_\theta = (K \cap (Ux)^\perp) \oplus \text{span}\{\cos \theta Ux + \sin \theta z\}.$$

It differs from K , and the two-dimensional projection calculation gives $\|P_{L_\theta} - P_K\| = \sin \theta \rightarrow 0$. If $R = 0$, the uniqueness already proved forces every range to be K . $\square$

Example 6. Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ have columns $e_1/4, e_2$. Its polar factor has columns e_1, e_2 , and $r = 3/4$. The isometry V with columns $e_1, (15e_2 + 8e_3)/17$ spans a different subspace, but

$$(F - V)^*(F - V) = \begin{pmatrix} 9/16 & 0 \\ 0 & 4/17 \end{pmatrix}.$$

It is therefore a minimizer for every metric in Theorem 2.

Remark 7. The complex-field qualification in the uniqueness criterion is necessary. Over $\mathbb{R}^2$, take $F = \text{diag}(1/4, 1)$. Here $r = 3/4$ and $R \neq 0$. Yet the first column of any orthogonal minimizer must be e_1 , by equality in the first part of the proof of Theorem 5. Its second column is then either e_2 or $-e_2$; the latter has error two. Thus I is the unique real minimizer. Theorems 2 and 4 themselves also hold over real Hilbert spaces.

Remark 8. If $K = \mathcal{H}$ is finite dimensional, every minimizer has range $\mathcal{H}$, even when the minimizing frame is not unique. In infinite dimension this fails. On $\mathcal{H} = \mathbb{C}e_0 \oplus \ell^2(\mathbb{N})$, take $F = 3I_{\mathbb{C}e_0} \oplus I_{\ell^2}$. Then $U = I$ and $r = 2$. For the unilateral shift S , the isometry $V = I_{\mathbb{C}e_0} \oplus S$ has proper range and $\|F - V\| = 2$, so it is a minimizer. Indeed $\|I - S\| \leq 2$, and unit alternating vectors on n successive coordinates give $\|(I - S)x_n\|^2 = 4 - 2/n$, proving equality.

5 Topology and the maximum metric

Proposition 9. *There is no metric on $\text{CR}(\mathbb{C}^2, \mathbb{C}^2)$ inducing the operator-norm topology for which the polar factor of every operator is a nearest partial isometry.*

Proof. For $\varepsilon > 0$, let $F_\varepsilon = \varepsilon P_1 + P_2$, as in Proposition 3. Then $F_\varepsilon \rightarrow P_2$ in operator norm and the polar factor of every F_ε is I . If such a metric d existed, P_2 would be an admissible competitor and

$$d(I, P_2) \leq d(I, F_\varepsilon) + d(F_\varepsilon, P_2) \leq 2d(F_\varepsilon, P_2) \rightarrow 0,$$

contrary to $I \neq P_2$. $\square$

The same example embeds into larger domain and ambient spaces of dimension at least two. The common rotation in Proposition 3 also avoids zero individual frame vectors. The metrics (2) prevent this rank-collapse convergence: their topology is deliberately stronger than the operator-norm topology, as in the geometry of [5].

Finiteness of p is essential for the same-kernel conclusion. Define the maximum metric by

$$d_{\infty, \kappa}(F, G) = \max\{\|F - G\|, \kappa\|P_F - P_G\|\}.$$

For $F = \text{diag}(3, 1/4)$, $U = I$, and $V = \text{diag}(1, 0)$, one has $d_{\infty, 1}(F, U) = d_{\infty, 1}(F, V) = 2$, although $P_U \neq P_V$. The bound $\|F - W\| \geq \|F\| - 1 = 2$ for every partial isometry W proves global minimality.

The results give explicit metrics and the full family of closest subspaces in the normalized setting of the source question. They do not impose Gabor or wavelet structure, concern Hilbert C^* -modules, or classify metrics outside (2). The polar construction, the kernel metric and the norm lower bound are known ingredients; no absolute priority claim is made.

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