

Finite Counterexamples to the Monotonicity Conjecture of Amir, Baldasso and Beilin for Median and Majority Dynamics

Alper Ferudun*

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Abstract

Amir, Baldasso and Beilin conjectured (Oberwolfach Report 4/2021, Conjecture 1; Stoch. Proc. Appl. 155 (2023), Conjecture 3.1) that in median dynamics on any locally finite graph (i.i.d. uniform opinions in $[0, 1]$, independent rate-one Poisson clocks, a ringing vertex takes the median of its neighbours' opinions, its own opinion being added when its degree is even) the probability that the opinion at a fixed vertex lies in $[0, \alpha]$ is non-decreasing in time for every $\alpha > \frac{1}{2}$. Equivalently, in majority dynamics started from i.i.d. Bernoulli(p) opinions, $f_p(t) = \mathbb{P}_p[\xi_t(x) = 1]$ is non-decreasing for every $p \geq \frac{1}{2}$. They proved this for complete graphs and for $\mathbb{Z}$. We show that the conjecture is false as stated. On a connected graph with six vertices and eight edges, all of odd degree, we determine f_p at one vertex for every p : with $r = p(1 - p)$,

$$f_p(t) - f_p(\infty) = r(2p - 1)e^{-t} \left[\frac{9}{16}rt - \frac{4+r}{4} - \frac{7}{24}re^{-2t} + r\left(\frac{71}{200} + \frac{7t}{80}\right)e^{-4t} \right].$$

Hence, for every $p \in (\frac{1}{2}, 1)$, $f_p(t) > f_p(\infty)$ for all $t > \frac{16}{9r} + \frac{26}{27}$, and f_p is eventually strictly decreasing; the conjecture fails at this vertex for every $\alpha \in (\frac{1}{2}, 1)$. The formula is obtained by a finite computation in exact rational arithmetic over the 64 configurations of the graph, and one instance is certified separately by rational bounds which do not use it. On a connected 3-regular graph on ten vertices, an exact closed form shows that $f_p(t) > f_p(\infty)$ for all large t , for every $p \in (\frac{1}{2}, 1)$, with a larger overshoot ($f(3) - f(\infty) > 3.47 \cdot 10^{-3}$ at $p = \frac{7}{10}$). All degrees in both examples are odd, so neither depends on the rule for ties at even degree. For infinite graphs we obtain only disconnected counterexamples. The conjecture for vertex-transitive graphs, in particular for $\mathbb{Z}^d$ ($d \geq 2$) and for regular trees, and for connected infinite graphs is not settled, and we have no certified counterexample among regular graphs of degree at least four. This is an unrefereed note.

1 Introduction

1.1 The conjecture

Let $G = (V, E)$ be a locally finite graph and write $N(v)$ for the set of neighbours of $v \in V$. In *median dynamics*, introduced by Damron and Sen [DS20] on the 3-regular tree and studied on general graphs by Amir, Baldasso and Beilin [ABB23], every vertex carries an opinion in $[0, 1]$; the initial opinions $\eta_0(v)$ are i.i.d. uniform, every vertex has an independent Poisson clock of rate one, and when the clock of v rings, $\eta(v)$ is replaced by the median of the opinions in the *pool* of v . The pool is $N(v)$ if $\deg v$ is odd and $N(v) \cup \{v\}$ if $\deg v$ is even, so that it has odd size [ABB23, (2.2)–(2.3)]. *Majority dynamics* is the same process with opinions in $\{0, 1\}$: a ringing vertex adopts the majority opinion of its pool.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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For $p \in [0, 1]$ let $\mathbb{P}_p$ refer to majority dynamics $(\xi_t)_{t \geq 0}$ started from i.i.d. Bernoulli(p) opinions, and for a vertex x put

$$f_p(t) = f_p^{G,x}(t) = \mathbb{P}_p[\xi_t(x) = 1].$$

By [ABB23, Proposition 2.2], $\xi_t^p(v) = \mathbf{1}_{[0,p]}(\eta_t(v))$ is majority dynamics from i.i.d. Bernoulli(p) opinions, so median dynamics couples majority dynamics for all p at once. Write $\mu_t^x(A) = \mathbb{P}[\eta_t(x) \in A]$ [ABB23, (3.1)].

In his abstract in Oberwolfach Report 4/2021 [Ami21], Amir states two conjectures obtained with Baldasso and Beilin. Conjecture 1 there (on p. 172) says that for every G , every $x \in V$ and every $\alpha > \frac{1}{2}$, “the function $t \rightarrow \mu_t(x)$ is monotone non-decreasing”. The set to which $\mu_t(x)$ is applied is missing in the printed text; Conjecture 2 of the same abstract is written with $\mu_t(x)([0, \alpha])$, and the source paper [ABB23] states the conjecture as follows.

Conjecture 1.1 ([ABB23, Conjectures 3.1 and 3.2]). *For every graph G , every $x \in V(G)$ and every $\alpha \leq \frac{1}{2}$, the function $t \mapsto \mu_t^x([0, \alpha])$ is non-increasing. Equivalently, for every $p \leq \frac{1}{2}$ the function $t \mapsto \mathbb{P}_p[\xi_t(x) = 1]$ is non-increasing.*

By the symmetry $\eta \mapsto 1 - \eta$ (Lemma 2.2), Conjecture 1.1 is the same statement as Conjecture 1 of [Ami21] read with $\mu_t(x)([0, \alpha])$, and also the same as:

$$\text{for every } G, x \text{ and } p \geq \frac{1}{2}, \text{ the function } t \mapsto f_p(t) \text{ is non-decreasing.} \quad (1)$$

The standing hypothesis of [ABB23, Section 2] is that G is a locally finite, possibly infinite, graph; finite graphs are included, and the complete graph K_N is one of the two cases proved there. Amir, Baldasso and Beilin proved Conjecture 1.1 for K_N [ABB23, Proposition 3.10] and for $\mathbb{Z}$ [ABB23, Proposition 3.11]; for $\mathbb{Z}$ their formula is $f_p(t) = 3p^2 - 2p^3 - e^{-t}p(1-p)(2p-1)$ [ABB23, (3.32)]. As a partial result for general graphs they quote the bound of Mossel, Neeman and Tamuz [MNT14]: $\mu_t^x([0, \alpha]) \leq \alpha$ for $\alpha \leq \frac{1}{2}$ [ABB23, Proposition 3.3], that is, $f_p(t) \geq p$ for $p \geq \frac{1}{2}$. They also showed that a stronger, *quenched* version, in which the update sequence is fixed and only the initial opinions are random, fails on $K_{3,m}$ [ABB23, Proposition 3.6]; they present this as refuting only the stronger version, and Conjecture 1.1, which averages over the clocks (the *annealed* version), was left open. Finally, for the variant in which ties at even degree are broken by a fair coin (“median dynamics with coin flips”, which yields zero-temperature Glauber dynamics), they expect Conjecture 1.1 to hold as well [ABB23, Section 5].

1.2 Results

We show that (1), and hence Conjecture 1.1 and Conjecture 1 of [Ami21], is false, already on small finite connected graphs.

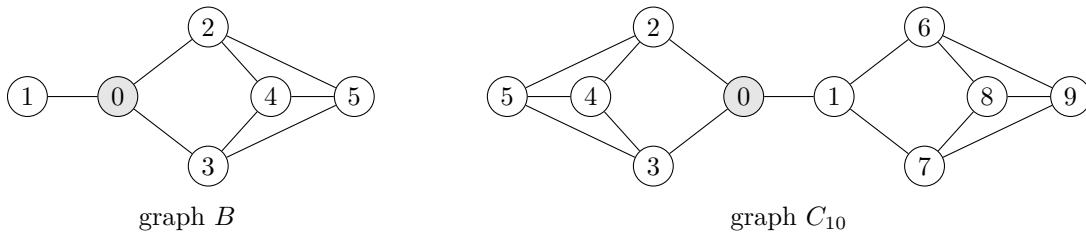


Figure 1: The graphs B and C_{10} ; the distinguished vertex $x = 0$ is shaded. In both, $\{2, 3, 4, 5\}$ spans K_4 minus the edge 23, and its two vertices of degree two in this subgraph, 2 and 3, are joined to x .

The graph B (Figure 1) has vertex set $\{0, \dots, 5\}$ and the eight edges $01, 02, 03, 24, 25, 34, 35, 45$: it is K_4 minus an edge on $\{2, 3, 4, 5\}$, whose two vertices $2, 3$ of degree two in it are joined to $x = 0$, together with a leaf 1 at x . It is connected and planar, and its degrees $(3, 1, 3, 3, 3, 3)$ are all odd. Theorem 3.1 gives f_p at x in closed form for every p (the formula of the abstract), and Corollary 3.2 deduces that for every $p \in (\frac{1}{2}, 1)$, $f_p(t) > f_p(\infty)$ for $t > \frac{16}{9r} + \frac{26}{27}$, where $r = p(1 - p)$, and that f_p is eventually strictly decreasing. Proposition 3.3 certifies one instance ($p = \frac{61}{100}$, $t_0 = \frac{223}{25}$, overshoot $9.39 \cdot 10^{-7}$) by rational bounds which do not use Theorem 3.1. Theorem 4.1 shows, again for every $p \in (\frac{1}{2}, 1)$, that $f_p(t) > f_p(\infty)$ for all large t , with a larger overshoot, on the connected planar 3-regular graph C_{10} of Figure 1, which is not vertex-transitive, and Proposition 5.1 gives an example with even degrees under the rule of [ABB23].

Corollary 1.2. *Conjecture 1 of [Ami21], equivalently Conjecture 1.1, is false. It fails on a finite connected graph, on a finite connected 3-regular graph, and on an infinite, disconnected, 3-regular graph; in each case for every level $\alpha \in (0, \frac{1}{2})$ in the form of Conjecture 1.1 and every $\alpha \in (\frac{1}{2}, 1)$ in the form of [Ami21]; and, since all degrees in B and C_{10} are odd, under every rule for ties at even degree, including the coin-flip variant of [ABB23, Section 5].*

Proof. Corollary 3.2 and Theorem 4.1, Lemma 2.2 and Remark 2.5, and Section 6.2 for the infinite graph. $\square$

The proofs of Theorems 3.1 and 4.1 are finite exact computations, not arguments that can be followed without a computer: for each of the 2^n initial configurations, $t \mapsto \mathbb{E}_s[\xi_t(x)]$ is computed in closed form in exact rational arithmetic (Lemma 2.6), and every coefficient is a polynomial of degree at most n in p . The program `check_note.py` of the package re-does all exact steps in a few seconds with the Python standard library (Section 7).

1.3 What is new, what is known, and what stays open

The question, the conjecture, the model and its conventions, the reduction of median to majority dynamics, the proved cases K_N and $\mathbb{Z}$ and the quenched counterexample are due to Amir, Baldasso and Beilin [ABB23, Ami21]. The energy argument and the uniformization of finite Markov chains used in Section 2 are standard tools. What is new, as far as our searches go (see “Scope and priority” at the end), are the explicit graphs B , C_{10} and A , the exact closed forms and the certificates. The natural corrected version of the conjecture, for vertex-transitive graphs (K_N and $\mathbb{Z}$ are vertex-transitive, our graphs are not), in particular for $\mathbb{Z}^d$ with $d \geq 2$ and for regular trees, is not addressed, and neither is a connected infinite counterexample (Section 6.3).

2 Reductions and the exact computation

Throughout this section $G = (V, E)$ is a finite graph with n vertices, $x \in V$ is fixed, and majority dynamics uses the pools of [ABB23] (own opinion added at even degree). Configurations are $s \in \{0, 1\}^V$; $|s| = \sum_v s(v)$, $\pi_p(s) = p^{|s|}(1 - p)^{n - |s|}$, and s^v is s with the opinion at v changed. A vertex v is *unstable* in s if the majority of its pool differs from $s(v)$; $U(s)$ is the set of unstable vertices. Majority dynamics is the continuous-time Markov chain on $\{0, 1\}^V$ with rate 1 from s to s^v for $v \in U(s)$; rings of stable vertices do not change the state.

2.1 Reductions

Lemma 2.1 (median and majority form). *For $\alpha \in (0, 1)$ and $t \geq 0$, $\mu_t^x([0, \alpha]) = \mu_t^x([0, \alpha]) = f_\alpha(t)$.*

Proof. By [ABB23, Proposition 2.2], whose proof rests on the identity [ABB23, (2.8)] that $\mathbf{1}_{[0,\alpha]}$ commutes with the median of a pool of odd size, $\mathbf{1}_{[0,\alpha]}(\eta_t)$ is majority dynamics started from i.i.d. Bernoulli(α) opinions; hence $\mu_t^x([0,\alpha]) = f_\alpha(t)$. The median of a pool of odd size is one of its entries, so by induction over the finitely many rings in $[0,t]$, $\eta_t(x)$ is one of the initial values $\eta_0(y)$, $y \in V$. Hence $\mathbb{P}[\eta_t(x) = \alpha] \leq \sum_{y \in V} \mathbb{P}[\eta_0(y) = \alpha] = 0$. $\square$

Lemma 2.2 (symmetry). $f_{1-p}(t) = 1 - f_p(t)$ for all $p \in [0, 1]$ and $t \geq 0$. Consequently the following are equivalent for fixed G and x : (i) $t \mapsto \mu_t^x([0, \alpha])$ is non-increasing for every $\alpha \leq \frac{1}{2}$; (ii) $t \mapsto \mu_t^x([0, \beta])$ is non-decreasing for every $\beta \geq \frac{1}{2}$; (iii) f_p is non-decreasing for every $p \geq \frac{1}{2}$.

Proof. Changing every opinion commutes with taking the majority of a pool of odd size. So, with the same clocks, $1 - \xi_t$ is majority dynamics started from $1 - \xi_0$, which is i.i.d. Bernoulli($1 - p$) if ξ_0 is i.i.d. Bernoulli(p). This gives the identity. With Lemma 2.1, $\mu_t^x([0, \alpha]) = f_\alpha(t) = 1 - f_{1-\alpha}(t) = 1 - \mu_t^x([0, 1 - \alpha])$ for $\alpha \in (0, 1)$, and the cases $\alpha = 0$ and $\beta = 1$ are constant functions; this gives the equivalences. $\square$

Lemma 2.3 (fixation and the limit). Assume that every vertex of G has odd degree, or that the pools are those of [ABB23]. Then every flip strictly lowers the number $H(s)$ of edges whose endpoints disagree. Consequently there are at most $|E|$ flips on every path, ξ_t is eventually constant almost surely, and $f_p(t) \rightarrow f_p(\infty) = \sum_s \pi_p(s) h(s)$ as $t \rightarrow \infty$, where $h(s) = s(x)$ if $U(s) = \emptyset$ and $h(s) = |U(s)|^{-1} \sum_{v \in U(s)} h(s^v)$ otherwise.

Proof. Let v have degree d , and let k of its neighbours disagree with v . If d is odd, the pool is $N(v)$ and v is unstable if and only if $k \geq \frac{d+1}{2}$. If d is even, the pool of [ABB23] is $N(v) \cup \{v\}$, of size $d + 1$, and its element v agrees with v ; so v is unstable if and only if $k \geq \frac{d}{2} + 1$. In both cases $k > \frac{d}{2}$. After the flip, v disagrees with exactly $d - k < k$ neighbours, and no other edge changes; so H drops by $2k - d \geq 1$. (Under the coin rule at even degree a vertex with $k = \frac{d}{2}$ may flip, and then H is unchanged; the lemma is used only for B and C_{10} , where all degrees are odd, and for the graph A of Section 5 with the pools of [ABB23].) Since $0 \leq H \leq |E|$, at most $|E|$ flips occur, so ξ_t is eventually constant, and $f_p(t) \rightarrow \mathbb{P}_p[\xi_\infty(x) = 1]$ by bounded convergence. In state s with $U(s) \neq \emptyset$ the next change of state is caused by the first ring among the $|U(s)|$ independent rate-one clocks of the unstable vertices; it occurs after an exponential time of rate $|U(s)|$ and the ringing vertex is uniform on $U(s)$. By the strong Markov property $h(s) = \mathbb{P}_s[\xi_\infty(x) = 1]$ satisfies the stated recursion, which terminates because H decreases along it. $\square$

Lemma 2.4 (an overshoot refutes monotonicity). If f_p is non-decreasing on $[0, \infty)$, then $f_p(t) \leq f_p(\infty)$ for all t . Hence, under the hypothesis of Lemma 2.3, a single $p > \frac{1}{2}$ and t_0 with $f_p(t_0) > f_p(\infty)$ refute (1), and with it Conjecture 1.1 and Conjecture 1 of [Ami21].

Proof. $f_p(t_0) \leq f_p(t)$ for $t \geq t_0$; let $t \rightarrow \infty$. The rest is Lemma 2.2. $\square$

Remark 2.5 (tie rules). If every degree of G is odd, every pool consists of neighbours only and has odd size. Then the rule of [ABB23] (own opinion added at even degree), the coin rule of zero-temperature Glauber dynamics, and the median dynamics with coin flips of [ABB23, Section 5] and [Ami21] all define the same process. This is the case for B and C_{10} .

2.2 Closed forms and certificates

For integers $k \geq 0$ and $i \geq 1$ let $E_{k,i}(t) = \frac{t^{i-1}}{(i-1)!} e^{-kt}$; its Laplace transform is $(z+k)^{-i}$. For $s \in \{0, 1\}^V$ let $g_s(t) = \mathbb{E}_s[\xi_t(x)]$, so that $f_p(t) = \sum_s \pi_p(s) g_s(t)$.

Lemma 2.6 (exact closed forms). Under the hypothesis of Lemma 2.3:

(a) With $u = |U(s)|$ and $m(s) = s(x)$,

$$g_s(t) = m(s)e^{-ut} + \sum_{v \in U(s)} \int_0^t e^{-u\tau} g_{s^v}(t - \tau) d\tau \quad (u \geq 1), \quad g_s \equiv m(s) \quad (u = 0).$$

(b) Every g_s is a finite sum $\sum_{k,i} c_s[k,i] E_{k,i}$ with $0 \leq k \leq n$, whose rational coefficients $c_s[k,i]$ do not depend on p . They are computed from (a) in order of increasing $H(s)$, using $E_{k,i} * E_{u,1} = E_{u,i+1}$ for $k = u$ and, for $k \neq u$ and $d = k - u$,

$$E_{k,i} * E_{u,1} = d^{-i} E_{u,1} - \sum_{l=1}^i d^{-(i+1-l)} E_{k,l}, \quad (2)$$

where $*$ is convolution on $[0, \infty)$.

(c) $f_p(t) = \sum_{k,i} c_{k,i}(p) E_{k,i}(t)$ with $c_{k,i}(p) = \sum_s \pi_p(s) c_s[k,i]$, a polynomial of degree at most n in p with rational coefficients. In particular, a claimed identity $c_{k,i}(p) = P(p)$ with a polynomial P of degree at most n holds for all p as soon as it holds at $n+1$ distinct values of p , and a coefficient that vanishes at $n+1$ distinct values of p vanishes identically.

Proof. (a) If none of the clocks of $U(s)$ rings in $[0, t]$, which has probability e^{-ut} , the state stays s . Otherwise the first such ring occurs at a time τ with density $ue^{-u\tau}$, at a vertex v uniform on $U(s)$ and independent of τ , and the state becomes s^v ; apply the strong Markov property. (b) In Laplace transforms (2) is the partial fraction decomposition of $(z+u)^{-1}(z+k)^{-i}$, which follows by induction on i from $\frac{1}{(z+u)(z+k)} = \frac{1}{d} \left(\frac{1}{z+u} - \frac{1}{z+k} \right)$; both sides of (2) are continuous, so they are equal. By Lemma 2.3, $H(s^v) < H(s)$ for $v \in U(s)$, so (a) determines all g_s recursively, starting from the configurations without unstable vertex; the exponents stay in $\{0, \dots, n\}$ since $u \leq n$. No p enters. (c) The first statement is immediate, and $\pi_p(s)$ has degree n in p . A polynomial of degree at most n with $n+1$ zeros is zero. $\square$

Lemma 2.7 (uniformization bounds). *Let Q be the generator of the chain, $\Lambda \geq \max_s |U(s)|$ (we take $\Lambda = n$), $P = I + Q/\Lambda$, which is a stochastic matrix with nonnegative entries, and $a = \Lambda t$. Let $m_k = \pi_p P^k \mathbf{1}_{\{s(x)=1\}} \in [0, 1]$ and, for integers K, J with $K+2 > a$ and $J+2 > a$,*

$$S_K = \sum_{k=0}^K \frac{a^k}{k!} m_k, \quad T_K = \frac{a^{K+1}/(K+1)!}{1 - a/(K+2)}, \quad L_J = \sum_{j=0}^J \frac{a^j}{j!}, \quad U_J = L_J + T_J.$$

Then $S_K/U_J \leq f_p(t) \leq (S_K + T_K)/L_J$. For rational p and t all these numbers are rational.

Proof. $e^{tQ} = e^{-a} e^{aP}$, so $f_p(t) = e^{-a} \sum_{k \geq 0} \frac{a^k}{k!} m_k$ with all terms nonnegative and $m_k \leq 1$. The tail of the exponential series after the index K is at most $\frac{a^{K+1}}{(K+1)!} \sum_{j \geq 0} \left(\frac{a}{K+2} \right)^j = T_K$, so $L_J \leq e^a \leq U_J$, and $e^{-a} S_K \leq f_p(t) \leq e^{-a} (S_K + T_K)$. $\square$

3 The graph B

Let B be the graph of Figure 1: $V = \{0, \dots, 5\}$, $E = \{01, 02, 03, 24, 25, 34, 35, 45\}$, $x = 0$. All degrees are odd, so by Remark 2.5 no tie rule enters. Write $r = p(1-p)$ and $w = 2p-1$.

Theorem 3.1. *For majority dynamics on B and every $p \in [0, 1]$,*

$$f_p(\infty) = p + rw\left(1 + \frac{14}{75}r\right) = \frac{1}{75}p^2(28p^3 - 70p^2 - 94p + 211), \quad (3)$$

$$f_p(t) - f_p(\infty) = rw e^{-t} \left[\frac{9}{16}rt - \frac{4+r}{4} - \frac{7}{24}re^{-2t} + r\left(\frac{71}{200} + \frac{7t}{80}\right)e^{-4t} \right]. \quad (4)$$

In the notation of Lemma 2.6(c), the nonzero coefficients are $c_{0,1} = f_p(\infty)$, $c_{1,1} = -\frac{1}{4}rw(4+r)$, $c_{1,2} = \frac{9}{16}r^2w$, $c_{3,1} = -\frac{7}{24}r^2w$, $c_{5,1} = \frac{71}{200}r^2w$ and $c_{5,2} = \frac{7}{80}r^2w$.

Proof. This is a finite exact computation. By Lemma 2.6(b) the 64 functions g_s are computed in exact rational arithmetic. Together they involve the six pairs $(k, i) \in \{(0, 1), (1, 1), (1, 2), (3, 1), (5, 1), (5, 2)\}$ and also $(4, 1)$ and $(6, 1)$: 16 of the configurations have a term in e^{-4t} , and two of them a term in e^{-6t} . These two pairs cancel in f_p : for every j , the sums of $c_s[4, 1]$ and of $c_s[6, 1]$ over the configurations with $|s| = j$ are zero. By Lemma 2.6(c) every coefficient $c_{k,i}(p)$ is a polynomial of degree at most 6 in p , and so is each claimed coefficient (they have degree 5). The computed coefficients agree with the claimed ones at the seven values $p = \frac{j}{8}$, $1 \leq j \leq 7$, and those of $(4, 1)$ and $(6, 1)$ vanish there; hence the six claimed identities hold for all p , and all other coefficients vanish identically. The two expressions in (3) are equal polynomials of degree 5. As a check, the bracket in (4) equals $-1 - \frac{14}{75}r$ at $t = 0$, so that $f_p(0) = p$; the absorption recursion of Lemma 2.3 gives the same $f_p(\infty)$; and the program `check_note.py` also forms the polynomials $\sum_j C_j[k, i] p^j (1-p)^{6-j}$ directly, where $C_j[k, i]$ is the sum of $c_s[k, i]$ over $|s| = j$, and finds the same six polynomials. $\square$

Corollary 3.2. *For every $p \in (\frac{1}{2}, 1)$ and every $t > \frac{16}{9r} + \frac{26}{27}$, $f_p(t) > f_p(\infty)$. Moreover $f'_p(0) = rw > 0$, and $f'_p(t) < 0$ for all sufficiently large t . Hence for every $p \in (\frac{1}{2}, 1)$ the function f_p is not non-decreasing, and at the vertex 0 of B , for every $\alpha \in (\frac{1}{2}, 1)$, $t \mapsto \mu_t^0([0, \alpha])$ is not non-decreasing, and for every $\alpha \in (0, \frac{1}{2})$, $t \mapsto \mu_t^0([0, \alpha])$ is not non-increasing.*

Proof. For $p \in (\frac{1}{2}, 1)$ we have $rw > 0$. In the bracket of (4) the last term is positive and $\frac{7}{24}re^{-2t} \leq \frac{7}{24}r$, so the bracket exceeds $\frac{9}{16}rt - 1 - \frac{r}{4} - \frac{7r}{24} = \frac{9}{16}rt - 1 - \frac{13}{24}r$, which is positive if and only if $t > \frac{16}{9r} + \frac{16 \cdot 13}{9 \cdot 24} = \frac{16}{9r} + \frac{26}{27}$. Differentiating (4),

$$f'_p(t) = rw e^{-t} \left[-\frac{9}{16}rt + \frac{4+r}{4} + \frac{9}{16}r + \frac{7}{8}re^{-2t} + r\left(\frac{7}{80} - \frac{71}{40} - \frac{7t}{16}\right)e^{-4t} \right];$$

at $t = 0$ the bracket is 1, and as $t \rightarrow \infty$ it tends to $-\infty$. The last statements follow from Lemmas 2.1, 2.2 and 2.4. $\square$

Since $r \leq \frac{1}{4}$, the threshold is at least $\frac{64}{9} + \frac{26}{27} > 8.07$; it is $\frac{60206}{7137} = 8.43575\dots$ for $p = \frac{61}{100}$. So f_p increases at first, consistently with the bound $f_p(t) \geq p$ of [MNT14] ([ABB23, Proposition 3.3]), rises above its limit and approaches it from above. The overshoot on B is small; the next statement certifies one instance without using Theorem 3.1.

Proposition 3.3. *Let $p = \frac{61}{100}$ and $t_0 = \frac{223}{25}$. On B , $f_p(\infty) = \frac{41541389119}{62500000000} = 0.664662225904$, and Lemma 2.7 with $\Lambda = 6$ and $K = J = 220$ gives rational numbers $L \leq f_p(t_0) \leq L'$ with $L' - L < 3 \cdot 10^{-65}$ and*

$$L - f_p(\infty) = 9.3891186962\dots \cdot 10^{-7} > 0.$$

Proof. The value of $f_p(\infty)$ is computed exactly by the absorption recursion of Lemma 2.3, and L, L' exactly by Lemma 2.7 ($a = \frac{1338}{25} < 222$); no floating-point number enters. The margin $L - f_p(\infty)$ is small, but it is a difference of two exactly known rationals. The enclosure agrees with Theorem 3.1 at t_0 . $\square$

4 A cubic counterexample

Let C_{10} be the graph of Figure 1: $V = \{0, \dots, 9\}$ and the 15 edges 01, 02, 03, 16, 17, 24, 25, 34, 35, 45, 68, 69, 78, 79, 89, with $x = 0$. It consists of an edge 01, a copy of K_4 minus the edge 23 on $\{2, 3, 4, 5\}$ whose vertices 2, 3 are joined to 0, and a copy of K_4 minus the edge 67 on $\{6, 7, 8, 9\}$ whose vertices 6, 7 are joined to 1; in other words, B with its leaf replaced by a mirror image of x with its own diamond. C_{10} is connected, planar and 3-regular; its automorphism group has order 32, with vertex orbits $\{0, 1\}$, $\{2, 3, 6, 7\}$, $\{4, 5, 8, 9\}$, so it is not vertex-transitive. Again $r = p(1 - p)$ and $w = 2p - 1$.

Theorem 4.1. *For majority dynamics on C_{10} :*

(a) For $p = \frac{7}{10}$, $f(\infty) = \frac{90794489017687}{115200000000000} = 0.78814660606\dots$, and Lemma 2.7 with $\Lambda = 10$ and $K = J = 150$ gives $f(3) \geq f(\infty) + 3.474 \cdot 10^{-3}$.

(b) For every $p \in [0, 1]$, $f_p(t) = \sum c_{k,i}(p) E_{k,i}(t)$, where the sum runs over the 19 pairs (k, i) of Table 1 (so $k \leq 10$ and $i \leq 3$), $c_{0,1}(p) = f_p(\infty) = p + wQ_{0,1}(r)$ and $c_{k,i}(p) = wQ_{k,i}(r)$ for $k \geq 1$, with the polynomials $Q_{k,i}$ of Table 1. For $k = 1$ only $i = 1, 2$ occur, and

$$\begin{aligned} c_{1,2}(p) &= \frac{r^2 w (80905r^2 + 5355504r + 5556600)}{1587600} \\ &= \frac{1}{1587600} p^2 (p-1)^2 (2p-1) (80905p^4 - 161810p^3 - 5274599p^2 + 5355504p + 5556600). \end{aligned} \quad (5)$$

(c) For every $p \in (\frac{1}{2}, 1)$, $(f_p(t) - f_p(\infty)) e^t / t \rightarrow c_{1,2}(p) > 0$ as $t \rightarrow \infty$; so $f_p(t) > f_p(\infty)$ for all sufficiently large t , and f_p is not non-decreasing.

Proof. (a) As in Proposition 3.3: $f(\infty)$ by the absorption recursion of Lemma 2.3 over the 1024 configurations, and the lower bound by Lemma 2.7 ($a = 30$), in exact rational arithmetic; the two-sided enclosure has width below $3 \cdot 10^{-55}$ and gives $f(3) - f(\infty) = 3.4740871122\dots \cdot 10^{-3}$.

(b) By Lemma 2.6(c), $c_{k,i}(p) = \sum_s \pi_p(s) c_s[k, i]$ with p -independent rationals $c_s[k, i]$, so each $c_{k,i}$ is a polynomial of degree at most 10 in p ; each entry of Table 1, as a function of p , has degree at most 9. The exact coefficients computed at the 11 distinct values $p = \frac{j}{12}$, $1 \leq j \leq 11$, agree with the polynomials of Table 1; hence every identity in (b) holds for all p . At these 11 values no pair (k, i) outside the table occurs, so all other coefficients, in particular $c_{1,i}$ for $i \geq 3$, vanish at 11 points and hence identically. (The functions g_s also contain the pair $(8, 3)$, whose coefficient in f_p is one of these; no g_s contains a term $t^2 e^{-t}$.) The two forms of (5) agree because $p^2(p-1)^2 = r^2$ and $80905(p^4 - 2p^3) - 5274599p^2 + 5355504p = 80905(r^2 - p^2) - 5274599p^2 + 5355504p = 80905r^2 + 5355504r$. The polynomials were also computed directly as $\sum_j C_j[k, i] p^j (1-p)^{10-j}$, without interpolation, with the same result.

(c) By (b), $f_p(t) - f_p(\infty) = c_{1,1}(p) e^{-t} + c_{1,2}(p) t e^{-t} + O(t^2 e^{-2t})$. For $p \in (\frac{1}{2}, 1)$ we have $r > 0$ and $w > 0$, and the quadratic $80905r^2 + 5355504r + 5556600$ has positive coefficients; so $c_{1,2}(p) > 0$. Apply Lemma 2.4. $\square$

Evaluated in floating point, the closed form gives a maximal overshoot $\max_t (f_p(t) - f_p(\infty))$ of about $6.2 \cdot 10^{-4}$, $1.5 \cdot 10^{-3}$, $3.47 \cdot 10^{-3}$ and $1.9 \cdot 10^{-3}$ for $p = 0.52, 0.55, 0.7, 0.8$.

5 An example with even degrees

The following example is an addition. Unlike B and C_{10} it depends on the even-degree rule of [ABB23] (own opinion added to the pool); we have not examined it under the coin rule, for which Lemma 2.3 fails. Let A have vertex set $\{0, \dots, 5\}$ and the ten edges 01, 02, 13, 14, 15, 23, 24, 25, 35, 45, with $x = 0$; the degrees are $(2, 4, 4, 3, 3, 4)$, and x has degree 2.

Proposition 5.1. *For majority dynamics on A with the pools of [ABB23] and every $p \in [0, 1]$, with $r = p(1 - p)$ and $w = 2p - 1$,*

$$f_p(\infty) = p + rw\left(1 + \frac{83}{50}r\right),$$

$$f_p(t) - f_p(\infty) = rw e^{-t} \left[\frac{2}{3}rt^2 - \frac{23}{18}rt - 1 - \frac{385}{216}r \right] - \frac{5}{54}r^2w e^{-4t} + r^2w \left(\frac{43}{200} + \frac{t}{5} \right) e^{-5t}.$$

Hence for every $p \in (\frac{1}{2}, 1)$ the leading term of $f_p(t) - f_p(\infty)$ is $\frac{2}{3}r^2w t^2 e^{-t} > 0$, and $f_p(t) > f_p(\infty)$ for all large t . For $p = \frac{7}{10}$ and $t_0 = \frac{27}{5}$, Lemma 2.7 with $\Lambda = 6$ and $K = J = 160$ gives $f(t_0) - f(\infty) \geq 4.776 \cdot 10^{-4}$, where $f(\infty) = \frac{1016603}{1250000}$.

Proof. As for Theorem 3.1: Lemma 2.3 holds with the pools of [ABB23]; the functions g_s involve the pairs $(0, 1)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(4, 1)$, $(5, 1)$ and $(5, 2)$ and also $(3, 1)$, whose coefficient in f_p vanishes (its sums over the configurations with $|s| = j$ are zero for every j). Every coefficient is a polynomial of degree at most 6 in p , and the computed coefficients agree with the claimed ones, and that of $(3, 1)$ with zero, at $p = \frac{j}{8}$, $1 \leq j \leq 7$. The instance is certified as in Proposition 3.3. $\square$

6 Discussion

6.1 A heuristic picture

The following description is a heuristic; it is not used in any proof. The terms with $k = 1$, which dominate $f_p(t) - f_p(\infty)$ for large t , come from configurations with exactly one unstable vertex, which persist for an exponential time of rate one. In B these are configurations in which x and its leaf, or x and one of the vertices 2, 3, hold an opinion that the rest of the diamond $\{2, 3, 4, 5\}$ has abandoned. For example, if x , its leaf and the vertex 2 hold 1 and 3, 4, 5 hold 0, only 2 is unstable; after it flips, only x is; so x keeps the opinion 1 for a time distributed as the sum of two independent exponential times of rate one, which contributes $(1 + t)e^{-t}$, and x ends with 0. Such configurations and their complements contribute to the coefficient $c_{1,2}$ with both signs; for $p > \frac{1}{2}$ the weighted sum is positive, so at late times x holds the opinion 1 more often than in the limit, and f_p approaches its limit from above. On $\mathbb{Z}$ and on K_N every vertex fixates at its first ring [ABB23, Section 3.1], and such delayed reversals do not occur.

6.2 Infinite graphs

Let G_∞ be the disjoint union of countably many copies of C_{10} . It is an infinite, locally finite, 3-regular graph, on which median dynamics is well defined [ABB23, Proposition 2.1]; it consists of independent copies of the finite process. The law of $\eta_t(x)$ for x in one copy depends only on that copy, so Theorem 4.1 shows that the conjecture fails on G_∞ . We do not have a connected infinite counterexample.

6.3 What stays open, and what was only tested

The examples do not touch vertex-transitive graphs. Both cases proved in [ABB23], K_N and $\mathbb{Z}$, are vertex-transitive, while B , C_{10} and A are not. The conjecture for vertex-transitive graphs, in particular for $\mathbb{Z}^d$ with $d \geq 2$ and for the regular trees, is a natural corrected version; we do not address it and make no conjecture about its answer. A connected infinite counterexample, and a certified counterexample among regular graphs of degree at least four, are also missing.

The following floating-point experiments are evidence only and prove nothing (a “violation” is an overshoot over the limit or a negative increment above 10^{-10} on a grid of $t \in [0, 30]$ and five values

of $p \in [0.55, 0.8]$, at all vertices). Violations were seen for 28 of the $13 + 224$ connected graphs with all degrees odd on 6 and 8 vertices, and for 0 of 5, 3 of 19 and 16 of 85 cubic graphs on 8, 10 and 12 vertices (largest overshoot about $8.7 \cdot 10^{-3}$; none of these graphs is vertex-transitive). Among the 2, 6, 16 and 59 connected 4-regular graphs (rule of [ABB23]) on 7, 8, 9 and 10 vertices, one graph on 10 vertices showed an apparent overshoot of about $7 \cdot 10^{-6}$, which we have not certified and do not claim; the 3 connected 5-regular graphs on 8 vertices and 59 of the 60 on 10 vertices (those found by random sampling) showed none. No violation was seen on 62 vertex-transitive graphs (among them all connected circulants of degree 3 to 5 on at most 14 vertices and the Petersen and Heawood graphs) and on three cubic vertex-transitive graphs with 18 vertices, and a grid scan of all connected graphs on at most 7 vertices found no counterexample on at most 5 vertices. None of this shows that B has the smallest possible order, or supports a conjecture for the vertex-transitive class.

7 Computations and reproducibility

Theorems 3.1 and 4.1 and Proposition 5.1 are proved by finite exact computations; Proposition 3.3 and Theorem 4.1(a) are certificates which use only Lemmas 2.3 and 2.7. Everything else is proved in the text. The reproducibility package has three parts.

The exact program of this note (`check_note.py`, Python standard library, about 3 seconds) implements Lemmas 2.3, 2.6 and 2.7 with rational numbers only. It checks the graph data (degrees, connectedness, automorphism groups and orbits); the energy decrease of Lemma 2.3 (and that under the coin rule A would have flips with H unchanged); the limits $f(\infty)$ by the absorption recursion; Theorem 3.1, Proposition 5.1 and all entries of Table 1 at the interpolation nodes and as directly expanded polynomials, the absence of $c_{1,i}$ for $i \geq 3$ on C_{10} , the identity in (5) and the arithmetic of Corollary 3.2; the two-sided enclosures of Lemma 2.7 for B , C_{10} and A (widths below $3 \cdot 10^{-65}$, $3 \cdot 10^{-55}$ and 10^{-57}); the formula [ABB23, (3.7)] for K_5 and the symmetry $f_{1-p} = 1 - f_p$; in floating point, the bound $f_p(t) \geq p$ of [MNT14] on a grid; and, as a control, that B without the edge 45 has $c_{1,2} < 0$ and no overshoot at $(\frac{61}{100}, \frac{223}{25})$. All checks pass.

The programs with which the results were first obtained (`original/`): exact certificate programs for B , C_{10} and A , the all- p proofs for B and C_{10} , a derivation of Theorem 3.1 with p symbolic in a computer algebra system, floating-point cross-checks of the closed forms against a matrix exponential (agreement to $6 \cdot 10^{-15}$ for B and $9 \cdot 10^{-15}$ for C_{10}), and the floating-point searches of Section 6.3, with their recorded outputs.

The programs of the second verification round (`independent_run_2/`, with their outputs): exact closed forms in the monomial basis with p symbolic, two exact certificates for each instance, the graph data and edge lists, the mutation test, and a Monte Carlo simulation (see the Verification paragraph below).

Verification. The results were first obtained with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first run compared the statement with the texts of [Ami21, ABB23], re-derived the reductions, the recursion, the partial fractions (2), the uniformization bound, the polynomial argument and the arithmetic of Corollary 3.2 and Theorem 4.1; re-derived Theorem 3.1 with p as a symbol; recomputed the coefficients of B at 13 rational values of p , those of C_{10} at $p = \frac{7}{10}$ and $c_{1,2}$ of C_{10} at 13 values by a third method, truncated Laurent expansions of the Laplace transforms at their poles; and recomputed $f(\infty)$, its own uniformization bounds and matrix exponentials to 30 digits for A , B and C_{10} . The second run computed all coefficient polynomials of B and C_{10} directly in p , without interpolation, and found them identical to ours; computed two-sided rational enclosures of $f(t_0)$ for B (width about $1.5 \cdot 10^{-65}$) and for C_{10} ; checked the model by a Monte Carlo simulation of the raw clock process, in which the

overshoot of C_{10} at $p = 0.7$, $t = 3$ appears at about 38 standard errors (that of B is far below Monte Carlo resolution); found, as a mutation test, that B without the edge 45 shows no overshoot at the same (p, t_0) ; scanned all connected graphs on at most 7 vertices; and ran the certificate programs of `original/` again, which reproduced their recorded outputs byte for byte. Both runs confirmed every claim. Their required corrections, which concern the wording of the proofs of Lemma 2.3 and Theorem 4.1(b), the page of Conjecture 1 in [Ami21], two references and the statements of scope, were made. The exact program of this note, `check_note.py`, was written afterwards and re-does every exact step, including the two-sided enclosures.

A further independent verification run, also AI-assisted, then checked the text of this note with programs of its own (`independent_run_2/` of the package). It read the sources again (Amir’s abstract in [Ami21], [ABB23] in arXiv v2, and the Crossref records of the four references); computed the closed forms of B , C_{10} and A in the basis $t^j e^{-kt}$ and directly as polynomials in a symbolic p , and found them identical to Theorem 3.1, Theorem 4.1(b), Table 1 and Proposition 5.1, including all constants used in Corollary 3.2; certified the overshoots of Proposition 3.3, Theorem 4.1(a) and Proposition 5.1 by two exact methods of its own, rational bounds for the exponentials in its closed forms and uniformization in integer arithmetic; checked the partial fractions (2), the widths stated for Lemma 2.7, the graph data and the edge lists of the text, the figure and the programs; ran the mutation test with B without the edge 45 and a Monte Carlo simulation of the raw clock process on C_{10} (overshoot at about 22 standard errors); and ran `check_note.py` and the certificate programs of `original/` again, with identical outputs. It confirmed every statement. Its required corrections were made: the proofs of Theorem 3.1 and Proposition 5.1 had said that the functions g_s involve only the pairs of the final formula, whereas some of them contain further pairs that cancel in f_p ; a decimal value after Corollary 3.2 was misprinted; and the coverage of the searches of Section 6.3 and the description of C_{10} in the abstract were made precise.

Scope and priority. The note answers Conjecture 1 of [Ami21], equivalently Conjectures 3.1 and 3.2 of [ABB23], in the negative, in the form stated there, for all locally finite graphs: since the conjecture is stated for every graph and the standing hypothesis of [ABB23] includes finite graphs, finite counterexamples suffice. The counterexamples are finite and connected, one of them is 3-regular, and they do not depend on the tie rule or on the level α ; for infinite graphs only disconnected counterexamples follow. The vertex-transitive case, in particular $\mathbb{Z}^d$ ($d \geq 2$) and regular trees, connected infinite graphs, and Conjecture 2 of [Ami21] (Conjectures 3.7 and 3.8 of [ABB23]) are not addressed. The question, the conjecture, the model and its conventions [ABB23, (2.2)–(2.3)], the reduction to majority dynamics [ABB23, Proposition 2.2 via (2.8)], the proved cases K_N and $\mathbb{Z}$ [ABB23, Propositions 3.10 and 3.11] and the quenched counterexample on $K_{3,m}$ [ABB23, Proposition 3.6] are due to Amir, Baldasso and Beilin. The bound $f_p(t) \geq p$ for $p \geq \frac{1}{2}$ is due to Mossel, Neeman and Tamuz [MNT14] ([ABB23, Proposition 3.3]) and is used here only as a consistency check. Median dynamics was introduced by Damron and Sen [DS20], who studied it on the 3-regular tree in connection with fixation, not the monotonicity of marginals in time. The energy argument and uniformization are standard. Only the graphs B , C_{10} and A , the closed forms and the certificates are, to our knowledge, not in print.

What was read: [ABB23] in the arXiv version v2 of 2 January 2023, whose arXiv record gives the journal reference; our numbering of its conjectures, propositions and equations is that of v2, and the journal version was not read. Amir’s abstract in [Ami21] (pp. 171–174) was read in full. Of [DS20] the arXiv abstract was read; [MNT14] was not read, and what we say about it is taken from [ABB23].

The searches were made on 20 October 2026, in two rounds: queries to the arXiv API for median dynamics and the median process, for majority dynamics together with monotonicity, marginals or conjectures, for zero-temperature Glauber dynamics with monotonicity, and for later papers of the

three proposers; the lists of works citing [ABB23] in Semantic Scholar (7 works) and OpenAlex (6, the same works); Crossref, which records 4 citations of the journal version; the zbMATH Open review of [ABB23] (Zbl 1515.60299), which presents the conjectures as open, with the partial results for K_n and $\mathbb{Z}$, and a zbMATH Open search for median and majority dynamics; eight web searches; and the list of problems in the mathematics release published by OpenAI in October 2026, which does not contain this conjecture. No printed answer to the question was found in these sources; by title and abstract, none of the citing works concerns the monotonicity of marginals in time. A search that finds nothing is not a proof of novelty; theses and talks were not checked, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A The closed form for C_{10}

Table 1 lists the coefficients of Theorem 4.1(b): $f_p(t) = \sum c_{k,i}(p) \frac{t^{i-1}}{(i-1)!} e^{-kt}$ at the vertex 0 of C_{10} , with $c_{0,1}(p) = p + wQ_{0,1}(r)$ and $c_{k,i}(p) = wQ_{k,i}(r)$ for $k \geq 1$, where $r = p(1-p)$ and $w = 2p - 1$. All coefficients with $k \geq 1$ are odd under $p \mapsto 1-p$, as Lemma 2.2 requires. At $p = \frac{7}{10}$, for instance, $c_{1,2} = \frac{1485516389}{20000000000}$.

(k, i)	$Q_{k,i}(r)$	(k, i)	$Q_{k,i}(r)$
$(0, 1)$	$\frac{r(738487r^3 + 3115584r^2 + 8087040r + 37324800)}{37324800}$		
$(1, 1)$	$-\frac{r(25075775r^3 + 1279240452r^2 + 753613875r + 250047000)}{250047000}$		
$(1, 2)$	$\frac{r^2(80905r^2 + 5355504r + 5556600)}{1587600}$	$(2, 1)$	$\frac{r^2(212089r^2 + 21344400r + 8467200)}{4233600}$
$(2, 2)$	$\frac{r^3(420 - 17r)}{560}$	$(3, 1)$	$\frac{5r^2(2r^2 + 54r + 27)}{162}$
$(4, 1)$	$\frac{r^2(1620701r^2 + 27945000r + 5400000)}{38880000}$	$(4, 2)$	$-\frac{r^2(12929r^2 + 2538000r + 432000)}{1296000}$
$(4, 3)$	$-\frac{r^3(10800 - 1151r)}{21600}$	$(5, 1)$	$\frac{r^2(2101r^2 - 24264r - 1890)}{10800}$
$(5, 2)$	$-\frac{7r^3}{5}$	$(6, 1)$	$-\frac{r^3(14050r + 11979)}{81000}$
$(6, 2)$	$-\frac{r^3(255r + 26)}{600}$	$(7, 1)$	$-\frac{433r^4}{6075}$
$(8, 1)$	$-\frac{736145r^4}{6322176}$	$(8, 2)$	$\frac{8335r^4}{112896}$
$(9, 1)$	$\frac{31655083r^4}{357210000}$	$(9, 2)$	$\frac{47057r^4}{1134000}$
$(10, 1)$	$\frac{11677r^4}{2332800}$		

Table 1: The 19 coefficients of the closed form for C_{10} (Theorem 4.1(b)).

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