

Weighted Switchings and Linear Corner Growth in Uniform Contingency Tables

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Abstract

Let X be uniformly distributed over nonnegative integer $n \times n$ matrices with both margin vectors $(3n, n, \dots, n)$. We prove that $\mathbb{E}X_{11} = \Theta(n)$, answering the linear-growth question in Barvinok's 2011 Oberwolfach contribution. More precisely, an explicit finite bound implies $\liminf_{n \rightarrow \infty} \mathbb{E}X_{11}/n \geq 2 - \sqrt{2}$. The argument is an exact weighted 2×2 switching identity, followed by two elementary moment inequalities. The same family stays at total variation distance at least $1/4096$ from every geometric law, even when its parameter depends on n . Thus geometric entry approximation does not hold for unrestricted growing margins. A general bound covers one exceptional row and column with margins $(L, t, \dots, t)$. We do not establish the precise limiting mean, a central limit theorem, or the full phase diagram.

1 The question and the finite bound

For integers $n \geq 2$ and $L, t \geq 0$, let $\Omega_n(L, t)$ be the finite set of nonnegative integer $n \times n$ matrices whose row and column margin vectors both equal $(L, t, \dots, t)$. The diagonal matrix shows that this set is nonempty. Throughout, uniform sampling assigns equal probability to each matrix in this set, not the factorial weights of the Fisher–Yates distribution.

Barvinok asks whether individual entries of uniform contingency tables admit asymptotically geometric distributions, and whether $\mathbb{E}X_{11}$ grows linearly for the margins $(3n, n, \dots, n)$ [1, p. 1353]. The corresponding deterministic maximum-entropy typical table has a linearly growing corner [2, §1.6]. That fact alone does not determine the expectation of the uniform table.

Theorem 1. *Let X be uniform on $\Omega_n(L, t)$, and write $A = X_{11}$. Then*

$$\mathbb{E}A \geq L - \beta_{n,t}, \quad \beta_{n,t} := \frac{n-1}{2(n+1)} \left(2t - 1 + \sqrt{(2t-1)^2 + 4(n+1)t} \right). \quad (1)$$

In particular, for $L = 3n$ and $t = n$,

$$\mathbb{E}A \geq 3n - \frac{n-1}{2(n+1)} \left(2n - 1 + \sqrt{8n^2 + 1} \right), \quad (2)$$

and

$$\frac{2n^2 + 4n}{4n + 1} \leq \mathbb{E}A \leq 3n. \quad (3)$$

Consequently $\mathbb{E}A = \Theta(n)$ and $\liminf_{n \rightarrow \infty} \mathbb{E}A/n \geq 2 - \sqrt{2}$.

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We use the convention $d_{\text{TV}}(P, Q) = \sup_E |P(E) - Q(E)|$. For $0 \leq q < 1$, let G_q denote the probability law $G_q(\{k\}) = (1 - q)q^k$ on the nonnegative integers.

Theorem 2. *For every $n \geq 2$, if $A = X_{11}$ for a uniform table with margins $(3n, n, \dots, n)$, then*

$$d_{\text{TV}}(\mathcal{L}(A), G_q) > 1/4096 \quad (0 \leq q < 1). \quad (4)$$

In particular, $\inf_{0 \leq q < 1} d_{\text{TV}}(\mathcal{L}(A), G_q) \geq 1/4096$ uniformly in n .

Here linear growth means order $\Theta(n)$; we do not assert $\mathbb{E}A/n \rightarrow 2 - \sqrt{2}$. The geometric obstruction concerns an unrestricted entry-law assertion, not a statement with fixed-base cloning or other regularity assumptions.

2 An exact weighted switching identity

For a table $X \in \Omega_n(L, t)$, put

$$a = X_{11}, \quad u_i = X_{i1}, \quad v_j = X_{1j}, \quad b_{ij} = X_{ij} \quad (i, j > 1).$$

Set

$$T = L - a, \quad Q = \sum_{i>1} u_i^2 + \sum_{j>1} v_j^2.$$

The constraints imply

$$\begin{aligned} \sum_{i>1} u_i &= \sum_{j>1} v_j = T, & \sum_{j>1} b_{ij} &= t - u_i, & \sum_{i>1} b_{ij} &= t - v_j, \\ \sum_{i,j>1} b_{ij} &= (n-1)t - T. \end{aligned} \quad (5)$$

Lemma 3. *Under the uniform measure on $\Omega_n(L, t)$,*

$$\mathbb{E} \left[(2t-1)T + (n-1)t - T^2 - Q \right] = \mathbb{E} \left[\mathbf{1}_{\{a=0\}} \sum_{i,j>1} b_{ij}(u_i+1)(v_j+1) \right] \geq 0. \quad (6)$$

Proof. Fix $i, j > 1$. The transformation

$$(a, b_{ij}, u_i, v_j) \mapsto (a-1, b_{ij}-1, u_i+1, v_j+1) \quad (7)$$

leaves all other entries unchanged. It is a bijection from the tables with $a, b_{ij} > 0$ onto the tables with $u_i, v_j > 0$. Both directions preserve the margins and nonnegativity.

If primed variables denote the image of a table, then

$$b_{ij}(u_i+1)(v_j+1) = (b'_{ij}+1)u'_i v'_j.$$

Summing these matching weights over the bijection and dividing by $|\Omega_n(L, t)|$ yields

$$\mathbb{E} \left[\mathbf{1}_{\{a>0\}} b_{ij}(u_i+1)(v_j+1) \right] = \mathbb{E}[(b_{ij}+1)u_i v_j]. \quad (8)$$

The factor b_{ij} handles the forbidden case $b_{ij} = 0$ on the left, while $u_i v_j$ handles the forbidden inverse moves.

Write

$$D = \sum_{i,j>1} b_{ij}(u_i + 1)(v_j + 1), \quad U = \sum_{i,j>1} (b_{ij} + 1)u_i v_j.$$

Equation (8), summed over i, j , gives $\mathbb{E}[\mathbf{1}_{\{a>0\}}D] = \mathbb{E}U$, hence

$$\mathbb{E}[D - U] = \mathbb{E}[\mathbf{1}_{\{a=0\}}D] \geq 0. \quad (9)$$

The boundary term is retained; we do not set it to zero.

The cubic terms cancel in $D - U$. Using (5),

$$\begin{aligned} D - U &= \sum_{i,j>1} b_{ij}(u_i + v_j + 1) - \sum_{i,j>1} u_i v_j \\ &= \sum_{i>1} u_i(t - u_i) + \sum_{j>1} v_j(t - v_j) + (n - 1)t - T - T^2 \\ &= (2t - 1)T + (n - 1)t - T^2 - Q. \end{aligned}$$

Together with (9), this proves the lemma. $\square$

The argument is a finite weighted bijection. It requires neither an independence approximation nor a mixing-time bound.

3 From side sums to corner growth

Proof of Theorem 1. Cauchy–Schwarz and the two side-sum constraints give

$$Q \geq \frac{2T^2}{n - 1}.$$

Therefore Lemma 3 implies

$$\frac{n + 1}{n - 1} \mathbb{E}[T^2] - (2t - 1)\mathbb{E}T - (n - 1)t \leq 0. \quad (10)$$

Let $m = \mathbb{E}T \geq 0$. Since $\mathbb{E}[T^2] \geq m^2$, the same inequality holds with $\mathbb{E}[T^2]$ replaced by m^2 . For $t > 0$ the resulting quadratic has one negative root and the positive root $\beta_{n,t}$ in (1). Thus $m \leq \beta_{n,t}$. If $t = 0$, every side entry is zero and $\beta_{n,0} = 0$, so the conclusion also holds. Since $a = L - T$, this proves (1).

Taking $L = 3n$ and $t = n$ gives (2). For the simpler finite lower bound, substitute $T = 3n - a$ directly in Lemma 3:

$$0 \leq \mathbb{E} \left[-2n^2 - 4n + (4n + 1)a - a^2 - Q \right] \leq -2n^2 - 4n + (4n + 1)\mathbb{E}a.$$

This proves the lower bound in (3); its upper bound is the deterministic inequality $a \leq 3n$. In particular the lower bound exceeds $n/2$. Finally, $\beta_{n,n}/n \rightarrow 1 + \sqrt{2}$, proving the liminf assertion. $\square$

Corollary 4. *Suppose integer sequences $t_n, L_n \geq 0$ satisfy $t_n/n \rightarrow C > 0$ and $L_n/n \rightarrow BC$. For a uniform table on $\Omega_n(L_n, t_n)$,*

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}X_{11}}{n} \geq C \left(B - 1 - \sqrt{1 + 1/C} \right).$$

If $B > 1 + \sqrt{1 + 1/C}$, the corner expectation has order $\Theta(n)$.

Proof. Divide (1) by n and take limits of its explicit right-hand side. The upper bound $X_{11} \leq L_n = O(n)$ completes the order statement in the supercritical case. $\square$

The corollary is a supercritical lower bound. It does not establish bounded expectations below the threshold or equality in the displayed limiting lower bound.

4 Why geometric approximation fails

Proof of Theorem 2. Put $M = 3n$. Suppose Y has law G_q and $\delta = d_{\text{TV}}(\mathcal{L}(A), \mathcal{L}(Y)) \leq 1/4096$. For any function $0 \leq h \leq M$ the layer-cake formula gives

$$|\mathbb{E}h(A) - \mathbb{E}h(Y)| \leq M\delta.$$

With $h(x) = \min(x, M)$ and $A \leq M$, it follows that

$$\mathbb{E}A \leq \mathbb{E} \min(Y, M) + M\delta \leq \mathbb{E}Y + M\delta. \quad (11)$$

If $q = 0$, this contradicts $\mathbb{E}A > n/2$ immediately.

For $0 < q < 1$, the event (M, ∞) gives $q^{M+1} \leq \delta \leq 1/4096$. Since $e^s - 1 \geq s$ for $s > 0$,

$$\mathbb{E}Y = \frac{1}{e^{-\log q} - 1} \leq \frac{1}{-\log q} \leq \frac{M+1}{\log 4096} < \frac{M+1}{8}.$$

For the last inequality, strict Jensen for $1/x$ on $[1, 2]$ gives $\log 2 > 2/3$, so $\log 4096 = 12 \log 2 > 8$. Equation (11) now yields

$$\mathbb{E}A < \frac{3n+1}{8} + \frac{3n}{4096} < \frac{n}{2} \quad (n \geq 2).$$

The final inequality is equivalent to $509n > 512$. This contradicts (3) and proves (4). $\square$

5 Relation to existing regimes and verification

Dittmer–Lyu–Pak [3] obtain phase-transition and geometric-limit results for growing exceptional blocks. Their introduction and §9.2 explicitly distinguish the single special row and column, and their simulations suggest a normal law in the supercritical single-corner regime. The present proof establishes linear mean growth and excludes geometric approximation; it does not prove a normal limit.

Lyu–Mukherjee [4, Corollary 1.30] prove geometric convergence for fixed-base cloned margins. Their Theorem 1.27 and Remark 1.28 also provide entry-mixture bounds for counting measure beyond tame margins. The bound depends on the size of the observed block; for the singleton exceptional corner its displayed error does not vanish. Our margin sequence has only one exceptional row and column, so it is not cloning a fixed base table. There is no contradiction with those results.

The accompanying standard-library Python program enumerates 24,535 matrices from 205 margin families and checks 103,262 weighted switching edges. It verifies the pointwise drift, the boundary correction, the moment inequalities, and a separate column-by-column counting recursion. Normal and optimized execution give identical results. These are finite diagnostics; the proofs are the bijection and inequalities above, not the computation.

This manuscript is unrefereed and makes no absolute priority claim. OpenAI Codex assisted with derivation, literature review, exact computational checks and manuscript preparation. No independent referee review or proof-assistant verification is claimed.

6 Finite sanity checks

The boundary term in Lemma 3 can be strictly positive. For $n = 2$ and $0 \leq L \leq t$, every table has the form

$$\begin{pmatrix} a & L-a \\ L-a & t-L+a \end{pmatrix}, \quad a = 0, 1, \dots, L,$$

and the uniform corner law is uniform on this interval. At $a = 0$ the raw down-weight is $(t-L)(L+1)^2$, so

$$\mathbb{E}[\mathbf{1}_{\{a=0\}}D] = (t-L)(L+1).$$

For example, $L = 2, t = 4$ gives the nonzero value 6. Thus omitting the boundary correction would produce a false identity even in dimension two.

For the source margins $(3n, n, \dots, n)$, direct enumeration gives the following small-dimensional checks:

n	$ \Omega_n(3n, n) $	$\mathbb{E}X_{11}$
2	3	5
3	70	227/35
4	17531	136044/17531

These values check the finite inequalities, but are not evidence for a particular limiting constant beyond what the analytic proof establishes.

References

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