

A Characteristic-Two Counterexample to Rigidity for a Punctured Projective Line

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Abstract

We construct a counterexample to the two-factor rigidity assertion in Conjecture 4 of Rosengarten's contribution to Oberwolfach Report 20/2021. Over $k = \mathbf{F}_2(s, t)$, let X be the projective line with the single degree-four closed point defined by $T^4 + sT^2 + t$ removed. We give an explicit nonzero morphism $X \times X \rightarrow U$ to a smooth connected commutative k -wound unipotent group of dimension three. The morphism is zero on both coordinate slices through infinity. The construction uses four rational functions and one polynomial identity, and all geometric hypotheses are verified directly. We also exhibit two choices of embeddings of the factors of one finite reduced algebra for which the imperfection of their compositum is respectively one and two. This identifies a failure of the embedding-independence assertion in Proposition 2.5 of arXiv:2307.04649v4. The example uses a closed point that is not geometrically irreducible and makes no claim about rigidity with that additional hypothesis.

1 The source assertion and the example

A smooth connected unipotent group U over a field k is *k-wound* if every k -morphism $\mathbf{A}_k^1 \rightarrow U$ is constant. The following assertion appears as Conjecture 4 on page 1107 of [1]: if x is one closed point of $\mathbf{P}_k^1$, $X = \mathbf{P}_k^1 \setminus \{x\}$, and $y_1, y_2 \in X(k)$, then any morphism $X \times X \rightarrow U$ to a wound unipotent group that is the identity on $\{y_1\} \times X$ and $X \times \{y_2\}$ is constant. The report states that the author has a proof in characteristic different from two and leaves the characteristic-two case conjectural. We use additive notation, with identity denoted by 0.

Let s, t be algebraically independent over $\mathbf{F}_2$, and set

$$k = \mathbf{F}_2(s, t), \quad q(T) = T^4 + sT^2 + t.$$

Define

$$U = \{(v_0, v_1, v_2, v_3) \in \mathbf{G}_a^4 : v_0 = v_0^2 + sv_1^2 + tv_2^2 + stv_3^2\}. \quad (1)$$

Theorem 1. *The polynomial q defines a single degree-four closed point x of $\mathbf{P}_k^1$. Put $X = \mathbf{P}_k^1 \setminus \{x\}$. The group U in (1) is smooth, geometrically connected, commutative, unipotent, and k -wound, of dimension three. For finite coordinates z, w on the two factors, put*

$$d = z + w, \quad e = zw, \quad B = e^2 + se + t, \quad Q = q(z)q(w).$$

The formulas

$$f(z, w) = \frac{1}{Q}(std^2, td^3, sed^2, dB) \quad (2)$$

define a nonconstant k -morphism $X \times X \rightarrow U$ satisfying

$$f(\infty, w) = f(z, \infty) = 0.$$

In particular, the source conjecture is false in characteristic two, even for a commutative target.

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The point x has separable degree two and inseparable degree two. It is a single closed point over k , but is not geometrically irreducible. The printed conjecture imposes no geometric irreducibility hypothesis on that point.

2 The closed point and the wound target

Lemma 2. *The polynomial q is irreducible over k .*

Proof. In $\mathbf{F}_2[s, t, T]$, the quotient by $(T^4 + sT^2 + t)$ is $\mathbf{F}_2[s, T]$, by eliminating t . Thus q is prime, hence irreducible, in this polynomial ring. As a monic polynomial in T over the UFD $\mathbf{F}_2[s, t]$, it is primitive, so Gauss's lemma gives irreducibility over $\mathbf{F}_2(s, t)$. Its degree is four. The polynomial $Y^2 + sY + t$ is separable and irreducible, and adjoining a square root of one of its roots supplies the remaining degree two. These observations also give the stated separable and inseparable degrees. $\square$

Lemma 3. *The group U has all the properties asserted in Theorem 1.*

Proof. The polynomial in (1) is additive, so its zero locus is a closed commutative subgroup of $\mathbf{G}_a^4$. Its derivative in v_0 is 1; hence this hypersurface is smooth of dimension three.

Over $k(c)$, where $c^2 = s$, write $Y = v_0 + cv_1$. Equation (1) becomes

$$v_0 = Y^2 + tv_2^2 + stv_3^2.$$

The three coordinates Y, v_2, v_3 are free, with inverse formulas

$$v_0 = Y^2 + tv_2^2 + stv_3^2, \quad v_1 = (Y + v_0)/c.$$

Both maps are additive polynomial maps. Therefore $U_{k(c)} \simeq (\mathbf{G}_a^3)_{k(c)}$, proving geometric connectedness and unipotence.

The extension $k/k^2 = \mathbf{F}_2(s, t)/\mathbf{F}_2(s^2, t^2)$ has basis $1, s, t, st$. It follows that

$$a_0^2 + sa_1^2 + ta_2^2 + sta_3^2 = 0, \quad a_i \in k, \quad \implies \quad a_0 = a_1 = a_2 = a_3 = 0. \quad (3)$$

Suppose a nonconstant morphism $\mathbf{A}_k^1 \rightarrow U$ existed. Its coordinates would be polynomials $g_i(T)$ in $k[T]$. Let $m > 0$ be their maximum degree and let a_i be the coefficient of T^m in g_i . At least one a_i is nonzero. Comparing degree $2m$ in

$$g_0 = g_0^2 + sg_1^2 + tg_2^2 + stg_3^2$$

gives the left side of (3) equal to zero, since $\deg g_0 \leq m < 2m$. This is a contradiction. Hence U is k -wound. $\square$

3 The global morphism

Proof of Theorem 1. The two lemmas verify the assertions about x and U . Let N_i denote the four numerators in (2). Direct expansion gives

$$Q = B^2 + s(e^2 + t)d^2 + td^4. \quad (4)$$

For completeness, $B^2 = e^4 + s^2e^2 + t^2$, while the remaining terms of $q(z)q(w)$ are

$$sz^2w^2(z^2 + w^2) + t(z^4 + w^4) + st(z^2 + w^2).$$

They equal the remaining terms in (4). Consequently,

$$\begin{aligned} N_0^2 + sN_1^2 + tN_2^2 + stN_3^2 &= s^2t^2d^4 + st^2d^6 + s^2te^2d^4 + std^2B^2 \\ &= std^2(B^2 + s(e^2 + t)d^2 + td^4) \\ &= N_0Q. \end{aligned}$$

Dividing by Q^2 proves that f takes values in U on the dense chart where z, w are finite and $q(z)q(w) \neq 0$.

Each numerator N_i has degree at most three separately in z and in w . This is strict compared with the degree four of each denominator factor. On the chart $u = 1/z$,

$$q(z) = u^{-4}(1 + su^2 + tu^4),$$

and the parenthetical factor is a unit at $u = 0$. Multiplication of each numerator by u^4 therefore gives a polynomial divisible by u , showing regularity and vanishing along $z = \infty$. The same argument applies along $w = \infty$. At their intersection, with $v = 1/w$, the expression

$$\frac{u^4v^4N_i(1/u, 1/v)}{(1 + su^2 + tu^4)(1 + sv^2 + tv^4)}$$

has numerator divisible by uv and denominator a unit. These charts cover all potential points of indeterminacy: the other denominator zeros are precisely the removed points. Thus f is globally regular and zero on both required slices. Its equation in U holds globally, since it holds on a dense open of the integral scheme $X \times X$.

Finally, $q(0) = t$ and $q(1) = 1 + s + t$ are nonzero in k , so $(0, 1) \in X(k) \times X(k)$. Formula (2) gives

$$f(0, 1) = \frac{(s, 1, 0, 1)}{1 + s + t} \neq 0.$$

Since f is zero on the infinity slices, it is nonconstant. □

4 The embedding issue in a later formulation

Theorem 1.2 of [2] presents a more general rigidity assertion using an imperfection invariant for finite reduced algebras. For a finite field extension M/k in characteristic two, write

$$\text{Imp}(M/k) = \log_2[M : kM^2].$$

For a product of finite extension fields, Definition 2.6 of that version chooses embeddings into a common field and applies this quantity to their compositum. Proposition 2.5 asserts independence of those choices. The following example shows that independence fails over a general base field.

Proposition 4. *There is a finite extension L/k such that two choices of embeddings of the factors of $L \times L$ have composita of imperfection one and two, respectively.*

Proof. Take algebraically independent a, c over $\mathbf{F}_2$ and put

$$E = \mathbf{F}_2(a, c), \quad s = c^2, \quad u = a^2, \quad t = u^2 + su, \quad k = \mathbf{F}_2(s, t).$$

Here s, t are algebraically independent: u is transcendental over $\mathbf{F}_2(s)$ and t is a nonconstant polynomial in u . Set

$$F = \mathbf{F}_2(s, u), \quad L = \mathbf{F}_2(s, a).$$

The polynomial $Y^2 + sY + t$ shows that F/k is separable quadratic, and L/F is purely inseparable quadratic. In fact L is the residue field of the degree-four point used above. The square subfields are

$$L^2 = \mathbf{F}_2(s^2, u), \quad E^2 = \mathbf{F}_2(u, s) = F.$$

It follows that $kL^2 = F = kE^2$. Since a, s are independent rational variables and a, c are independent rational variables, respectively,

$$[L : F] = 2, \quad [E : F] = 4.$$

Thus $\text{Imp}(L/k) = 1$ and $\text{Imp}(E/k) = 2$.

There are two k -embeddings $\sigma, \tau : L \hookrightarrow E$ given by

$$\sigma(s) = \tau(s) = s, \quad \sigma(a) = a, \quad \tau(a) = a + c.$$

Indeed, the substitution $a \mapsto a + c, c \mapsto c$ is an automorphism of E , and

$$(a + c)^4 + c^2(a + c)^2 = a^4 + c^2a^2 = t.$$

Using σ for both factors of $L \times L$ gives compositum L . Using σ and τ gives E , because $c = \tau(a) + \sigma(a)$ belongs to that compositum. The two values are therefore different, as claimed. $\square$

This calculation concerns Proposition 2.5 and Definition 2.6 of the specific version cited. It prevents that definition from providing an embedding-independent bound for the boundary in our example. The geometric counterexample in Theorem 1 is independently proved and does not rely on accepting or rejecting other results in [2]. A formulation that includes all conjugates of the boundary may assign a different invariant; its validity is not addressed here.

5 Scope and reproducibility

Theorem 1 gives a negative answer to the full universal statement recorded as OWR-7155441-002 in the frozen *UnsolvedMath* version 1.6.0. No result in characteristic different from two is asserted. Nor does the example address the variant in which the removed point is required to be geometrically irreducible.

The numerator identity can be verified by direct expansion as above. The accompanying script independently computes it both in a polynomial ring modulo two and by XOR-convolution of monomial supports. It also checks the degree bounds, the displayed rational value, the embedding formulas, and a deliberately corrupted numerator tuple. These exact identities supplement the written proofs of the field and geometric claims; they are not finite-field sampling or proof-assistant certification. The search and its output are retained in the source package.

AI assistance was used in developing the construction and manuscript. The author is responsible for the content. This unrefereed preprint does not claim independent review or absolute priority. The literature check is current to 27 October 2026.

References

- [1] Z. Rosengarten, Rigidity and Unirational Groups, in *Algebraic Groups*, Oberwolfach Report 20/2021, pp. 1106–1108, especially Conjecture 4, p. 1107. [doi:10.4171/OWR/2021/20](https://doi.org/10.4171/OWR/2021/20).
- [2] Z. Rosengarten, *Rigidity And Unirational Groups*, [arXiv:2307.04649v4](https://arxiv.org/abs/2307.04649v4), 19 July 2026, especially Theorem 1.2, Proposition 2.5 and Definition 2.6.
- [3] UlamAI, *UnsolvedMath*, version 1.6.0, record OWR-7155441-002. <https://huggingface.co/datasets/ulamai/UnsolvedMath>.