

Parameter Holes in Bessel Wave Packet Systems

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Abstract

Let a wave packet system consist of dilations, translations and modulations of a nonzero function in $L^2(\mathbb{R})$, indexed by an arbitrary discrete subset of the affine Weyl–Heisenberg parameter group. We prove that a Bessel upper bound forces this parameter set to miss a left translate of every compact set. In particular, every lower density in the sense of Czaja–Kutyniok–Speegle vanishes and the lower Beurling dimension is zero. No product structure or additional regularity of the generator is assumed. The proof averages translated counting measures along explicit left Følner sets. Hypothetical relative denseness produces a positive Haar limit, which transfers the discrete Bessel bound to the continuous coefficient integral. Time–frequency orthogonality makes that integral infinite. We also make explicit the necessary nonzero-generator qualification in the original conjecture.

1 The parameter set problem

For $x > 0$ and $y, z \in \mathbb{R}$, define the unitary operators

$$D_x f(t) = \sqrt{x} f(xt), \quad T_y f(t) = f(t - y), \quad M_z f(t) = e^{2\pi i z t} f(t).$$

Following Czaja, Kutyniok and Speegle [2], equip $G = \mathbb{R}_+ \times \mathbb{R}^2$ with the multiplication

$$(x, y, z)(a, b, c) = (xa, ay + b, z/a + c). \quad (1)$$

For discrete $M \subset G$, the system generated by $\psi \in L^2(\mathbb{R})$ is $\{D_x T_y M_z \psi : (x, y, z) \in M\}$. It is Bessel with bound $B < \infty$ if

$$\sum_{\lambda \in M} |\langle f, \pi(\lambda) \psi \rangle|^2 \leq B \|f\|_2^2 \quad (f \in L^2(\mathbb{R})), \quad \pi(x, y, z) = D_x T_y M_z. \quad (2)$$

A lower frame bound is not assumed.

The conjecture in [1, p. 514] asks whether a Bessel wave packet system necessarily has lower parameter dimension zero. The integer-modulation case $M = B_0 \times \mathbb{Z}$ is treated in [1, 2]. Here we prove a stronger statement for arbitrary parameter sets.

Theorem 1. *Let $0 \neq \psi \in L^2(\mathbb{R})$ and let $M \subset G$ be discrete. If (2) holds, then for every compact $K \subset G$ there is $a \in G$ such that $M \cap aK = \emptyset$.*

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The group law and side of translation are essential. In the conventions of [2], put $Q_h = [e^{-h}, e^h] \times [-h, h]^2$ and, for $A > 0$, define

$$D_A^-(M) = \liminf_{h \rightarrow \infty} \inf_{a \in G} \frac{\#(M \cap aQ_h)}{h^A}, \quad (3)$$

$$\dim^-(M) = \inf\{A > 0 : D_A^-(M) < \infty\}. \quad (4)$$

Corollary 2. *Under the hypotheses of Theorem 1, $D_A^-(M) = 0$ for every $A > 0$ and $\dim^-(M) = 0$.*

Proof. For each $h > 0$, apply Theorem 1 to the compact set Q_h . The infimum in (3) is zero at every scale. $\square$

The nonzero condition cannot be dropped. Section 5.1 gives a parameter set of lower dimension three, while every zero-generated family is Bessel. The affirmative conclusion resolves the nondegenerate conjecture, with the necessary qualification stated explicitly.

2 Covariance and local counting bounds

Direct substitution gives

$$\pi(x, y, z)\pi(a, b, c) = e^{2\pi izb/a} \pi(xa, ay + b, z/a + c). \quad (5)$$

Thus π is a projective unitary representation. Each left translate gM has the same Bessel bound: the vectors $\pi(g\lambda)\psi$ differ from $\pi(g)\pi(\lambda)\psi$ only by scalars of modulus one, so apply (2) to $\pi(g)^*f$. Strong continuity follows first on continuous compactly supported functions and then on L^2 by density and unitarity.

Lemma 3. *Under the hypotheses of Theorem 1, for every compact $K \subset G$ there is $C_K < \infty$ such that $\sup_{a \in G} \#(M \cap aK) \leq C_K$.*

Proof. Since $\langle \psi, \pi(e)\psi \rangle = \|\psi\|_2^2 > 0$, choose a relatively compact open identity neighborhood U with

$$|\langle \psi, \pi(u)\psi \rangle| \geq \frac{1}{2} \|\psi\|_2^2 \quad (u \in U).$$

For $\lambda = au \in M \cap aU$, covariance gives

$$|\langle \pi(a)\psi, \pi(\lambda)\psi \rangle| = |\langle \psi, \pi(u)\psi \rangle|.$$

Testing (2) at $f = \pi(a)\psi$ yields

$$\#(M \cap aU) \frac{\|\psi\|_2^4}{4} \leq B \|\psi\|_2^2, \quad \#(M \cap aU) \leq \frac{4B}{\|\psi\|_2^2}.$$

Cover K by finitely many left translates $b_j U$ and sum the bounds for the sets $ab_j U$. $\square$

This also supplies local finiteness and countability if those are not included in the convention for a discrete set. It imposes no minimum separation. The local coefficient argument is standard; see [2, Theorem 3].

3 Averaging to Haar measure

In coordinates $s = \log x$, multiplication becomes

$$(s, y, z)(u, b, c) = (s + u, e^u y + b, e^{-u} z + c). \quad (6)$$

Both left and right multiplication have Jacobian determinant one. A Haar measure is therefore $d\mu = ds dy dz = dx dy dz/x$. In logarithmic coordinates set

$$F_T = [-T, T] \times [-e^{2T}, e^{2T}]^2, \quad \mu(F_T) = 8Te^{4T}.$$

Lemma 4. *For each fixed $a = (s_0, y_0, z_0) \in G$ and all sufficiently large T ,*

$$\frac{\mu(aF_T \triangle F_T)}{\mu(F_T)} \leq \frac{|s_0|}{T} + (|y_0| + |z_0|)e^{-T}.$$

In particular $(F_n)_{n \geq 1}$ is a left Følner sequence.

Proof. Left multiplication sends (s, y, z) to $(s + s_0, y + e^s y_0, z + e^{-s} z_0)$. The scale intervals overlap in length $2T - |s_0|$. On an overlapping slice, the spatial intervals shift by at most $e^T |y_0|$ and $e^T |z_0|$. With $L = e^{2T}$, their relative overlap is at least

$$\left(1 - \frac{|y_0|e^T}{2L}\right) \left(1 - \frac{|z_0|e^T}{2L}\right) \geq 1 - \frac{(|y_0| + |z_0|)e^T}{2L}$$

once both factors are nonnegative. Multiply by the relative scale overlap and use $\mu(aF_T \triangle F_T) = 2(\mu(F_T) - \mu(aF_T \cap F_T))$. $\square$

Define positive measures by

$$\nu_n(\varphi) = \frac{1}{\mu(F_n)} \int_{F_n} \sum_{\lambda \in M} \varphi(g\lambda) d\mu(g), \quad \varphi \in C_c(G). \quad (7)$$

For $\text{supp } \varphi \subset K$, Lemma 3 gives

$$\left| \sum_{\lambda \in M} \varphi(g\lambda) \right| \leq C_K \|\varphi\|_\infty \quad (8)$$

uniformly in g , since contributing points lie in $g^{-1}K$. The sum is measurable by countability. Hence (7) is a locally bounded positive functional on $C_c(G)$ and defines a Radon measure.

Lemma 5. *A subsequence of (ν_n) converges vaguely to $c\mu$, $c \geq 0$. If a compact $K \subset G$ satisfies $M \cap aK \neq \emptyset$ for all $a \in G$, then $c > 0$.*

Proof. The measures are uniformly bounded on each compact set. As G is locally compact and second countable, take a compact exhaustion and countable uniformly dense test-function families on its members. A diagonal subsequence has convergent integrals on those families. Local bounds extend the limits to a positive locally bounded functional on $C_c(G)$; Riesz representation gives a Radon measure η . This is vague convergence for all compactly supported continuous tests along a single subsequence.

For fixed a , change variables $k = ag$ in (7). By left Haar invariance and (8),

$$|\nu_n(\varphi \circ L_a) - \nu_n(\varphi)| \leq C_{\text{supp } \varphi} \|\varphi\|_\infty \frac{\mu(aF_n \triangle F_n)}{\mu(F_n)} \longrightarrow 0.$$

Thus η is left invariant. Uniqueness of Haar measure, with the zero measure included separately, gives $\eta = c\mu$, $c \geq 0$.

Under the additional hypothesis, choose $0 \leq \varphi \in C_c(G)$ with $\varphi \geq 1$ on K . Applying the hypothesis with $a = g^{-1}$ gives $gM \cap K \neq \emptyset$ for every g . Therefore $\nu_n(\varphi) \geq 1$ and $\eta(\varphi) \geq 1$, so $c > 0$. $\square$

This nonvanishing step excludes escape of mass. The averaged measures need not be probability measures and can have infinite total mass.

4 The continuous coefficient obstruction

Lemma 6. *For $f, \psi \in L^2(\mathbb{R})$ and fixed $x > 0$,*

$$\int_{\mathbb{R}^2} |\langle f, D_x T_y M_z \psi \rangle|^2 dy dz = \|f\|_2^2 \|\psi\|_2^2. \quad (9)$$

If both vectors are nonzero, then

$$\int_G |\langle f, \pi(g)\psi \rangle|^2 d\mu(g) = \infty. \quad (10)$$

Proof. Put $u = D_x^{-1}f$. Up to a phase of modulus one the coefficient is the Fourier transform of $u(t)\overline{\psi(t-y)}$, evaluated at z . The product is in $L^1(\mathbb{R})$ for every y by Cauchy–Schwarz. Moreover

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u(t)|^2 |\psi(t-y)|^2 dt dy = \|u\|_2^2 \|\psi\|_2^2 < \infty.$$

It is consequently in $L^2(\mathbb{R})$ for almost every y . Plancherel in z and Tonelli in t, y give (9). Integrating against dx/x then gives

$$\int_G |\langle f, \pi(g)\psi \rangle|^2 d\mu(g) = \|f\|_2^2 \|\psi\|_2^2 \int_0^\infty \frac{dx}{x} = \infty$$

when both vectors are nonzero. $\square$

The time–frequency orthogonality identity and the failure of square integrability for noncompact dilation groups are known. A broader metaplectic version is [3, Theorem 5.2]; these continuous facts are not claimed as new.

Proof of Theorem 1. Suppose a compact K satisfies $M \cap aK \neq \emptyset$ for every a . Lemma 5 gives a vague subsequential limit $c\mu$ with $c > 0$. Fix $f \in L^2(\mathbb{R})$ and put $H_f(g) = |\langle f, \pi(g)\psi \rangle|^2$, a nonnegative continuous function. For $0 \leq \chi \leq 1$ with $\chi \in C_c(G)$, each system indexed by gM has the same Bessel bound, so

$$\nu_n(\chi H_f) = \frac{1}{\mu(F_n)} \int_{F_n} \sum_{\lambda \in M} \chi(g\lambda) H_f(g\lambda) d\mu(g) \leq B \|f\|_2^2.$$

Vague convergence applies to χH_f and yields $c \int_G \chi H_f d\mu \leq B \|f\|_2^2$. Choose compactly supported continuous cutoffs $\chi_j \uparrow 1$. Monotone convergence gives

$$c \int_G H_f d\mu \leq B \|f\|_2^2. \quad (11)$$

For nonzero f , Lemma 6 contradicts (11). This proves the theorem. $\square$

5 Scope and relation to density arguments

Averaging point sets against invariant measures and recovering Haar measure are classical methods; see the hull and covolume framework of Enstad–Raum [4, Section 2]. Their Bessel estimate in Theorem 3.12 uses Assumption 3.4, placing the reproducing-kernel Hilbert space isometrically in $L^2(G)$ and imposing localization. Here the coefficient map does not land in $L^2(G)$. Instead, hypothetical relative denseness forces (11), which is then contradicted.

Related coherent-system results [5, 6] assume square-integrable projective representations. The Bessel obstruction in [7, Section 5] concerns product systems with additional Fourier-side and modulation-covering hypotheses. Theorem 1 requires none of those restrictions.

Remark 7. The averaging sets F_T do not replace the source’s density boxes. Indeed Q_h is not left Følner. For $a = (0, 1, 0)$ in logarithmic coordinates and all sufficiently large h ,

$$\frac{\mu(aQ_h \cap Q_h)}{\mu(Q_h)} = \frac{h + \log(2h) - 1 + e^{-h}/(2h)}{2h}.$$

This follows by integrating $\max(0, 1 - e^s/(2h))$ over $[-h, h]$ and dividing by $2h$. The relative symmetric difference tends to one, not zero. The conclusion about (3) instead follows from empty translates at each scale.

The theorem concerns lower dimension and is consistent with the examples of upper dimension three in [2]. It does not characterize all Bessel sets or construct dual frames.

5.1 The zero generator

Take $M_0 = \{(e^n, m, k) : n, m, k \in \mathbb{Z}\}$. For $a = (x, y, z)$, membership of (e^n, m, k) in aQ_h is equivalent to

$$n \in [\log x - h, \log x + h], \quad |m - e^n y/x| \leq h, \quad |k - xz/e^n| \leq h.$$

Each successive interval has length $2h$ and contains between $2h - 1$ and $2h + 1$ integers. Uniformly in a , for $h > 1$,

$$(2h - 1)^3 \leq \#(M_0 \cap aQ_h) \leq (2h + 1)^3.$$

Thus $D_3^-(M_0) = 8$ and $\dim^-(M_0) = 3$, while the zero-generated family is Bessel. The displayed source conjecture therefore needs the nonzero qualification. This exception is not the substantive contribution and is consistent with the explicit zero exception in the introduction of the 2005 author preprint of [2] and the nonzero hypothesis of its Theorem 10.

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