

Normalized Maps of Arbitrary Arity to Wound Unipotent Groups

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Abstract

For every integer $n \geq 2$, we construct a field k of characteristic two, a single closed point $x \in \mathbf{P}_k^1$, and a nonconstant morphism $(\mathbf{P}_k^1 \setminus \{x\})^n \rightarrow U$ to a smooth connected commutative k -wound unipotent group, zero on every coordinate slice through infinity. The residue field $L = k(x)$ always satisfies $[L : kL^2] = 2$. The construction uses a determinant and the parameter-parity decomposition of a product of additive polynomials. For $n \geq 3$, it disproves the primitivity-bound assertion in Conjecture 5 of Oberwolfach Report 20/2021, under either the printed index or its logarithmic normalization. The case $n = 2$ recovers the author's earlier counterexample; the present result gives arbitrarily many factors at the same residue-field index. The fields and targets vary with n , and the boundary points are not geometrically irreducible.

1 The statement and its scope

A smooth connected unipotent group over k is k -wound if every k -morphism $\mathbf{A}_k^1 \rightarrow U$ is constant. We use additive notation for the groups constructed below.

Conjecture 5 of Rosengarten's contribution to [1], printed page 1108, concerns maps from products of open subschemes $X_i \subset \mathbf{P}_k^1$ to wound unipotent groups. If the map is the identity on every chosen coordinate slice, the conjecture asserts constancy when the number of factors exceeds the primitivity of the union of the complementary divisors. The preceding definition prints the index $[L : kL^P]$, whereas its discussion of generators and closed points uses the logarithm of that index. Our examples with $n \geq 3$ exceed both thresholds.

Theorem 1. *For every integer $n \geq 2$, there are a field k of characteristic two, a single closed point $x \in \mathbf{P}_k^1$ with residue field L satisfying $[L : kL^2] = 2$, a smooth geometrically connected commutative k -wound unipotent group U of dimension $2^n - 1$, and a nonconstant k -morphism*

$$f : X^n \longrightarrow U, \quad X = \mathbf{P}_k^1 \setminus \{x\},$$

such that f is zero on every slice obtained by setting one coordinate equal to $\infty \in X(k)$.

Corollary 2. *Conjecture 5 of [1] is false under either normalization described above. More generally, no finite bound depending only on the index $[L : kL^2]$ can bound the number of factors in such nonconstant normalized maps, uniformly over fields, one-point complements, and wound targets.*

The two-factor case was given explicitly in [3]. The new content here is a uniform determinant construction for every number of factors and the fixed-index consequence. The base field in our construction has degree of imperfection n . Consequently this does not contradict a rigidity assertion whose strict threshold is the degree of imperfection of the base field, such as the formulation in [2, Theorem 1.3]. Nor does it refute the descending-central-series assertion discussed after the source conjecture: our targets are commutative.

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2 The field, point and target

Fix $n \geq 2$, and take algebraically independent elements $a_1, \dots, a_n$ over $\mathbf{F}_2$. Put

$$k = \mathbf{F}_2(a_1, \dots, a_n), \quad q(T) = T^{2^n} + \sum_{i=1}^{n-1} a_i T^{2^{n-i}} + a_n. \quad (1)$$

Lemma 3. *The polynomial q is irreducible over k . If L is its root field, then $[L : kL^2] = 2$. The corresponding closed point has separable degree 2^{n-1} and inseparable degree two.*

Proof. Eliminating a_n identifies

$$\mathbf{F}_2[a_1, \dots, a_n, T]/(q) \simeq \mathbf{F}_2[a_1, \dots, a_{n-1}, T].$$

Thus q is prime, hence irreducible, in the polynomial ring on the left. It is monic and primitive as a polynomial in T , so Gauss's lemma gives irreducibility over k .

For a root α we can write

$$L = \mathbf{F}_2(a_1, \dots, a_{n-1}, \alpha), \quad a_n = \alpha^{2^n} + \sum_{i=1}^{n-1} a_i \alpha^{2^{n-i}},$$

where $a_1, \dots, a_{n-1}, \alpha$ are algebraically independent. These formulas give

$$kL^2 = \mathbf{F}_2(a_1, \dots, a_{n-1}, \alpha^2),$$

and the asserted index follows. Moreover, $q(T) = h(T^2)$ with $h'(Y) = a_{n-1} \neq 0$. The irreducible polynomial h has degree 2^{n-1} , proving the separable and inseparable degree statements. $\square$

For $\epsilon \in \{0, 1\}^n$, write $a^\epsilon = \prod_i a_i^{\epsilon_i}$ and let $\mathbf{0}$ denote the zero index. Define

$$U = \left\{ (v_\epsilon)_{\epsilon \in \{0,1\}^n} \in \mathbf{G}_a^{2^n} : v_{\mathbf{0}} = \sum_{\epsilon \in \{0,1\}^n} a^\epsilon v_\epsilon^2 \right\}. \quad (2)$$

Lemma 4. *The group U is smooth, geometrically connected, commutative, unipotent, and k -wound, of dimension $2^n - 1$.*

Proof. The defining polynomial is additive. Its derivative in $v_{\mathbf{0}}$ is 1, so U is a smooth commutative closed subgroup of the indicated vector group, of dimension $2^n - 1$.

Over $k(c)$ with $c^2 = a_1$, let $e_1 = (1, 0, \dots, 0)$ and $Y = v_{\mathbf{0}} + cv_{e_1}$. Equation (2) becomes

$$v_{\mathbf{0}} = Y^2 + \sum_{\epsilon \neq \mathbf{0}, e_1} a^\epsilon v_\epsilon^2.$$

Thus Y and the $2^n - 2$ remaining coordinates are free. Recover $v_{\mathbf{0}}$ from this formula and $v_{e_1} = (Y + v_{\mathbf{0}})/c$. These are inverse additive polynomial maps. Hence U becomes a vector group, proving geometric connectedness and unipotence.

The elements a^ϵ form a k^2 -basis of k . Therefore $\sum_\epsilon a^\epsilon c_\epsilon^2 = 0$ with $c_\epsilon \in k$ forces every c_ϵ to vanish. If a nonconstant map $\mathbf{A}_k^1 \rightarrow U$ existed, its coordinates would be polynomials $g_\epsilon(T)$. Let $m > 0$ be their maximum degree and c_ϵ their coefficients in degree m . Comparison of degree $2m$ in their defining equation gives $\sum_\epsilon a^\epsilon c_\epsilon^2 = 0$, a contradiction since some c_ϵ is nonzero. This proves woundness. $\square$

3 The determinant and the normalized morphism

Put $A = \prod_{i=1}^n a_i$ and $Q = \prod_{j=1}^n q(z_j)$. For $1 \leq i \leq n$ set

$$b_1 = 0, \quad b_i = 2^{i-2} \ (i \geq 2), \quad \Delta = \det(z_j^{b_i})_{1 \leq i, j \leq n}.$$

In characteristic two, Δ is the sum of its permutation monomials. Their exponent vectors are distinct, so $\Delta \neq 0$.

Because every z_j has even exponent in AQ , there is a unique decomposition

$$AQ = \sum_{\epsilon \in \{0,1\}^n} a^\epsilon H_\epsilon^2, \quad H_\epsilon \in \mathbf{F}_2[a_1, \dots, a_n, z_1, \dots, z_n]. \quad (3)$$

This supplies an explicit formula: select each monomial whose parameter exponents have parity ϵ , subtract ϵ from those exponents, and divide all exponents by two. No extension of the coefficient field is involved.

Lemma 5. *The zero-index term in (3) satisfies $H_0 = A\Delta$.*

Proof. A term of AQ has even parameter exponents exactly when the selected terms of Q use each a_i an odd number of times. There are n factors and n parameters. Each factor contributes at most one parameter. Hence every parameter must occur exactly once, and no leading term $z_j^{2^n}$ can occur.

Set $d_i = 2^{n-i}$ for $i < n$ and $d_n = 0$. Summing over the possible assignments of parameters to factors, the zero-parity component is

$$A^2 \sum_{\sigma \in S_n} \prod_{j=1}^n z_j^{d_{\sigma(j)}} = A^2 \det(z_j^{d_i}) = A^2 \Delta^2.$$

Indeed the d_i are a permutation of the $2b_i$, and determinant signs are trivial in characteristic two. Frobenius is injective on the polynomial ring, so $H_0 = A\Delta$. $\square$

Define

$$N_\epsilon = \Delta H_\epsilon, \quad f_\epsilon = \frac{N_\epsilon}{Q}. \quad (4)$$

In particular $N_0 = A\Delta^2$. Multiplying (3) by Δ^2 gives the exact identity

$$\sum_{\epsilon} a^\epsilon N_\epsilon^2 = A\Delta^2 Q = N_0 Q. \quad (5)$$

Proof of Theorem 1. Lemmas 3 and 4 verify the point and target hypotheses. Identity (5), divided by Q^2 , shows that (4) takes values in U on the finite-coordinate chart of X^n .

For every j and ϵ the separate degrees satisfy

$$\deg_{z_j} \Delta = 2^{n-2}, \quad \deg_{z_j} H_\epsilon \leq 2^{n-1}, \quad \deg_{z_j} N_\epsilon \leq 3 \cdot 2^{n-2} < 2^n.$$

The middle bound follows directly from the parity extraction in (3). At a point where a subset S of coordinates is infinite, put $w_j = 1/z_j$ for $j \in S$ and multiply numerator and denominator by $\prod_{j \in S} w_j^{2^n}$. The resulting numerator is polynomial and divisible by every w_j . Each corresponding denominator factor is

$$1 + \sum_{i=1}^{n-1} a_i w_j^{2^n - 2^{n-i}} + a_n w_j^{2^n},$$

which is a unit along $w_j = 0$. Other denominator zeros are exactly the removed points. This proves regularity on all of X^n , including intersections of infinity divisors, and zero restriction to every required slice. The equation in U holds globally since it holds on a dense open of the integral variety X^n .

Finally, $f_0 = A\Delta^2/Q$ is nonzero in the rational function field. Together with the zero slice restrictions, this proves that f is nonconstant. $\square$

Proof of Corollary 2. Take every $X_i = X$ and every chosen point to be infinity. The source uses a union of complementary divisors in $\mathbf{P}^1$, so the union is the single point x , with reduced coordinate algebra L . Its printed index is two and its logarithm is one. For every $n \geq 3$, Theorem 1 contradicts either threshold. Since n is unbounded while the index stays two, the uniform no-bound assertion follows. $\square$

4 The three-factor example and related formulations

For $n = 3$, write $(a_1, a_2, a_3) = (s, t, u)$. Then

$$q(T) = T^8 + sT^4 + tT^2 + u, \quad \Delta = (z_1 + z_2)(z_1 + z_3)(z_2 + z_3),$$

and

$$f_0 = \frac{stu(z_1 + z_2)^2(z_1 + z_3)^2(z_2 + z_3)^2}{(z_1^8 + sz_1^4 + tz_1^2 + u)(z_2^8 + sz_2^4 + tz_2^2 + u)(z_3^8 + sz_3^4 + tz_3^2 + u)}.$$

The other seven coordinates are given by the finite operation (3) followed by (4). This is already a counterexample under both source normalizations. For $n = 2$, the construction recovers [3], after ordering the coordinates by their parameter monomials.

The later formulation [2, Theorem 1.2] uses a disjoint boundary algebra and an invariant defined through composita of embedded residue fields. The embedding-independence assertion in [2, Proposition 2.5 and Definition 2.6] is addressed by the separate explicit field example in [3]. Our proof of Corollary 2 needs no finite-product invariant: the source union is a single point and its field index is computed directly. We make no assertion about a corrected invariant using all conjugates, geometric-irreducibility variants, other characteristics, or unrelated applications of the cited work. In particular the fields, points and targets vary with n .

The construction was found by an exact polynomial search in the three-factor case and then proved uniformly by Lemma 5. The accompanying code checks the full identities for $n = 2, 3, 4$, compares the small cases with a second polynomial implementation, and detects deliberately corrupted coordinates. These finite checks supplement the written proof. This unrefereed manuscript was developed with AI assistance in algebraic exploration, source comparison and exposition; mathematical acceptance here is based on the explicit proof and its detailed self-audit. No independent review, formal proof-assistant verification or absolute-priority certification is claimed.

References

- [1] Z. Rosengarten, Rigidity and Unirational Groups, in *Algebraic Groups*, Oberwolfach Reports **18** (2021), Report 20/2021, pp. 1106–1108, especially Conjecture 5, p. 1108. [doi:10.4171/OWR/2021/20](https://doi.org/10.4171/OWR/2021/20).
- [2] Z. Rosengarten, *Rigidity And Unirational Groups*, [arXiv:2307.04649v4](https://arxiv.org/abs/2307.04649v4), 19 July 2026.
- [3] A. Ferudun, *A Characteristic-Two Counterexample to Rigidity for a Punctured Projective Line*, unrefereed preprint, 27 October 2026. [doi:10.5281/zenodo.23683405](https://doi.org/10.5281/zenodo.23683405).