

Free Hyperplane Arrangements with Exponents 1, 2 and at Most One 3 Are Supersolvable: A Proof of a Conjecture of Tohăneanu

Alper Ferudun*

October 12, 2026

Abstract

Ş. O. Tohăneanu conjectured (Oberwolfach Report 5/2021, p. 284; Conjecture 4.1 of the preprint arXiv:1707.07091v3) that every central free hyperplane arrangement over a field of characteristic zero whose exponents are 1's, 2's and at most one 3 is supersolvable. We prove this conjecture in every rank. The preprint was withdrawn by its author on 16 January 2022; no result of it is used here, but the strategy of our proof is that of the preprint and of the Oberwolfach abstract. Two cases follow from a theorem of DiPasquale (Math. Ann. 385 (2023)) and are proved again: exponents 1 and 2 only, and one exponent 3 together with a codimension-two flat in four hyperplanes; rank three was also known. New, to our knowledge and relative to the literature we found, is the case without such a flat in rank at least 4, and therefore the conjecture in general. With two exponents 3 the statement is false: the non-Fano arrangement (exponents $(1, 3, 3)$) and nine hyperplanes of rank four (exponents $(1, 2, 3, 3)$) are free and not supersolvable. No computation enters the proof; the computations reported are tests. The book of Orlik and Terao was not accessible to us and was not read. No proof and no counterexample of the conjecture was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The conjecture

Let K be a field, V a K -vector space of dimension k , and $S = \text{Sym}(V^*) = K[x_1, \dots, x_k]$. An *arrangement* in V is a finite set $\mathcal{A}$ of hyperplanes of V , that is, of linear subspaces of codimension one; so all arrangements in this note are *central*. For $H \in \mathcal{A}$ we fix $\alpha_H \in V^*$ with $H = \ker \alpha_H$, and we put $n = |\mathcal{A}|$ and $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$. The *rank* $r(\mathcal{A})$ is the codimension of $\bigcap_{H \in \mathcal{A}} H$, and $\mathcal{A}$ is *essential* if $r(\mathcal{A}) = k$.

The *module of logarithmic derivations* of $\mathcal{A}$ is

$$D(\mathcal{A}) = \{\theta \in \text{Der}_K(S) : \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

A derivation $\theta = \sum_j P_j \partial_j$, where $\partial_j = \partial/\partial x_j$, is *homogeneous of degree d* if all P_j are homogeneous polynomials of degree d . Thus the *Euler derivation* $\theta_E = \sum_j x_j \partial_j$, which lies in $D(\mathcal{A})$, has degree 1. The arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S -module. Then $D(\mathcal{A})$ has a basis of k homogeneous derivations; their degrees form a multiset of k non-negative integers which does not depend on the basis, the *exponents* $\text{exp}(\mathcal{A})$. We write $(1, 2^{k-2}, 3)$ for the multiset consisting of one 1, $k-2$ entries 2 and one 3, and so on.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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The *intersection lattice* $L(\mathcal{A})$ is the set of all intersections of hyperplanes of $\mathcal{A}$, including V , ordered by reverse inclusion. An element $X \in L(\mathcal{A})$ is *modular* if $X + Y \in L(\mathcal{A})$ for all $Y \in L(\mathcal{A})$. An essential arrangement $\mathcal{A}$ of rank k is *supersolvable* if $L(\mathcal{A})$ contains a chain $V = X_0 < X_1 < \cdots < X_k = 0$ of modular elements.

These are the definitions of Tohăneanu’s Oberwolfach abstract [Toh21, p. 283] and of the preprint [Toh20, Section 1] on which it is based; the degree of a derivation is defined in the preprint. In both texts K has characteristic zero and the arrangements are central and essential. The abstract contains the following conjecture.

Conjecture (Tohăneanu [Toh21, p. 284]). Every central free hyperplane arrangement with “exponents 1’s, 2’s, and at most one 3” is supersolvable.

In the preprint it is Conjecture 4.1, stated for arrangements with $\exp(\mathcal{A}) = (1, 2^{k-2}, 3)$. The conjecture is record OWR-4990374-015 of the UnsolvedMath collection [ula26], where it is listed as open.

1.2 Results

Theorem 1.1 (the conjecture). *Let K be a field of characteristic zero, and let $\mathcal{A}$ be a central arrangement in K^k which is free and whose exponents are 0’s, 1’s, 2’s and at most one 3. Then $\mathcal{A}$ is supersolvable.*

Exponents 0 occur exactly when $\mathcal{A}$ is not essential, and supersolvability then refers to the essential arrangement $\mathcal{A}/C$ in V/C , where $C = \bigcap_{H \in \mathcal{A}} H$ (Sections 1.3 and 2.1).

To state the result on which the proof rests, let $L_2(\mathcal{A})$ be the set of all $Z \in L(\mathcal{A})$ of codimension two, and let m_Z be the number of hyperplanes of $\mathcal{A}$ containing Z . We call Z *multiple* if $m_Z \geq 3$, a *triple point* if $m_Z = 3$, and a *quadruple point* if $m_Z = 4$. An essential arrangement is *irreducible* if it is not a product of two arrangements in spaces of positive dimension. The *relation space* is $F(\mathcal{A}) = \{r \in K^{\mathcal{A}} : \sum_H r_H \alpha_H = 0\}$, and $\mathcal{A}$ is *formal* if $F(\mathcal{A})$ is spanned by its elements with at most three non-zero coordinates [BT94, Section 3]. Details are in Section 2.1.

Theorem 1.2 (core statement). *Let K be any field, and let $\mathcal{A}$ be an essential, irreducible, formal arrangement of rank $k \geq 2$ with $2k$ hyperplanes such that*

$$m_Z \leq 4 \text{ for all } Z \in L_2(\mathcal{A}) \quad \text{and} \quad \sum_{Z \in L_2(\mathcal{A})} \binom{m_Z - 1}{2} = k + 1.$$

Then $\mathcal{A}$ is supersolvable. More precisely, let $k \geq 3$, and for $k = 3$ suppose that $\mathcal{A}$ is not the arrangement of the six lines joining the pairs of four points of the projective plane, no three of them collinear. Then there is a triple point Z_0 , contained in the hyperplanes H_0, H_1, H_2 of $\mathcal{A}$, such that H_1 and H_2 contain no other multiple element of $L_2(\mathcal{A})$. For such Z_0 the subspace $X = \bigcap \{H \in \mathcal{A} : H \neq H_1, H_2\}$ is a line and a modular element of $L(\mathcal{A})$, we have $\mathcal{A}_X = \mathcal{A} \setminus \{H_1, H_2\}$, and the arrangement $\mathcal{A}_X/X$ in V/X satisfies the same hypotheses in rank $k - 1$.

In Theorem 1.2 freeness does not occur. An essential free arrangement with exponents $(1, 2^{k-2}, 3)$ satisfies its hypotheses (Proposition 3.2); the analogous statements for exponents $(1, 2^{k-1})$ are Theorem 5.2(a) and Corollary 5.3(a). Theorem 1.1 follows by a reduction to essential, irreducible arrangements (Section 5). The arrangement excluded for $k = 3$, which we call the *complete quadrilateral*, is supersolvable (Lemma 5.1).

1.3 Conventions that could change the answer

The degree of a derivation is the degree of its coefficients, so the Euler derivation has degree 1 [Toh20, Section 1]; with the other usual convention all exponents would be smaller by one. “At most one 3” refers to the multiset of exponents: $(1, 1, 2, 3)$ and $(1, 3)$ are admitted, $(1, 2, 3, 3)$ is not, and with two exponents 3 the statement is false (Section 7). An essential free arrangement has exactly one exponent 1 if it is irreducible, and at least two otherwise (Lemma 2.3); so the 1’s of the conjecture admit reducible arrangements, and Conjecture 4.1 of the preprint is the irreducible case. The sources assume that $\mathcal{A}$ is essential; Theorem 1.1 also covers arrangements which are not. It is claimed for characteristic zero, the setting of the conjecture; see Remark 6.5. Affine arrangements, multi-arrangements and other exponents are not part of the conjecture and are not treated.

1.4 What was known, what is not new, and what is new

The conjecture and the preprint. The conjecture is due to Ş. O. Tohăneanu. According to the abstract [Toh21, p. 284] it was verified in the preprint [Toh20] when the exponents are only 1’s and 2’s, for inductively free arrangements, and for free arrangements of rank 3, 4 or 5. The preprint was withdrawn by its author on 16 January 2022 (version 4); the withdrawal notice names Sections 4.2.1 and 4.2.2 and case (a) of the proof of Proposition 4.8 of version 3. We record these facts only. No result of the preprint is used in our proofs; in particular the case of exponents 1 and 2 is proved here.

The strategy of this note is that of the preprint and of the abstract, and we credit it to them: the formality of free arrangements as the tool; the modular coatom obtained from two hyperplanes which lie in a single dependent triple ([Toh20, Lemma 2.2], [Toh21, item iii, p. 284]); induction on the rank; the counts $n = 2k$ and $u + 3v = k + 1$ [Toh20, Lemma 4.2]; and the pigeonhole count for exponents $(1, 2^{k-1})$ [Toh20, Lemmas 3.1 and 3.2]. The bound $m_Z \leq \max_i d_i + 1$ is [Toh20, Lemma 2.1] and, according to [Toh21, p. 284], [Sch03, Lemma 5.2]; Lemma 2.5 generalizes it.

Not new, and proved again. For exponents $(1, 2^{k-1})$, and for exponents $(1, 2^{k-2}, 3)$ when there is a quadruple point ($v = 1$), supersolvability follows from a theorem of DiPasquale [DiP23, Proposition 6.1 with Corollary 4.10]; the derivation is in Remark 6.4. The combinatorial analogue of Theorem 5.2(a) and of the induction step is in [DGT14, Theorems 5.6 and 5.11]. Rank three is covered by the classification [Toh16, Theorem 2.4], as noted in [Toh20], and the question whether free arrangements with exponents 1 and 2 are supersolvable is posed in the introduction of [Toh16]. For arrangements whose underlying matroid is regular, freeness is equivalent to supersolvability by a recent preprint of Liang and Wang [LW26], of which we read the abstract and the introduction. This does not give the conjecture, because the matroid of a free arrangement with these exponents is not known to be regular in advance; when there is no quadruple point it turns out to be graphic (Remark 6.1), and with a quadruple point it is not regular. Further attributions, to [Yuz93, BT94, MMR24, BEZ90], are given after Lemmas 4.1 to 4.3. The numbers of [DiP23, DGT14, Toh16, MMR24] are those of their arXiv versions.

New, to our knowledge and relative to the literature we found, is the case without a quadruple point ($v = 0$) in rank $k \geq 4$, and therefore the conjecture in general; the other cases are proved again. The new case is Lemma 4.3(ii): there are $k + 1$ triple points, their $k + 1$ relations lie in the relation space, which has dimension k , and the support of a linear dependency among them leads to a triple point two of whose three hyperplanes contain no other multiple flat. The same count of dimensions gives $v \leq 1$, and the induction runs inside the classes $\mathcal{C}_k$ and $\mathcal{C}'_k$ of Definition 3.1, so that one proof covers all cases. We claim no priority; see “Scope and priority” at the end of Section 9.

2 Preliminaries

2.1 Notation

Throughout, K is a field, $V = K^k$, and $\mathcal{A}$ is an arrangement in V with n hyperplanes and forms α_H as in Section 1.1.

Ranks and flats. For $\mathcal{B} \subseteq \mathcal{A}$ let $r(\mathcal{B}) = \dim \text{span}\{\alpha_H : H \in \mathcal{B}\} = \text{codim} \bigcap_{H \in \mathcal{B}} H$. For $X \in L(\mathcal{A})$ let $r(X) = \text{codim } X$ and $\mathcal{A}_X = \{H \in \mathcal{A} : X \subseteq H\}$; then $X = \bigcap_{H \in \mathcal{A}_X} H$ and $L(\mathcal{A}_X) = \{Y \in L(\mathcal{A}) : Y \supseteq X\}$. For two distinct hyperplanes H, H' of $\mathcal{A}$ there is exactly one $Z \in L_2(\mathcal{A})$ with $H, H' \in \mathcal{A}_Z$, namely $Z = H \cap H'$. Hence

$$|\mathcal{A}_Z \cap \mathcal{A}_{Z'}| \leq 1 \text{ for } Z \neq Z' \text{ in } L_2(\mathcal{A}), \quad \sum_{Z \in L_2(\mathcal{A})} \binom{m_Z}{2} = \binom{n}{2}. \quad (1)$$

The *degree* $\deg(H)$ of $H \in \mathcal{A}$ is the number of multiple $Z \in L_2(\mathcal{A})$ with $H \in \mathcal{A}_Z$. The *hypergraph* of $\mathcal{A}$ has the vertex set $\mathcal{A}$ and the hyperedges $\mathcal{A}_Z$, Z multiple. It is *connected* if $\mathcal{A}$ is not the disjoint union of two non-empty subsets such that every hyperedge lies in one of them.

Products and irreducibility. For arrangements $\mathcal{A}_1$ in V_1 and $\mathcal{A}_2$ in V_2 , the product $\mathcal{A}_1 \times \mathcal{A}_2$ is the arrangement $\{H \oplus V_2 : H \in \mathcal{A}_1\} \cup \{V_1 \oplus H : H \in \mathcal{A}_2\}$ in $V_1 \oplus V_2$. Let $\mathcal{A}$ be essential. Then $\mathcal{A}$ is *reducible* if $\mathcal{A} = \mathcal{A}' \sqcup \mathcal{A}''$ with non-empty $\mathcal{A}', \mathcal{A}''$ and $r(\mathcal{A}') + r(\mathcal{A}'') = k$, and *irreducible* otherwise. In the reducible case $V^* = U' \oplus U''$, where U' and U'' are spanned by the forms of $\mathcal{A}'$ and of $\mathcal{A}''$; with V' and V'' the annihilators of U'' and U' we get $V = V' \oplus V''$ and $\mathcal{A} = \mathcal{A}'|_{V'} \times \mathcal{A}''|_{V''}$, a product of essential arrangements, where $\mathcal{A}'|_{V'} = \{H \cap V' : H \in \mathcal{A}'\}$. By induction, every essential arrangement is a product of irreducible essential arrangements of positive rank.

Arrangements which are not essential. Let $C = \bigcap_{H \in \mathcal{A}} H$. Then $\mathcal{A}/C = \{H/C : H \in \mathcal{A}\}$ is an essential arrangement in V/C , and $Y \mapsto Y/C$ is a bijection from $L(\mathcal{A})$ to $L(\mathcal{A}/C)$ which is compatible with sums of subspaces. We call $\mathcal{A}$ *supersolvable* if $\mathcal{A}/C$ is supersolvable; equivalently, if $L(\mathcal{A})$ contains a chain $V = X_0 < \dots < X_r = C$, $r = r(\mathcal{A})$, of modular elements.

The relation space. We have $\dim F(\mathcal{A}) = n - r(\mathcal{A})$. For $Z \in L_2(\mathcal{A})$ let $F_Z = F(\mathcal{A}) \cap K^{\mathcal{A}_Z}$ be the space of relations supported on $\mathcal{A}_Z$; then $\dim F_Z = m_Z - 2$. A relation with exactly three non-zero coordinates involves three forms, any two of which are independent; they span a plane, so the relation lies in one F_Z . No non-zero relation has fewer non-zero coordinates, and each F_Z is spanned by relations among three forms. Hence

$$\mathcal{A} \text{ is formal} \iff F(\mathcal{A}) = \sum_{Z \in L_2(\mathcal{A})} F_Z$$

[BT94, Remark 3.5]. If Z is a triple point, then $F_Z = K\rho_Z$, and all three coordinates of ρ_Z on $\mathcal{A}_Z$ are non-zero. Formality is not determined by the lattice $L(\mathcal{A})$ in general: [Yuz93, Example 2.2] gives two arrangements of rank three with isomorphic intersection lattices, one formal and the other not (a footnote to the introduction of [Yuz93] gives the priority to an example of Ziegler). This is why we do not call Theorem 1.2 a combinatorial statement.

2.2 Cited theorems

(T1) *Terao's factorization theorem* [Ter81], in the form of [PT20, Theorem 3.9]: if $\mathcal{A}$ is free with exponents $d_1, \dots, d_k$, then $\chi(\mathcal{A}, t) = \prod_i (t - d_i)$, where $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}$ and μ is the Möbius function of $L(\mathcal{A})$ ($\mu(V) = 1$ and $\mu(X) = -\sum_{Y \supseteq X} \mu(Y)$). We use only the coefficient of t^{k-2} . Since $\mu(H) = -1$ for $H \in \mathcal{A}$, we have $\mu(Z) = m_Z - 1$ for $Z \in L_2(\mathcal{A})$, and therefore

$$\sum_{Z \in L_2(\mathcal{A})} (m_Z - 1) = \sum_{i < j} d_i d_j. \quad (2)$$

We did not read [Ter81]. The theorem is cited as [OT92, Theorem 4.137] in [BT94, p. 63] and as [OT92, Theorem 4.61] in [Toh21, Toh20]. We apply it only in characteristic zero.

(T2) *Saito's criterion* [PT20, Theorem 3.7], where it is stated for an arbitrary field; cited as [OT92, Theorem 4.19] in [Toh20]. Homogeneous derivations $\theta_1, \dots, \theta_k \in D(\mathcal{A})$ form a basis of $D(\mathcal{A})$ if and only if $\det(\theta_i(x_j)) = cQ(\mathcal{A})$ with $c \in K \setminus \{0\}$. In particular the exponents of a free arrangement sum to n .

Theorem 2.1 (Yuzvinsky [Yuz93, Corollary 2.5]). *Let $\mathcal{A}$ be an essential arrangement over any field. If $\mathcal{A}$ is free, then $\mathcal{A}$ is formal.*

In [Yuz93] the field is arbitrary, the rank is assumed to be at least three (in rank at most two every arrangement is formal), the proof uses sheaf cohomology on the intersection lattice, and formal arrangements are defined, following Falk and Randell, as the 3-generated ones [Yuz93, Definition 2.1]: the dependencies of length at most three among the forms impose the same linear conditions as all dependencies. This is the definition of Section 2.1 in dual form. Appendix A contains a complete proof of Theorem 2.1 which follows Brandt and Terao [BT94, Proposition 4.13 and Theorem 4.15]; [DiP23, Corollary 4.10] is a further published proof.

2.3 Four elementary lemmas

Lemma 2.2 (rank two). *Let $Z \in L_2(\mathcal{A})$. Then $\mathcal{A}_Z$ is free with exponents $(0^{k-2}, 1, m_Z - 1)$.*

Proof. Choose coordinates such that $Z = \{x_1 = x_2 = 0\}$ and x_1 is the form of one hyperplane of $\mathcal{A}_Z$; then all forms of $\mathcal{A}_Z$ lie in $\text{span}(x_1, x_2)$. Put $Q = Q(\mathcal{A}_Z)$, $\theta_1 = x_1\partial_1 + x_2\partial_2$, $\theta_2 = (Q/x_1)\partial_2$ and $\theta_j = \partial_j$ for $j \geq 3$. We have $\theta_1(\alpha) = \alpha$ for $\alpha \in \text{span}(x_1, x_2)$; $\theta_2(x_1) = 0$; for another form $\alpha = ax_1 + bx_2$ of $\mathcal{A}_Z$, $\theta_2(\alpha) = bQ/x_1$ is divisible by α ; and $\theta_j(\alpha) = 0$ for $j \geq 3$. The matrix $(\theta_i(x_j))$ is triangular with the diagonal $x_1, Q/x_1, 1, \dots, 1$, so its determinant is Q . By (T2) the θ_i form a basis of $D(\mathcal{A}_Z)$, and their degrees are $1, m_Z - 1, 0, \dots, 0$. $\square$

Lemma 2.3 (exponents 0 and 1). *Let $\mathcal{A}$ be essential and free. Then no exponent is 0, and the number of exponents equal to 1 is 1 if $\mathcal{A}$ is irreducible and at least 2 if $\mathcal{A}$ is reducible.*

Proof. The number of exponents 0 is $\dim_K D(\mathcal{A})_0$. A derivation of degree 0 is $\partial_v : \alpha \mapsto \alpha(v)$ for some $v \in V$, and $\partial_v(\alpha_H) \in \alpha_H S$ means $\alpha_H(v) = 0$. So $D(\mathcal{A})_0 \cong \bigcap_H H = 0$. Hence all exponents are at least 1, and the number of exponents 1 is $\dim_K D(\mathcal{A})_1$. A derivation θ of degree 1 is determined by the endomorphism $N = \theta|_{V^*}$ of V^* , and $\theta \in D(\mathcal{A})$ if and only if every α_H is an eigenvector of N . Suppose this holds. The α_H span V^* , so V^* is the direct sum of the eigenspaces E_c of N ; each α_H lies in one of them, and each E_c is spanned by the α_H it contains. Thus $\mathcal{A} = \bigsqcup_c \{H : \alpha_H \in E_c\}$ with ranks adding up to k . If $\mathcal{A}$ is irreducible, only one eigenvalue occurs, N is a scalar and θ is a multiple of θ_E ; so $\dim D(\mathcal{A})_1 = 1$. If $\mathcal{A} = \mathcal{A}' \sqcup \mathcal{A}''$ is reducible, with $V^* = U' \oplus U''$ as in Section 2.1, the endomorphism which is the identity on U' and zero on U'' gives an element of $D(\mathcal{A})_1$ which is not a multiple of θ_E . $\square$

Lemma 2.4 (products). *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. Then $\mathcal{A}$ is free if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are free, and then $\exp(\mathcal{A})$ is the union of the multisets $\exp(\mathcal{A}_1)$ and $\exp(\mathcal{A}_2)$. The empty arrangement in K^c is free with exponents 0^c .*

This is [OT92, Proposition 4.28], as cited in [HR15, Proposition 2.4], where $K = \mathbb{C}$; for a product with an empty arrangement see also [Yuz93, (1.1)]. The following proof works over every field.

Proof. Let $S_i = \text{Sym}(V_i^*)$, so that $S = S_1 \otimes_K S_2$ and $\text{Der}_K(S) = (S \otimes V_1) \oplus (S \otimes V_2)$. Write $\theta = \theta' + \theta''$ accordingly. For $H \in \mathcal{A}_1$ the form α_H lies in V_1^* , so $\theta(\alpha_H) = \theta'(\alpha_H)$; similarly for $\mathcal{A}_2$. Hence $D(\mathcal{A}) = D' \oplus D''$, where D' consists of the $\theta' \in S \otimes V_1$ with $\theta'(\alpha_H) \in \alpha_H S$ for all $H \in \mathcal{A}_1$. Expanding the coefficients of θ' in the monomials y^μ of S_2 , $\theta' = \sum_\mu \theta'_\mu y^\mu$ with $\theta'_\mu \in S_1 \otimes V_1$, we get $\theta'(\alpha_H) = \sum_\mu \theta'_\mu(\alpha_H) y^\mu$, which lies in $\alpha_H S = \bigoplus_\mu \alpha_H S_1 y^\mu$ if and only if all θ'_μ lie in $D(\mathcal{A}_1)$. So $D' = D(\mathcal{A}_1) \otimes_K S_2$, likewise $D'' = S_1 \otimes_K D(\mathcal{A}_2)$, and $D(\mathcal{A}) \cong (D(\mathcal{A}_1) \otimes_K S_2) \oplus (S_1 \otimes_K D(\mathcal{A}_2))$ as graded modules. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are free, homogeneous bases of $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$ together give a homogeneous basis of $D(\mathcal{A})$. Conversely, if $D(\mathcal{A})$ is free, its graded direct summand $D(\mathcal{A}_1) \otimes_K S_2$ is a finitely generated graded projective S -module, hence free; modulo the ideal $\mathfrak{n}$ of S generated by V_2^* , a homogeneous basis of it gives a basis of $(D(\mathcal{A}_1) \otimes_K S_2)/\mathfrak{n}(D(\mathcal{A}_1) \otimes_K S_2) = D(\mathcal{A}_1)$ over $S/\mathfrak{n} = S_1$, with the same degrees. The last statement is clear. $\square$

Lemma 2.5 (sub-arrangements). *Let $\mathcal{B} \subseteq \mathcal{A}$ be arrangements in K^k , both free, with exponents $e_1 \leq \dots \leq e_k$ and $d_1 \leq \dots \leq d_k$. Then $e_i \leq d_i$ for all i .*

Proof. We have $D(\mathcal{A}) \subseteq D(\mathcal{B})$. Let $\theta_1, \dots, \theta_k$ and $\varphi_1, \dots, \varphi_k$ be homogeneous bases of $D(\mathcal{A})$ and $D(\mathcal{B})$ with degrees d_i and e_i , and write $\theta_j = \sum_l f_{jl} \varphi_l$; the f_{jl} are unique, hence homogeneous of degree $d_j - e_l$. Suppose $e_i > d_i$. For $j \leq i \leq l$ we have $d_j \leq d_i < e_i \leq e_l$, so $f_{jl} = 0$. Thus $\theta_1, \dots, \theta_i$ lie in the free module $S\varphi_1 \oplus \dots \oplus S\varphi_{i-1}$ of rank $i-1$, and are linearly dependent over S , a contradiction. $\square$

3 The numerical conditions

Definition 3.1. Let $\mathcal{A}$ be an essential arrangement of rank k .

- $\mathcal{A} \in \mathcal{C}'_k$ ($k \geq 1$) if: (C1) $\mathcal{A}$ is irreducible; (C2) $\mathcal{A}$ is formal; (C3') $n = 2k - 1$; (C4') $m_Z \leq 3$ for all $Z \in L_2(\mathcal{A})$; (C5') the number of triple points is $k - 1$.
- $\mathcal{A} \in \mathcal{C}_k$ ($k \geq 2$) if: (C1); (C2); (C3) $n = 2k$; (C4) $m_Z \leq 4$ for all $Z \in L_2(\mathcal{A})$; (C5) $\sum_{Z \in L_2(\mathcal{A})} \binom{m_Z - 1}{2} = k + 1$.

For $\mathcal{A} \in \mathcal{C}_k$ we write u for the number of triple points and v for the number of quadruple points. By (C4) and (C5),

$$u + 3v = k + 1. \quad (3)$$

The arrangements of Theorem 1.2 are the members of $\mathcal{C}_k$.

Proposition 3.2. *Let K have characteristic zero, and let $\mathcal{A}$ be essential and free. If $\exp(\mathcal{A}) = (1, 2^{k-1})$ with $k \geq 1$, then $\mathcal{A} \in \mathcal{C}'_k$. If $\exp(\mathcal{A}) = (1, 2^{k-2}, 3)$ with $k \geq 2$, then $\mathcal{A} \in \mathcal{C}_k$.*

Proof. (C1): exactly one exponent is 1, so $\mathcal{A}$ is irreducible by Lemma 2.3. (C2) is Theorem 2.1. (C3), (C3'): the exponents sum to n by (T2). (C4), (C4'): for $Z \in L_2(\mathcal{A})$ the largest exponent of $\mathcal{A}_Z$ is $m_Z - 1$ (Lemma 2.2), and Lemma 2.5 for $\mathcal{A}_Z \subseteq \mathcal{A}$ gives $m_Z - 1 \leq d_k$, that is $m_Z \leq 4$, respectively $m_Z \leq 3$. (C5), (C5'): subtracting (2) from the second part of (1), and using $\binom{m}{2} - (m-1) = \binom{m-1}{2}$ and $n = \sum_i d_i$, we get

$$\sum_{Z \in L_2(\mathcal{A})} \binom{m_Z - 1}{2} = \binom{n}{2} - \sum_{i < j} d_i d_j = \frac{1}{2} \left(\sum_i d_i^2 - \sum_i d_i \right) = \sum_i \binom{d_i}{2}. \quad (4)$$

The right-hand side is $(k-2) + 3 = k+1$ for $(1, 2^{k-2}, 3)$ and $k-1$ for $(1, 2^{k-1})$; in the second case every multiple Z is a triple point and contributes 1. $\square$

The characteristic enters only through (T1), that is, through (4); see Remark 6.5. The identity (3) for free arrangements is [Toh20, Lemma 4.2].

4 Three lemmas on formal arrangements

Lemma 4.1. *Let $\mathcal{A}$ be essential, irreducible and formal, of rank $k \geq 2$. Then*

- (a) $\deg(H) \geq 1$ for every $H \in \mathcal{A}$;
- (b) *the hypergraph of $\mathcal{A}$ is connected.*

Proof. (a) Suppose that $H \in \mathcal{A}_Z$ for no multiple Z . Then every F_Z lies in the coordinate hyperplane $\{r_H = 0\}$ of $K^{\mathcal{A}}$: this is clear if $H \notin \mathcal{A}_Z$, and if $H \in \mathcal{A}_Z$, then $m_Z = 2$ and $F_Z = 0$. Hence so does $F(\mathcal{A}) = \sum_Z F_Z$, which means that α_H is not in the span of the other forms. So $r(\mathcal{A} \setminus \{H\}) = k - 1$, and $\mathcal{A} = \{H\} \sqcup (\mathcal{A} \setminus \{H\})$ shows that $\mathcal{A}$ is reducible; note that $\mathcal{A} \setminus \{H\} \neq \emptyset$ because $n \geq k \geq 2$.

(b) Let $\mathcal{A} = \mathcal{A}' \sqcup \mathcal{A}''$ such that every hyperedge lies in $\mathcal{A}'$ or in $\mathcal{A}''$. Let F' be the sum of the F_Z with $\mathcal{A}_Z \subseteq \mathcal{A}'$, and F'' the sum of those with $\mathcal{A}_Z \subseteq \mathcal{A}''$; all other F_Z are zero. Then $F' \subseteq K^{\mathcal{A}'}$, $F'' \subseteq K^{\mathcal{A}''}$ and $F(\mathcal{A}) = F' \oplus F''$, so that $F(\mathcal{A}) \cap K^{\mathcal{A}'} = F'$. As $F(\mathcal{A}) \cap K^{\mathcal{A}'}$ is the relation space of $\mathcal{A}'$, we get $r(\mathcal{A}') = |\mathcal{A}'| - \dim F'$, and in the same way $r(\mathcal{A}'') = |\mathcal{A}''| - \dim F''$. Adding, $r(\mathcal{A}') + r(\mathcal{A}'') = n - \dim F(\mathcal{A}) = k$. As $\mathcal{A}$ is irreducible, $\mathcal{A}'$ or $\mathcal{A}''$ is empty. $\square$

Part (a) is stated in [Yuz93, p. 239, (4.3)'] and is [BT94, Proposition 3.9(ii)]; it is the first step of the proof of [Toh20, Lemma 3.2]; see also [DiP23, Corollary 4.13].

Lemma 4.2. *Let $\mathcal{A}$ be essential and formal of rank $k \geq 2$, and let Z_0 be a triple point, $\mathcal{A}_{Z_0} = \{H_0, H_1, H_2\}$, such that $\deg(H_1) = \deg(H_2) = 1$. Put $X = \bigcap \{H \in \mathcal{A} : H \neq H_1, H_2\}$. Then*

- (a) X is a line, $X \in L(\mathcal{A})$, and $\mathcal{A}_X = \mathcal{A} \setminus \{H_1, H_2\}$;
- (b) X is modular in $L(\mathcal{A})$;
- (c) every $Y \in L(\mathcal{A}_X)$ which is modular in $L(\mathcal{A}_X)$ is modular in $L(\mathcal{A})$; hence, if $\mathcal{A}_X$ is supersolvable, then so is $\mathcal{A}$;
- (d) the multiple elements of $L_2(\mathcal{A}_X)$ are the multiple elements of $L_2(\mathcal{A})$ other than Z_0 , with the same multiplicities; the arrangement $\mathcal{A}_X$ is formal; and if $\mathcal{A}$ is irreducible, then so is $\mathcal{A}_X/X$.

Proof. Write α_i for the form of H_i ($i = 0, 1, 2$) and r_i for the coordinate r_{H_i} of $r \in K^{\mathcal{A}}$.

(a) Let ρ be a generator of F_{Z_0} , with coordinates $a_0, a_1, a_2 \neq 0$ on H_0, H_1, H_2 . By hypothesis Z_0 is the only multiple element of $L_2(\mathcal{A})$ which lies in H_1 or in H_2 . So for every $Z \neq Z_0$ all elements of F_Z have $r_1 = r_2 = 0$. By formality every $r \in F(\mathcal{A})$ has the form $r = t\rho + f$ with $t \in K$ and $f \in \sum_{Z \neq Z_0} F_Z$; hence $(r_1, r_2) = t(a_1, a_2)$, and the linear form $\lambda(r) = a_2 r_1 - a_1 r_2$ vanishes on $F(\mathcal{A})$. The map $\iota : V \rightarrow (K^{\mathcal{A}})^*$ which sends v to the linear form $r \mapsto \sum_H r_H \alpha_H(v)$ is injective, because $\mathcal{A}$ is essential; its image lies in the annihilator of $F(\mathcal{A})$; and both have dimension k . So $\lambda = \iota(v_0)$ for some $v_0 \in V$: we have $\alpha_1(v_0) = a_2$, $\alpha_2(v_0) = -a_1$, and $\alpha_H(v_0) = 0$ for all $H \neq H_1, H_2$. Thus $0 \neq v_0 \in X$, and $v_0 \notin H_1 \cup H_2$. Since $\alpha_2 \in \text{span}(\alpha_0, \alpha_1)$, the forms α_H with $H \neq H_1, H_2$ span V^* together with α_1 ; so $r(\mathcal{A} \setminus \{H_1, H_2\}) \geq k - 1$ and $\dim X \leq 1$. Hence $X = Kv_0$, and $\mathcal{A}_X = \mathcal{A} \setminus \{H_1, H_2\}$.

(b) Let $Y \in L(\mathcal{A})$. If $\mathcal{A}_Y \subseteq \mathcal{A}_X$, then $Y \supseteq X$ and $X + Y = Y$. Otherwise Y lies in H_1 or in H_2 , so $X \not\subseteq Y$ and $\dim(X + Y) = \dim Y + 1$. Put $W = \bigcap (\mathcal{A}_Y \cap \mathcal{A}_X) \in L(\mathcal{A})$, with $W = V$ if $\mathcal{A}_Y \cap \mathcal{A}_X = \emptyset$; then $W \supseteq X + Y$. If $\mathcal{A}_Y \setminus \mathcal{A}_X$ is a single hyperplane H , then $Y = W \cap H$ and $\dim W \leq \dim Y + 1$. If $\mathcal{A}_Y \supseteq \{H_1, H_2\}$, then $Y \subseteq H_1 \cap H_2 = Z_0 \subseteq H_0$, so $H_0 \in \mathcal{A}_Y \cap \mathcal{A}_X$ and $W \subseteq H_0$; on $W \cap H_1$ the forms α_0 and α_1 vanish, hence so does α_2 ; therefore $Y = W \cap H_1$, and again $\dim W \leq \dim Y + 1$. In both cases $W = X + Y$.

(c) Let $Y \in L(\mathcal{A}_X)$ be modular in $L(\mathcal{A}_X)$, and let $Y' \in L(\mathcal{A})$. By (b), $X + Y' \in L(\mathcal{A})$; it contains X , so it lies in $L(\mathcal{A}_X)$. As $X \subseteq Y$, we get $Y + Y' = Y + (X + Y') \in L(\mathcal{A}_X) \subseteq L(\mathcal{A})$. If

$V = X_0 < \cdots < X_{k-1} = X$ is a chain of modular elements of $L(\mathcal{A}_X)$, then together with $X_k = 0$ it is a chain of modular elements of $L(\mathcal{A})$.

(d) The elements of $L_2(\mathcal{A}_X)$ are the $Z \in L_2(\mathcal{A})$ with $Z \supseteq X$, that is, with $\mathcal{A}_Z \subseteq \mathcal{A}_X$, and for them $(\mathcal{A}_X)_Z = \mathcal{A}_Z$. A multiple $Z \in L_2(\mathcal{A})$ with $\mathcal{A}_Z \not\subseteq \mathcal{A}_X$ lies in H_1 or in H_2 , so it is Z_0 . This proves the first statement. Formality: $F(\mathcal{A}_X) = F(\mathcal{A}) \cap K^{\mathcal{A}_X}$, and an element $r = t\rho + f$ of $F(\mathcal{A})$ as in (a) has $r_1 = ta_1$, so it lies in $K^{\mathcal{A}_X}$ if and only if $t = 0$. Hence $F(\mathcal{A}_X) = \sum_{Z \neq Z_0} F_Z$, and for multiple $Z \neq Z_0$ the space F_Z is the space of relations of $\mathcal{A}_X$ supported on $(\mathcal{A}_X)_Z$. Irreducibility: the forms of $\mathcal{A}_X/X$ are those of $\mathcal{A}_X$. Let $\mathcal{A}_X = \mathcal{B}' \sqcup \mathcal{B}''$ with non-empty $\mathcal{B}', \mathcal{B}''$ and $r(\mathcal{B}') + r(\mathcal{B}'') = k-1$, and let $H_0 \in \mathcal{B}'$. Then $r(\mathcal{B}' \cup \{H_1, H_2\}) \leq r(\mathcal{B}') + 1$, because $\alpha_2 \in \text{span}(\alpha_0, \alpha_1)$. So $k = r(\mathcal{A}) \leq r(\mathcal{B}' \cup \{H_1, H_2\}) + r(\mathcal{B}'') \leq k$, and $\mathcal{A} = (\mathcal{B}' \cup \{H_1, H_2\}) \sqcup \mathcal{B}''$ is reducible. $\square$

Parts (a) and (b) are the mechanism of [Toh20, Lemma 2.2], stated there for free arrangements and proved in coordinates. Part (b) also follows from the criterion of [BEZ90] quoted in [MMR24, Lemma 2.1], and the formality of $\mathcal{A}_X$ from [MMR24, Corollary 5.2] (if X is modular and $\mathcal{A}$ is formal, then $\mathcal{A}_X$ is formal); the proofs above are direct. In general a localization of a formal arrangement need not be formal (the example after Definition 2.3 in [Yuz93]), so the statement on formality in (d) is needed for the induction.

Lemma 4.3. *Let $k \geq 3$.*

- (i) *If $\mathcal{A} \in \mathcal{C}'_k$, then some triple point lies in two hyperplanes of degree 1.*
- (ii) *If $\mathcal{A} \in \mathcal{C}_k$, then $v \leq 1$, and unless $(k, v) = (3, 0)$, some triple point lies in two hyperplanes of degree 1.*

Proof. In both cases $\deg(H) \geq 1$ for all H and the hypergraph is connected (Lemma 4.1), and two hyperedges share at most one vertex (1). Let s be the number of hyperplanes of degree 1.

(i) There are $k-1$ hyperedges, all of size three, so $\sum_H \deg(H) = 3(k-1)$. As the $2k-1-s$ hyperplanes of degree at least 2 contribute at least 2 each, $3(k-1) \geq s + 2(2k-1-s)$, that is, $s \geq k+1 > k-1$. Each hyperplane of degree 1 contains exactly one triple point, and there are $k-1$ triple points; so two of them contain the same one.

(ii) By (C2) and (C3),

$$k = n - k = \dim F(\mathcal{A}) = \dim \sum_Z F_Z \leq \sum_Z (m_Z - 2) = u + 2v. \quad (5)$$

With (3) this gives $v = (u + 3v) - (u + 2v) \leq 1$.

Case $v = 1$. Then $u = k-2$ and $\sum_H \deg(H) = 3(k-2) + 4 = 3k-2 \geq s + 2(2k-s)$, so $s \geq k+2$. Suppose that no triple point lies in two hyperplanes of degree 1. The only multiple flat in a hyperplane of degree 1 is a triple point or the quadruple point; so $s \leq (k-2) + 4$. Hence equality holds, and all four hyperplanes through the quadruple point have degree 1. Then no other hyperedge meets the hyperedge of the quadruple point, and as $n = 2k > 4$, this contradicts the connectedness.

Case $v = 0$, $k \geq 4$. There are $u = k+1$ triple points Z , with generators ρ_Z of F_Z , and these $k+1$ vectors lie in $F(\mathcal{A})$, which has dimension k . Choose a non-trivial linear dependency $\sum_Z c_Z \rho_Z = 0$, and put

$$T = \{Z : c_Z \neq 0\}, \quad t = |T|, \quad W = \bigcup_{Z \in T} \mathcal{A}_Z, \quad w = |W|.$$

Every $H \in W$ contains at least two members of T : if it contained exactly one, Z , the H -coordinate of the dependency would be $c_Z(\rho_Z)_H \neq 0$. Counting the pairs (H, Z) with $Z \in T$ and $H \in \mathcal{A}_Z$ gives

$$3t \geq 2w. \quad (6)$$

Moreover $t \geq 4$: let $Z_1 \in T$ and $\mathcal{A}_{Z_1} = \{G_1, G_2, G_3\}$; each G_a contains a second member $Z'_a \neq Z_1$ of T , and $\mathcal{A}_{Z'_a} \cap \mathcal{A}_{Z_1} = \{G_a\}$ by (1), so Z_1, Z'_1, Z_2, Z'_2 are four different members of T .

Let c be the number of pairs (H, Z) with $H \in W$, $Z \notin T$ a triple point, and $H \in \mathcal{A}_Z$. The triple points in the $2k-w$ hyperplanes outside W are not in T . There are $k+1-t$ triple points outside T , with $3(k+1-t)$ incidences in total, of which c are with hyperplanes in W . So $\sum_{H \notin W} \deg(H) = 3(k+1-t) - c$. Let s' be the number of hyperplanes outside W of degree 1. Then $3(k+1-t) - c \geq s' + 2(2k-w-s')$, that is, $s' \geq k-3+3t-2w+c$, and with (6)

$$s' - (k+1-t) \geq (4t-2w) + c - 4 \geq t + c - 4. \quad (7)$$

If $t \geq 5$ or $c \geq 1$, the right-hand side is at least 1. In the remaining case $t = 4$, $c = 0$, no triple point outside T lies in a hyperplane of W ; so every hyperedge lies in W or in $\mathcal{A} \setminus W$, and $W = \mathcal{A}$ by connectedness; then there is no triple point outside T at all, and $k+1 = t = 4$, contradicting $k \geq 4$. Hence

$$s' \geq (k+1-t) + 1.$$

In particular $s' \geq 1$, so $W \neq \mathcal{A}$ and T is not the set of all triple points (if it were, we would have $W = \mathcal{A}$ by Lemma 4.1(a), hence $s' = 0$); thus $k+1-t \geq 1$. Each of the s' hyperplanes outside W of degree 1 contains exactly one triple point, which is not in T . As there are more such hyperplanes than triple points outside T , some triple point lies in two hyperplanes of degree 1. $\square$

Part (i) is [Toh20, Lemmas 3.1 and 3.2]. The inequality between the second and the fifth term of (5) is [BT94, Remark 3.7]; in the equality case, which is (i) and the case $v = 1$, it is the first assertion of [DiP23, Proposition 6.1]. For $k = 3$ and $v = 0$ the conclusion of (ii) fails: the four triple points of the complete quadrilateral form a dependency with $t = 4$, $w = 6$, $c = 0$, and all six lines have degree 2.

5 Proofs of the theorems

In an essential arrangement the elements V and 0 of $L(\mathcal{A})$ and all hyperplanes of $\mathcal{A}$ are modular, since $H + Y$ is H or V . So essential arrangements of rank at most 2 are supersolvable.

Lemma 5.1 (the complete quadrilateral). *Let $\mathcal{A} \in \mathcal{C}_3$ with $v = 0$. Then $\mathcal{A}$ consists of the six lines joining the pairs of four points of the projective plane $\mathbb{P}(V)$, no three of which are collinear; in suitable coordinates $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z)$. Every triple point of $\mathcal{A}$ is modular, and $\mathcal{A}$ is supersolvable.*

Proof. We regard the hyperplanes of $\mathcal{A}$ as lines and the elements of $L_2(\mathcal{A})$ as points of $\mathbb{P}(V)$. By (C3)–(C5), $\mathcal{A}$ has six lines, exactly four triple points and no other multiple point. A line contains at most two triple points, since each of them needs two of the five other lines, and these pairs are disjoint by (1). There are $4 \cdot 3 = 12 = 6 \cdot 2$ incidences between triple points and lines, so every line contains exactly two triple points. Two triple points lie on at most one common line, and there are six lines and $\binom{4}{2} = 6$ pairs of triple points. So every pair of triple points is joined by exactly one line of $\mathcal{A}$, these are all lines of $\mathcal{A}$, and no three of the four points are collinear. Four such points are the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, $(1:1:1)$ in suitable coordinates, which gives the polynomial.

Let P be a triple point, and let $R \neq P$ be another point of $L_2(\mathcal{A})$. If R is a triple point, it is joined to P by a line of $\mathcal{A}$. If R is a double point, it is the intersection of two lines of $\mathcal{A}$ which have no triple point in common; so they contain all four triple points, and one of them passes through P . In both cases the subspace $P + R$ of V is a hyperplane of $\mathcal{A}$. Further $P + H \in \{H, V\}$ for $H \in \mathcal{A}$. So P is modular, and $V < H < P < 0$, for any $H \in \mathcal{A}$ through P , is a chain of modular elements. $\square$

Theorem 5.2. *The arrangements of Definition 3.1 are supersolvable. More precisely:*

- (a) *Every $\mathcal{A} \in \mathcal{C}'_k$ ($k \geq 1$) is supersolvable.*
- (b) *Every $\mathcal{A} \in \mathcal{C}_k$ ($k \geq 2$) is supersolvable. If $k \geq 3$ and $\mathcal{A}$ is not a complete quadrilateral, there is a triple point Z_0 as in Lemma 4.2, and for every such triple point the arrangement $\mathcal{A}_X/X$ lies in $\mathcal{C}_{k-1}$.*

Proof. Induction on k ; ranks 1 and 2 are settled. Let $k \geq 3$. If $\mathcal{A} \in \mathcal{C}_3$ and $v = 0$, then $\mathcal{A}$ is a complete quadrilateral and supersolvable by Lemma 5.1. In all other cases Lemma 4.3 gives a triple point Z_0 with $\mathcal{A}_{Z_0} = \{H_0, H_1, H_2\}$ and $\deg(H_1) = \deg(H_2) = 1$. Let Z_0 be any such triple point. By Lemma 4.2(a) and (b), $X = \bigcap \{H \neq H_1, H_2\}$ is a modular line with $\mathcal{A}_X = \mathcal{A} \setminus \{H_1, H_2\}$. Let $\mathcal{B} = \mathcal{A}_X/X$, an essential arrangement of rank $k - 1 \geq 2$ in V/X . Its forms are those of $\mathcal{A}_X$, so $F(\mathcal{B}) = F(\mathcal{A}_X)$, and $Y \mapsto Y/X$ identifies $L(\mathcal{A}_X)$ with $L(\mathcal{B})$. By Lemma 4.2(d), $\mathcal{B}$ is irreducible and formal, and its multiple elements of codimension two are those of $\mathcal{A}$ except Z_0 , with the same multiplicities. Therefore: if $\mathcal{A} \in \mathcal{C}'_k$, then $\mathcal{B}$ has $2k - 3 = 2(k - 1) - 1$ hyperplanes, all $m_Z \leq 3$ and $k - 2$ triple points, that is, $\mathcal{B} \in \mathcal{C}'_{k-1}$; if $\mathcal{A} \in \mathcal{C}_k$, then $\mathcal{B}$ has $2(k - 1)$ hyperplanes, all $m_Z \leq 4$, and $\sum_Z \binom{m_Z - 1}{2} = (k + 1) - 1 = (k - 1) + 1$, that is, $\mathcal{B} \in \mathcal{C}_{k-1}$. By induction $\mathcal{B}$ is supersolvable; hence so is $\mathcal{A}_X$, and $\mathcal{A}$ is supersolvable by Lemma 4.2(c). $\square$

Proof of Theorem 1.2. The hypotheses say that $\mathcal{A} \in \mathcal{C}_k$. The statements are Theorem 5.2(b), Lemma 5.1 and Lemma 4.2(a), (b). $\square$

Corollary 5.3. *Let K have characteristic zero, and let $\mathcal{A}$ be essential and free.*

- (a) *If $\exp(\mathcal{A}) = (1, 2^{k-1})$, then $\mathcal{A}$ is supersolvable.*
- (b) *If $\exp(\mathcal{A}) = (1, 2^{k-2}, 3)$, $k \geq 2$, then $\mathcal{A}$ is supersolvable.*

Proof. Proposition 3.2 and Theorem 5.2. $\square$

Part (a) is the statement of [Toh20, Section 3]; it also follows from [DiP23] (Remark 6.4). Part (b) is [Toh20, Conjecture 4.1].

Proof of Theorem 1.1. Let $C = \bigcap_{H \in \mathcal{A}} H$, and let V_1 be a complement of C in V . Then $\mathcal{A} = \mathcal{A}_1 \times \emptyset_C$, where $\mathcal{A}_1 = \{H \cap V_1 : H \in \mathcal{A}\}$ is an essential arrangement in V_1 , linearly isomorphic to $\mathcal{A}/C$, and $\emptyset_C$ is the empty arrangement in C . By Lemma 2.4, $\mathcal{A}_1$ is free and $\exp(\mathcal{A})$ is the union of $\exp(\mathcal{A}_1)$ and $0^{\dim C}$; by Lemma 2.3, $\mathcal{A}_1$ has no exponent 0. So the exponents 0 of $\mathcal{A}$ are exactly $\dim C$ in number, and we may assume that $\mathcal{A}$ is essential.

Write $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_c$ with irreducible essential arrangements $\mathcal{A}_l$ of rank $r_l \geq 1$ (Section 2.1). By Lemma 2.4 each $\mathcal{A}_l$ is free, and $\exp(\mathcal{A})$ is the union of the $\exp(\mathcal{A}_l)$. By Lemma 2.3 each $\exp(\mathcal{A}_l)$ contains exactly one 1 and no 0; its other entries are 2's and, for at most one l , a single 3. So $\exp(\mathcal{A}_l)$ is $(1, 2^{r_l-1})$ or $(1, 2^{r_l-2}, 3)$, and $\mathcal{A}_l$ is supersolvable by Corollary 5.3.

Finally a product of supersolvable arrangements is supersolvable: the elements of $L(\mathcal{A}_1 \times \mathcal{A}_2)$ are the subspaces $X_1 \oplus X_2$ with $X_i \in L(\mathcal{A}_i)$, and $(X_1 \oplus X_2) + (Y_1 \oplus Y_2) = (X_1 + Y_1) \oplus (X_2 + Y_2)$; so $X_1 \oplus X_2$ is modular if X_1 and X_2 are, and a chain of modular elements $X_a^1 \oplus V_2$ ($0 \leq a \leq r_1$), followed by $0 \oplus X_b^2$ ($1 \leq b \leq r_2$), is obtained from chains for $\mathcal{A}_1$ and $\mathcal{A}_2$. $\square$

6 Remarks

Remark 6.1 (what the induction produces). Let $\mathcal{A} \in \mathcal{C}_k$, for instance an essential free arrangement with exponents $(1, 2^{k-2}, 3)$ in characteristic zero. Each step of the induction removes two hyperplanes and one triple point, and keeps the quadruple point if there is one. If $v = 1$, the induction ends with four concurrent lines in rank 2; if $v = 0$, it ends with a complete quadrilateral in rank 3. So the proof gives a chain of modular elements $V = X_0 < X_1 < \dots < X_k = 0$ with $|\mathcal{A}_{X_l} \setminus \mathcal{A}_{X_{l-1}}| = 2$ for $l \geq 4$, and the sizes of the steps $|\mathcal{A}_{X_l} \setminus \mathcal{A}_{X_{l-1}}|$, $l = 1, \dots, k$, are

$$1, 3, 2, 2, \dots, 2 \quad \text{if } v = 1, \quad 1, 2, 3, 2, \dots, 2 \quad \text{if } v = 0.$$

In other words, $\mathcal{A}$ arises from a pencil of four lines or from a complete quadrilateral by adding, $k-2$ or $k-3$ times, one dimension and two hyperplanes which form a triple point with an existing hyperplane and contain no other multiple flat. For $\mathcal{A} \in \mathcal{C}'_k$ the sizes are $1, 2, 2, \dots, 2$.

In terms of matroids each step is a parallel connection with a line of three points: as $\alpha_2 \in \text{span}(\alpha_0, \alpha_1)$ with both coefficients non-zero (Lemma 4.2) and α_1 is not in the span of the forms of $\mathcal{A}_X$, a subset of $\mathcal{A}$ which contains exactly one of H_1, H_2 is independent if and only if its other members are, and a subset which contains both is independent if and only if its other members do not include H_0 and are independent together with H_0 . The forests of a graph to which a triangle is glued along an edge obey the same rule. Hence the matroid of $\mathcal{A}$ is the cycle matroid of the complete graph K_4 with $k-3$ triangles glued successively along edges if $v = 0$, and of a triangle with $k-2$ further triangles glued in this way if $\mathcal{A} \in \mathcal{C}'_k$, $k \geq 2$ (compare [DGT14, Theorem 5.6]).

Remark 6.2 (the converse). Every $\mathcal{A} \in \mathcal{C}_k$ is free with exponents $(1, 2^{k-2}, 3)$, and every $\mathcal{A} \in \mathcal{C}'_k$ is free with exponents $(1, 2^{k-1})$. Indeed, $\mathcal{A}$ is supersolvable by Theorem 5.2, and supersolvable arrangements are free by the theorem of Jambu and Terao [JT84], which we use as it is cited in [Toh21, p. 284]; we did not read [JT84]. By Lemma 2.3 exactly one exponent is 1 and the other $k-1$ exponents are at least 2; by (T2) they sum to $2k-1$, respectively $2k-2$. So, for an essential irreducible arrangement of rank $k \geq 2$ in characteristic zero, being free with exponents $(1, 2^{k-2}, 3)$ is equivalent to membership in $\mathcal{C}_k$. This remark rests on the cited theorem and is not used elsewhere.

Remark 6.3 (a question of the preprint). Let s be the number of hyperplanes which lie in exactly one dependent triple. In [Toh20, Remark 4.7] the inequality $s \geq u + 1$ is stated for free arrangements of rank 4 and 5 with exponents $(1, 2, 2, 3)$ and $(1, 2, 2, 2, 3)$, and it is asked whether it can be proved in rank at least 6. The graphic arrangement of the complete graph K_4 with a triangle glued along an edge, which is the arrangement of Section 8, has rank 4, is free with exponents $(1, 2, 2, 3)$, and has $u = 5$ and $s = 2$.

Remark 6.4 (the cases covered by DiPasquale's theorem). Let $\mathcal{A}$ be essential, irreducible and free of rank $k \geq 3$, with exponents $(1, 2^{k-1})$, or with exponents $(1, 2^{k-2}, 3)$ and $v = 1$. By Proposition 3.2, $\sum_Z (m_Z - 2)$ equals $k-1 = n-k$ in the first case and $u+2v = k = n-k$ in the second. As $\mathcal{A}$ is formal, $F(\mathcal{A}) = \sum_Z F_Z$ has dimension $n-k = \sum_Z \dim F_Z$, so this sum is direct. In the notation of [DiP23, Sections 4–6], where $R_2(\mathcal{A}) = F(\mathcal{A})$ and $R_3(\mathcal{A}_Y)$ is the kernel of $\bigoplus_{Z \in L_2(\mathcal{A}_Y)} F_Z \rightarrow F(\mathcal{A}_Y)$, this gives $R_i(\mathcal{A}_Y) = 0$ for all $i \geq 3$ and all $Y \in L(\mathcal{A})$; hence the modules $\mathcal{S}^i(\mathcal{A})$ of [DiP23, Definition 5.1], which are free of rank $\sum_Y \dim R_i(\mathcal{A}_Y)$ (Y of codimension i), vanish for $i > 2$. Since free arrangements are totally formal [DiP23, Corollary 4.10], $\mathcal{A}$ is a TF_2 arrangement as defined at the beginning of Section 6 of that paper, which works over an arbitrary field. By [DiP23, Proposition 6.1], an irreducible TF_2 arrangement of rank $r \geq 3$ is free if and only if it has exactly $r-1$ multiple elements of codimension two, and if and only if it is supersolvable. So Corollary 5.3(a) and the case $v = 1$ of Corollary 5.3(b) follow from [DiP23]. For $v = 0$ there are $k+1$ multiple elements of codimension two; so, by the same proposition, $\mathcal{A}$ is not TF_2 , and the proposition does not give its supersolvability. The proof in [DiP23]

that $r - 1$ multiple flats imply supersolvability adds the multiple flats one at a time; Lemma 4.1 and the induction for $\mathcal{C}'_k$ and for $v = 1$ are arguments of the same kind. For arrangements which are formal but not assumed free, Theorem 5.2 is not literally contained in [DiP23, Proposition 6.1], whose hypothesis is total formality. So, relative to the literature we found, what is new is the case $v = 0$ in rank $k \geq 4$, and therefore the conjecture in general; the other cases are proved again.

Remark 6.5 (the field). The conjecture is stated for fields of characteristic zero, and Theorem 1.1 is claimed in this setting. Our own arguments do not use the characteristic: Lemmas 2.2 to 2.5, Section 4, Lemma 5.1, Theorem 5.2 and Appendix A hold over every field over which Saito's criterion (T2) holds. The characteristic was restricted only because of (T1), and Proposition B.1 in Appendix B proves the two consequences of (T1) and (T2) which enter Proposition 3.2, namely $\sum_i d_i = n$ and (4), without the factorization theorem. Therefore Proposition 3.2, Corollary 5.3 and Theorem 1.1 hold over every field for which Saito's criterion holds; in [PT20, Theorem 3.7] the criterion is stated for an arbitrary field, with a reference to Terao for positive characteristic. The exponents of an arrangement defined over the integers may depend on the characteristic [PT20, Example 3.3]; the statement concerns the exponents over the given field. Theorem 1.2 holds over every field without any proviso; for tests over finite fields see Section 9.

7 The hypotheses, the limits of the argument, and sharpness

Where each hypothesis is used. Freeness enters through the reduction in the proof of Theorem 1.1 (Lemmas 2.3 and 2.4) and through Proposition 3.2, that is, through (T1), (T2), Lemmas 2.3 and 2.5 and Theorem 2.1. The exponents enter only through the numerical conditions (C3)–(C5). Irreducibility is used in Lemma 4.1 and is inherited by Lemma 4.2(d). Formality is used in Lemma 4.1, in Lemma 4.2(a) and (d), and in (5). Neither can be dropped in Theorem 1.2: the second verification run found ten hyperplanes of the Coxeter arrangement B_5 , with one quadruple and three triple points, which satisfy all hypotheses except formality, and eight hyperplanes in $\mathbb{Q}^4$ which satisfy all hypotheses except irreducibility; neither arrangement is supersolvable.

Why the argument stops outside the range of the conjecture. For a formal arrangement the number of independent linear dependencies among the local relations is $\sum_Z (m_Z - 2) - (n - k)$; it is the dimension of the space of dependencies among the dependencies of length three, which Yuzvinsky relates to the second of his obstructions to freeness [Yuz93, Section 4; see Remark 4.13]. If $\mathcal{A}$ is essential and free with exponents d_i in characteristic zero, then, since $m - 2 \leq \binom{m-1}{2}$ for $m \geq 2$, (4) and $n = \sum_i d_i$ give

$$\sum_Z (m_Z - 2) - (n - k) \leq \sum_i \binom{d_i}{2} - \sum_i (d_i - 1) = \sum_i \binom{d_i - 1}{2}. \quad (8)$$

The bound is 0 if all exponents are at most 2, and 1 if exactly one exponent is 3 and the others are at most 2: this is the range of the conjecture. With no dependency the count of Lemma 4.3(i) applies; with one dependency its support can be controlled as in Lemma 4.3(ii). For two exponents 3 the bound is 2, and the statement fails: in the non-Fano arrangement below, six local relations lie in a relation space of dimension 4, there are two independent dependencies, and every line has degree 2 or 3, so that Lemma 4.2 has nothing to start from. For exponents $(1, 2^{k-2}, 4)$ the bound is 3; here $n = 2k + 1$ and $\sum_Z \binom{m_Z - 1}{2} = k + 4$, and if all multiple flats are triple points the count $3u \geq s + 2(n - s)$ gives only $s \geq k - 10$. We make no statement about these exponents.

Sharpness. The hypothesis “at most one 3” cannot be replaced by “at most two 3’s”, in rank three and in rank four.

(a) *The non-Fano arrangement $\mathcal{A}$* with $Q(\mathcal{A}) = xyz(x - y)(x - z)(y - z)(x + y - z)$ over a field of characteristic different from 2; it is the example given in [Toh21, p. 284] and [Toh20, Section 4.1].

It is free with exponents $(1, 3, 3)$: the derivations

$$\theta_E, \quad (x + y - z)(x(x - z)\partial_x + y(y - z)\partial_y), \quad x(x - z)(2y - z)\partial_x + y(y - z)(2x - z)\partial_y$$

lie in $D(\mathcal{A})$, and the determinant of their coefficients is $2Q(\mathcal{A})$; this basis is the one of [PT20, Example 3.3] after the substitution $z \mapsto -z$. It has six triple points and no other multiple point, the degrees of its seven lines are $3, 3, 2, 2, 3, 3, 2$, and no element of $L_2(\mathcal{A})$ is modular; so it is not supersolvable.

(b) *Nine hyperplanes in rank four*, in characteristic zero, found by the second verification run in the Coxeter arrangement B_4 :

$$x_4, \quad x_3 - x_4, \quad x_3 + x_4, \quad x_2 - x_3, \quad x_2 - x_4, \quad x_2 + x_4, \quad x_2 + x_3, \quad x_1 - x_2, \quad x_1 - x_3.$$

The first seven hyperplanes form a non-Fano arrangement in the coordinates x_2, x_3, x_4 , and the last two form a triple point with $x_2 - x_3$. The arrangement is irreducible, has seven triple points and no quadruple point, and $\chi(\mathcal{A}, t) = (t - 1)(t - 2)(t - 3)^2$. It is free with exponents $(1, 2, 3, 3)$: the derivations

$$\theta_E, \quad (x_1 - x_2)(x_1 - x_3)\partial_1, \quad \sum_{i=1}^4 x_i^3 \partial_i, \quad x_4^2(x_2 + x_3 - x_1)\partial_1 + x_3 x_4^2 \partial_2 + x_2 x_4^2 \partial_3 + x_2 x_3 x_4 \partial_4$$

lie in $D(\mathcal{A})$, and the determinant of their coefficients is $Q(\mathcal{A})$. It is not supersolvable: its only modular element of codimension three is the line $X = \{x_2 = x_3 = x_4 = 0\}$; a chain of modular elements would pass through X and give such a chain for $\mathcal{A}_X/X$, which is the non-Fano arrangement. The first verification run found 180 such sub-arrangements of B_4 , among them $x_1, x_2, x_1 + x_2, x_1 \pm x_3, x_1 \pm x_4, x_3 \pm x_4$; for the verification of the data of this section see Section 9.

8 A worked example

Let $\mathcal{A}$ be the arrangement in $V = K^4$, $\text{char } K = 0$, with the eight forms

$$\alpha_1 = x_1, \quad \alpha_2 = x_2, \quad \alpha_3 = x_3, \quad \alpha_4 = x_1 - x_2, \quad \alpha_5 = x_1 - x_3, \quad \alpha_6 = x_2 - x_3, \quad \alpha_7 = x_4, \quad \alpha_8 = x_1 - x_4,$$

and let $H_i = \ker \alpha_i$. It is the graphic arrangement of the graph with the vertices $0, \dots, 4$ and the eight edges $\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{0, 4\}, \{1, 4\}$, in this order: the complete graph on $\{0, 1, 2, 3\}$ with the triangle $\{0, 1, 4\}$ glued along the edge $\{0, 1\}$. With $x_0 = 0$, the edge $\{i, j\}$ gives the form $x_i - x_j$.

Flats and degrees. The arrangement is essential of rank 4 and irreducible, and $n = 8 = 2k$. A set of three forms $x_i - x_j$ is dependent if and only if the edges form a triangle, so the multiple elements of $L_2(\mathcal{A})$ are the five triple points

$$Z_1: \{H_1, H_2, H_4\}, \quad Z_2: \{H_1, H_3, H_5\}, \quad Z_3: \{H_2, H_3, H_6\}, \quad Z_4: \{H_4, H_5, H_6\}, \quad Z_0: \{H_1, H_7, H_8\},$$

with the relations $\rho_{Z_1} = e_1 - e_2 - e_4$, $\rho_{Z_2} = e_1 - e_3 - e_5$, $\rho_{Z_3} = e_2 - e_3 - e_6$, $\rho_{Z_4} = e_4 - e_5 + e_6$, $\rho_{Z_0} = e_1 - e_7 - e_8$, where $e_i = e_{H_i}$. So $u = 5$, $v = 0$ and $\sum_Z \binom{m_Z - 1}{2} = 5 = k + 1$. The degrees of $H_1, \dots, H_8$ are $3, 2, 2, 2, 2, 2, 1, 1$; thus $s = 2$. The relation space has dimension $n - k = 4$, and $\rho_{Z_1}, \rho_{Z_2}, \rho_{Z_3}, \rho_{Z_0}$ are linearly independent; so $\mathcal{A}$ is formal and $\mathcal{A} \in \mathcal{C}_4$.

Freeness. The derivations

$$\theta_E, \quad \sum_{i=1}^4 x_i^2 \partial_i, \quad \sum_{i=1}^4 x_i^3 \partial_i, \quad x_4(x_1 - x_4)\partial_4$$

lie in $D(\mathcal{A})$, and the determinant of their coefficients is $-x_4(x_1 - x_4) \cdot x_1 x_2 x_3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = -Q(\mathcal{A})$. By (T2), $\mathcal{A}$ is free with exponents $(1, 2, 2, 3)$, and (4) reads $5 = 0 + 1 + 1 + 3$.

The dependency of Lemma 4.3. The five relations lie in a space of dimension 4, and the only dependency is $\rho_{Z_1} - \rho_{Z_2} + \rho_{Z_3} + \rho_{Z_4} = 0$. So $T = \{Z_1, \dots, Z_4\}$ consists of the four triangles of K_4 , $W = \{H_1, \dots, H_6\}$, $t = 4$, $w = 6$, and (6) reads $12 \geq 12$. The triple point Z_0 outside T lies in $H_1 \in W$, so $c = 1$; the hyperplanes outside W are H_7, H_8 , both of degree 1, so $s' = 2$. Inequality (7) reads $s' - (k + 1 - t) = 1 \geq t + c - 4 = 1$: there are more hyperplanes of degree 1 outside W than triple points outside T , and Z_0 lies in two of them.

The modular line and the chain. Lemma 4.2 applies to Z_0 with $H_0 = H_1$ and the two hyperplanes H_7, H_8 of degree 1: $\rho = e_1 - e_7 - e_8$, $\lambda(r) = -r_7 + r_8$, and $v_0 = (0, 0, 0, -1)$ satisfies $\alpha_7(v_0) = -1$, $\alpha_8(v_0) = 1$ and $\alpha_i(v_0) = 0$ for $i \leq 6$. So $X = H_1 \cap \dots \cap H_6 = Ke_4$ is a modular line, and $\mathcal{A}_X/X$ is the arrangement $x_1x_2x_3(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ in K^3 : the complete quadrilateral, the exceptional case of Theorem 1.2. Its triple point $\{x_1 = x_2 = 0\}$ is modular (Lemma 5.1). The resulting chain of modular elements of $L(\mathcal{A})$ is

$$V < \{x_1 = 0\} < \{x_1 = x_2 = 0\} < \{x_1 = x_2 = x_3 = 0\} = Ke_4 < 0,$$

with $|\mathcal{A}_{X_l}| = 1, 3, 6, 8$ for $l = 1, \dots, 4$ and steps 1, 2, 3, 2, as in Remark 6.1 for $v = 0$.

9 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation. The computations of this section are *tests*: a search that finds no counterexample is not a proof. They are exact (rational arithmetic, or arithmetic in a finite field). The reproducibility package contains the programs, the command lines and the recorded outputs: the programs with which the result was first tested (`original/`), those of three independent verification runs (`verification_run_A/`, `verification_run_B/`, `independent_run_2/`), and the program of this note (`writing_stage/`). Freeness with prescribed exponents is decided in both directions, by four different implementations which all rest on (T2): an answer “free” comes with a basis, and an answer “not free with these exponents” is the failure of a necessary condition. Supersolvability is decided from the definition. The verification report of the package lists every test with its numbers.

Exhaustive enumerations. Table 1 lists all subsets of size $2k$, up to signed permutations of the coordinates, of four sets of hyperplanes: the 13 lines with normals in $\{0, \pm 1\}^3$, the 40 hyperplanes with normals in $\{0, \pm 1\}^4$, and the Coxeter arrangements B_5 and B_6 .

ground set	k	subsets of size $2k$	orbits	ess., irred.	free	$(k+1, 0)$	$(k-2, 1)$	size $2k-1$
$\{0, \pm 1\}^3$	3	1,716	106	106	26	9	17	26
$\{0, \pm 1\}^4$	4	76,904,685	415,920	410,655	2,929	913	2,016	2,056
B_5	5	3,268,760	3,071	2,363	134	66	68	78
B_6	6	1,251,677,700	86,764	59,779	726	382	344	333

Table 1: Exhaustive enumerations. “Orbits”: orbits of subsets of size $2k$; “ess., irred.”: essential irreducible orbits; “free”: orbits which are free with exponents $(1, 2^{k-2}, 3)$, split by (u, v) in the next two columns; last column: orbits of subsets of size $2k - 1$ which are free with exponents $(1, 2^{k-1})$ (for $k = 3$ from the second and third runs). All free arrangements found are supersolvable.

Every free arrangement found is formal, has all $m_Z \leq 4$ and $v \leq 1$, and, except for the nine orbits of complete quadrilaterals, has a triple point with two hyperplanes of degree 1 whose line X is a modular coatom. Among the essential irreducible orbits with (C3)–(C5), or with (C3')–(C5'), the formal ones are supersolvable and the others are not. The table was computed with the original

programs, which discard the sets that are not essential or not irreducible before the freeness test (by Lemma 2.3 they are not free with these exponents). The second verification run reproduced it with its own program, orbit by orbit. The third run reproduced its first three rows with a third program, which decides freeness for every orbit and discards nothing.

Further tests. None of the following tests found a deviation from a statement of this note. The first verification run tested 966,449 sets of hyperplanes over $\mathbb{Q}$ and over $\mathbb{F}_p$, $p \leq 11$, in ranks 3 to 6, and Theorem 1.2 on all 36,288,252 sets formed by the standard basis and six further vectors of $\mathbb{F}_2^6$. The second run tested Theorem 1.1 with reducible and non-essential arrangements on the orbits of all subsets with at most 6, 7, 10 and 11 hyperplanes of the four ground sets, and random arrangements over $\mathbb{Q}$ and over four prime fields. The third run decided freeness for 60,000 random sets of hyperplanes of rank 4 over $\mathbb{Q}$; it tested Theorem 5.2 on sets which contain the coordinate hyperplanes and satisfy the numerical conditions, 201,696 of them in $\mathbb{Q}^4$ and all 87,616 of them in $\mathbb{F}_4^4$ and $\mathbb{F}_5^4$; and it tested the identities of Proposition B.1, Theorem 2.1 and Lemmas 2.3 and 2.5 for arbitrary exponents on 558,955 sets of hyperplanes over the fields with 2, 3, 4, 5, 7, 8 and 9 elements. The statements on hypergraphs which the proof of Lemma 4.3 uses were enumerated for $k \leq 6$ in (i), for $k \leq 6$ with a quadruple point and for $k \leq 5$ without one, and further up to isomorphism: the only configuration without a triple point carrying two hyperplanes of degree 1 is that of the complete quadrilateral. Every printed statement of Sections 7 and 8 was verified by the program `check_note.py` of this note and again by the third run. Outside the range of the conjecture the same programs find free arrangements which are not supersolvable (Section 7).

Limits of the tests. The enumerations cover the ground sets named here and in the verification report and no others; ranks 7 and higher were tested only at random. In some families the freeness of the members of the classes was decided for a sample only (one member in eight in ranks 5 and 6 of the first run; one set in twenty among the 201,696 sets of the third run). Arrangements over number fields were tested only through reductions to finite fields. The original library stores its linear systems in 64-bit integers and stops with an overflow error when the coefficients of the forms have size about 10^5 ; it returns no wrong answer in that case, and all its runs have coefficients of a few units.

Verification. The proof was first written with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs.

The first two runs did not see each other and examined the first written version of the proof. The first run compared the statement with [Toh21, Toh20], read [BT94], derived every step of Sections 2 to 5 and of Appendix A again, and proved Saito’s criterion and Lemma 2.4 over arbitrary fields for itself. The second run recomputed the enumerations, read and ran the original programs, searched for counterexamples, and examined the literature; it found the theorem of DiPasquale. Neither found a mathematical error or a gap. All corrections required by them were made, with one exception: a further citation search with MathSciNet and Google Scholar, which we could not make.

The third run examined the final text, and first what had been written after the first two runs: the proof of Proposition B.1 in its present form, Remarks 6.4 and 6.5, the references to [MMR24] and [Yuz93], Section 8 and the explicit bases of Sections 7 and 8. It derived every proof of Sections 2 to 5 and of both appendices again, line by line; compared the statement with the Oberwolfach abstract, with version 3 of the preprint and with the withdrawal notice; and compared the citations of [Yuz93, BT94, DiP23, DGT14, Toh16, MMR24, PT20, HR15] with the cited texts. It found no wrong theorem, proposition or lemma and no gap. It required twelve corrections of the text and of the package, which were made; none changes a statement or a proof. It found the preprint [LW26]; the last paragraph of Remark 6.1 was added at this revision and has been examined by this run only. It also ran the quick part of the package and the program of this note again from the archive, and all

recorded outputs were reproduced. No proof depends on a computation.

Scope and priority. The note proves the conjecture stated in [Toh21, p. 284] and, for irreducible arrangements, as Conjecture 4.1 in [Toh20]: over a field of characteristic zero, in every rank, for central arrangements, including reducible and non-essential ones (Theorem 1.1); for other fields see Remark 6.5. The conjecture and the strategy of the proof are due to Tohăneanu [Toh21, Toh20]. The preprint [Toh20] was withdrawn by its author on 16 January 2022; the withdrawal notice names Sections 4.2.1 and 4.2.2 and case (a) of the proof of Proposition 4.8; no result of the preprint is used here. New, to our knowledge and relative to the literature we found, is the case without a quadruple point ($v = 0$) in rank $k \geq 4$, and therefore the conjecture in general; the other cases are proved again: they follow from [DiP23, Proposition 6.1 and Corollary 4.10] and, in rank three, from [Toh16, Theorem 2.4]. The other attributions, to [DGT14, Yuz93, BT94, MMR24, BEZ90, Sch03, LW26], are in Sections 1.4 and 4. We claim no priority.

What was read: [Toh21] (pp. 283–286 and p. 228) in the publisher’s file; version 3 of [Toh20] completely, with the withdrawal notice of version 4; [Yuz93] and [BT94] completely, in the publishers’ files; and, in their arXiv versions, to which our numbers refer, the parts of [DiP23, DGT14, Toh16, MMR24, PT20, HR15] which are cited and the abstract and the introduction of [LW26] (the sections are listed in the verification report). The journal versions were not compared. Not read: [OT92], which was not accessible to us, and [Ter81], [JT84], [Sch03], [BEZ90], [BH98]; what we say about them is taken from the papers named where they are cited.

The searches were made on 11 October 2026, at each stage of the work, and repeated in part on 12 October 2026 by the third verification run: queries to the arXiv (by topic, and the list of the arXiv papers of the author of the conjecture), zbMATH Open, OpenAlex (which lists no work citing [Toh20]), Crossref, Semantic Scholar (at the first stage), MathOverflow, and nine web searches. The texts found which cite the preprint are [Toh21] and one paper which only lists it. MathSciNet and Google Scholar were not searched, and the author of the conjecture was not contacted. No proof and no counterexample of the conjecture was found in the literature accessible to us; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A Free arrangements are formal

We prove Theorem 2.1, following Brandt and Terao [BT94, Proposition 4.13 and Theorem 4.15], in the case $k = 2$ of their Theorem 4.15: the module M below is their $D_2(\mathcal{A})$, the modules M_Z are their $D_2(\mathcal{A}_Z)$, and the injectivity of $M \rightarrow \bigoplus_Z M_Z$ is the vanishing of their $K_3(\mathcal{A})$. We use three standard facts on a finitely generated graded module $N \neq 0$ over $S = K[x_1, \dots, x_k]$: the Auslander–Buchsbaum formula $\text{pd } N + \text{depth } N = k$; the inequality $\text{depth } N \leq \dim S/\mathfrak{p}$ for every associated prime $\mathfrak{p}$ of N [BT94, Theorem 4.14]; and, in Appendix B, the theorem of Hilbert and Serre that $(1 - t)^{\dim N}$ times the Hilbert series of N is a polynomial. These facts are standard (see for instance [BH98], which we did not consult for this note).

Proposition A.1. *Let $\mathcal{A}$ be an essential arrangement in $V = K^k$ over any field K . If $D(\mathcal{A})$ is a free S -module, then $F(\mathcal{A}) = \sum_{Z \in L_2(\mathcal{A})} F_Z$.*

Proof. Write $\mathcal{A} = \{H_1, \dots, H_n\}$, $\alpha_i = \alpha_{H_i}$, $E = K^n$ with coordinate functions $e_1^*, \dots, e_n^*$, and $F = F(\mathcal{A}) \subseteq E$. Let $\iota : V \rightarrow E^*$, $\iota(v) = \sum_i \alpha_i(v) e_i^*$. It is injective, because $\mathcal{A}$ is essential, and its image

is the annihilator of F : it lies in the annihilator, and both have dimension k . So $E^*/\iota(V) = F^*$ by restriction of linear forms to F . Let $\bar{e}_i \in F^*$ be the restriction of e_i^* .

We have $\text{Der}_K(S) = S \otimes_K V$, and $\varepsilon = 1 \otimes \iota : S \otimes V \rightarrow S \otimes E^* = S^n$ is the map $\theta \mapsto (\theta(\alpha_i))_i$. Let $N_0 = \bigoplus_i \alpha_i S \subseteq S^n$. By definition $D(\mathcal{A}) = \varepsilon^{-1}(N_0)$, so there is an exact sequence of finitely generated graded S -modules, with maps of degree zero,

$$\begin{aligned} 0 \rightarrow D(\mathcal{A}) \rightarrow S \otimes V \rightarrow S^n/N_0 \rightarrow M \rightarrow 0, \\ M := S^n/(\varepsilon(S \otimes V) + N_0) = (S \otimes F^*)/\langle \alpha_i \otimes \bar{e}_i : 1 \leq i \leq n \rangle. \end{aligned} \quad (9)$$

Here V and F^* are in degree 0; the relations $\alpha_i \otimes \bar{e}_i$ have degree 1, so $M_0 = F^*$.

Projective dimension. The module $S^n/N_0 = \bigoplus_i S/\alpha_i S$ has projective dimension 1. If $D(\mathcal{A})$ is free, the image I of $S \otimes V$ in S^n/N_0 has projective dimension at most 1, and the exact sequence $0 \rightarrow I \rightarrow S^n/N_0 \rightarrow M \rightarrow 0$ gives $\text{pd } M \leq 2$. If $M = 0$, then $F^* = M_0 = 0$ and there is nothing to prove. Let $M \neq 0$; then $\text{depth } M \geq k - 2$.

Localization. Let $\mathfrak{p}$ be a prime ideal of S and $\mathcal{A}_{\mathfrak{p}} = \{i : \alpha_i \in \mathfrak{p}\}$. For $i \notin \mathcal{A}_{\mathfrak{p}}$ the form α_i is a unit in $S_{\mathfrak{p}}$, so $1 \otimes \bar{e}_i = 0$ in $M_{\mathfrak{p}}$. Therefore

$$M_{\mathfrak{p}} = (S_{\mathfrak{p}} \otimes F^*/U)/\langle \alpha_i \otimes \bar{e}_i : i \in \mathcal{A}_{\mathfrak{p}} \rangle, \quad U = \text{span}\{\bar{e}_i : i \notin \mathcal{A}_{\mathfrak{p}}\}, \quad F^*/U = (F \cap K^{\mathcal{A}_{\mathfrak{p}}})^*, \quad (10)$$

because the subspace of F on which all $\bar{e}_i$ with $i \notin \mathcal{A}_{\mathfrak{p}}$ vanish is $F \cap K^{\mathcal{A}_{\mathfrak{p}}}$.

Associated primes. If $M_{\mathfrak{p}} \neq 0$, then $F \cap K^{\mathcal{A}_{\mathfrak{p}}} \neq 0$ by (10): the forms α_i , $i \in \mathcal{A}_{\mathfrak{p}}$, are linearly dependent. As they are pairwise non-proportional, $\mathfrak{p}$ contains two linearly independent linear forms, and $\text{ht } \mathfrak{p} \geq 2$. Now let $\mathfrak{p}$ be an associated prime of M . Then $k - 2 \leq \text{depth } M \leq \dim S/\mathfrak{p} = k - \text{ht } \mathfrak{p}$, so $\text{ht } \mathfrak{p} = 2$. Hence $\mathfrak{p} = (\alpha_i, \alpha_j)$ for two independent forms of $\mathcal{A}$, which is the ideal $I(Z)$ of $Z = H_i \cap H_j \in L_2(\mathcal{A})$. The linear forms in $I(Z)$ are the forms vanishing on Z , so $\mathcal{A}_{\mathfrak{p}} = \mathcal{A}_Z$, and $F \cap K^{\mathcal{A}_Z} = F_Z \neq 0$ means that Z is multiple.

The local modules. For a multiple $Z \in L_2(\mathcal{A})$ let

$$M_Z = (S \otimes F_Z^*)/\langle \alpha_i \otimes \bar{e}_i|_{F_Z} : H_i \in \mathcal{A}_Z \rangle.$$

Restriction $F^* \rightarrow F_Z^*$ induces a surjection $\pi_Z : M \rightarrow M_Z$: a generator $\alpha_i \otimes \bar{e}_i$ is mapped to a generator if $H_i \in \mathcal{A}_Z$, and to 0 otherwise, since $\bar{e}_i$ vanishes on $F_Z \subseteq K^{\mathcal{A}_Z}$ for $H_i \notin \mathcal{A}_Z$. By (10) for $\mathfrak{p} = I(Z)$, where $F \cap K^{\mathcal{A}_{\mathfrak{p}}} = F_Z$, the map π_Z becomes an isomorphism after localization at $I(Z)$. Let N' be the kernel of $M \rightarrow \bigoplus_Z M_Z$, $m \mapsto (\pi_Z(m))_Z$, the sum over the multiple Z . Then $N'_{\mathfrak{p}} = 0$ for $\mathfrak{p} = I(Z)$, Z multiple, that is, for every prime which can be associated to M . An associated prime $\mathfrak{p}$ of N' is associated to M and satisfies $N'_{\mathfrak{p}} \neq 0$. So N' has no associated prime, and $N' = 0$.

Degree zero. The injective map $M \rightarrow \bigoplus_Z M_Z$ is in degree 0 the map $F^* \rightarrow \bigoplus_Z F_Z^*$, $\lambda \mapsto (\lambda|_{F_Z})_Z$. Its injectivity says that no non-zero linear form on F vanishes on all F_Z , that is, $F = \sum_Z F_Z$. $\square$

B The identity (4) without the factorization theorem

Proposition B.1. *Let K be a field over which Saito's criterion (T2) holds, and let $\mathcal{A}$ be an essential free arrangement over K with exponents $d_1, \dots, d_k$. Then*

$$\sum_i d_i = n \quad \text{and} \quad \sum_{Z \in L_2(\mathcal{A})} \binom{m_Z - 1}{2} = \sum_i \binom{d_i}{2}.$$

Proof. For a finitely generated graded S -module N let $h_N(t) = \sum_d \dim_K N_d t^d$. We have $h_S = (1-t)^{-k}$, $h_{S/\alpha S} = (1-t)^{1-k}$ for a linear form $\alpha \neq 0$, and $h_{D(\mathcal{A})} = \sum_i t^{d_i} (1-t)^{-k}$. By (9),

$$h_M(t) = \frac{P(t)}{(1-t)^k}, \quad P(t) = n(1-t) - k + \sum_i t^{d_i}. \quad (11)$$

Let $Z \in L_2(\mathcal{A})$ be multiple and $m = m_Z$. The arrangement $\mathcal{A}_Z$ is not essential, but the image of $V \rightarrow (K^{\mathcal{A}_Z})^*$, $v \mapsto (\alpha_H(v))_{H \in \mathcal{A}_Z}$, is again the annihilator of $F(\mathcal{A}_Z) = F_Z$: both have dimension 2. So the sequence (9) for $\mathcal{A}_Z$,

$$0 \rightarrow D(\mathcal{A}_Z) \rightarrow S \otimes V \rightarrow \bigoplus_{H \in \mathcal{A}_Z} S/\alpha_H S \rightarrow M_Z \rightarrow 0,$$

is exact, with the module M_Z of Appendix A. By Lemma 2.2, $D(\mathcal{A}_Z)$ is free with exponents $(0^{k-2}, 1, m-1)$. Hence

$$h_{M_Z}(t) = \frac{m(1-t) - 2 + t + t^{m-1}}{(1-t)^k} = \frac{R_m(t)}{(1-t)^{k-2}}, \quad R_m(t) = \sum_{j=0}^{m-3} (m-2-j)t^j, \quad (12)$$

where we used $(1-t)^2 R_m(t) = (1-t) \sum_{i=0}^{m-3} (1-t^{i+1}) = (m-2) - (m-1)t + t^{m-1}$; note that $R_m(1) = \binom{m-1}{2}$.

By the proof of Proposition A.1 the map $M \rightarrow \bigoplus_Z M_Z$ is injective; let C be its cokernel. We claim that $C_{\mathfrak{p}} = 0$ for every prime $\mathfrak{p}$ of height at most 2. The description (10) holds for M_Z as well, so $(M_Z)_{\mathfrak{p}} \neq 0$ implies that $\mathfrak{p}$ contains three forms of $\mathcal{A}_Z$, hence $\mathfrak{p} \supseteq I(Z)$, and $\mathfrak{p} = I(Z)$ if $\text{ht } \mathfrak{p} \leq 2$. If $\mathfrak{p}$ is not of the form $I(Z)$ with Z multiple, then all $(M_Z)_{\mathfrak{p}}$ vanish. If $\mathfrak{p} = I(Z_1)$, then $(M_Z)_{\mathfrak{p}} = 0$ for $Z \neq Z_1$, and $M_{\mathfrak{p}} \rightarrow (M_{Z_1})_{\mathfrak{p}}$ is an isomorphism. In both cases $C_{\mathfrak{p}} = 0$. Hence $\dim C \leq k-3$, and $C = 0$ if $k \leq 2$. By the theorem of Hilbert and Serre, $(1-t)^{k-3} h_C(t)$ is a polynomial if $k \geq 3$. Since $h_C = \sum_Z h_{M_Z} - h_M$, we get from (11) and (12) that the polynomial

$$(1-t)^2 \sum_Z R_{m_Z}(t) - P(t) = (1-t)^k h_C(t)$$

is divisible by $(1-t)^3$. Therefore $P(1) = 0$, $P'(1) = 0$ and $P''(1) = 2 \sum_Z R_{m_Z}(1) = 2 \sum_Z \binom{m_Z-1}{2}$. As $P'(1) = -n + \sum_i d_i$ and $P''(1) = \sum_i d_i(d_i-1)$, this is the claim. $\square$

The proof uses (T2) only through Lemma 2.2. The first verification run obtained the same identity from (11) and (12) by the associativity formula for multiplicities; the third run tested both identities over seven finite fields (Section 9).

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