

# The Klee–Novik Complexes $B(i, d)$ Triangulate $S^i \times B^{d-i-1}$

Alper Ferudun\*

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## Abstract

For  $0 \leq i \leq d - 2$ , Klee and Novik defined  $B(i, d)$  as the subcomplex of the boundary of the  $d$ -dimensional cross-polytope generated by the facets whose  $xy$ -words have at most  $i$  switches, and asked whether  $B(i, d)$  is a combinatorial triangulation of  $S^i \times B^{d-i-1}$ . We show that it is, for all  $0 \leq i \leq d - 2$ . Klee and Novik observed that  $B(i, d)$  collapses onto the boundary of the  $(i + 1)$ -dimensional cross-polytope and noted that it is therefore a disc bundle over  $S^i$ . We show that  $|B(i, d)|$  is in fact a product. That sphere is a join factor of the boundary of the  $d$ -dimensional cross-polytope, so it has a regular neighbourhood that is a product, and  $|B(i, d)|$  together with an outer collar is another regular neighbourhood of it; uniqueness of regular neighbourhoods gives the result. Consequently  $\partial B(i, d)$  is PL homeomorphic to  $S^i \times S^{d-i-2}$ , and a conjecture of Cohen, Klee and Pannell holds. This is an unrefereed note.

## 1 Introduction

Let  $C_d^*$  be the boundary complex of the  $d$ -dimensional cross-polytope on the vertices  $x_1, y_1, \dots, x_d, y_d$ , where  $x_j$  and  $y_j$  are antipodal. Its facets correspond to words  $u = u_1 \cdots u_d$  in the letters  $x, y$ , and the *switch set* of  $u$  is  $S(u) = \{j \in [d - 1] : u_j \neq u_{j+1}\}$ . Klee and Novik [KN12] defined  $B(i, d)$  as the subcomplex of  $C_d^*$  generated by the facets with at most  $i$  switches, and  $C(i, d)$  as the subcomplex generated by the other facets. For  $0 \leq i \leq d - 2$  they proved that  $B(i, d)$  is a combinatorial manifold with boundary with the homology of  $S^i$ , that  $C(i, d) \cong B(d - i - 2, d)$ , and that  $\partial B(i, d)$  is homeomorphic to  $S^i \times S^{d-i-2}$  [KN12, Thm. 1.2]. Klee’s Oberwolfach abstract on this joint work with Novik, which has the same title as [KN12], ends with the following question and calls it “very natural (yet strangely elusive)” [Kle11, p. 372].

**Problem 1.1** ([Kle11, Question 4]). *Is  $B(i, d)$  a combinatorial triangulation of  $S^i \times B^{d-i-1}$ , that is, is  $|B(i, d)|$  PL homeomorphic to  $S^i \times B^{d-i-1}$ ?*

This is record OWR-4798-013 of the UnsolvedMath dataset [ula26]. The case  $i = 0$  is trivial, and the case  $i = 1$  is in the proof of [KN12, Thm. 1.2(e)]; Wang and Zheng [WZ20, Lemma 2.2] record that  $|B(0, d)|$  and  $|B(1, d)|$  are homeomorphic to  $B^{d-1} \times S^0$  and  $B^{d-2} \times S^1$ . Cohen, Klee and Pannell [CKP14] treated  $i = d - 2$  and the complexes  $B(1, 4)$ ,  $B(1, 5)$  by explicit moves.

**Theorem 1.2.** *For  $d \geq 2$  and  $0 \leq i \leq d - 2$ ,  $|B(i, d)|$  is PL homeomorphic to  $S^i \times B^{d-i-1}$ . So the answer to Problem 1.1 is yes.*

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\*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.  
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Klee and Novik remark [KN12, Remark 3.7] that  $B(i, d)$  collapses onto  $B(i, i+1) = C_{i+1}^*$  and that the results of [RS72, Ch. 3] then make  $B(i, d)$  a disc bundle over  $S^i$ . What we add is that  $|B(i, d)|$  is in fact a product, because  $C_d^* \cong C_{i+1}^* * C_{d-i-1}^*$  is a join. Every vertex of  $C_d^*$  lies in  $\partial B(i, d)$  (Remark 3.4), so  $|B(i, d)|$  itself is not a regular neighbourhood of  $|C_{i+1}^*|$ . But  $|B(i, d)|$  together with an outer collar, which exists because  $C(i, d) \cong B(d-i-2, d)$  is again a combinatorial manifold, is a regular neighbourhood of the join factor  $|C_{i+1}^*|$  in the sphere  $|C_d^*|$ , and the standard regular neighbourhood of a join factor is a product. For  $i \leq d-4$  one could use Zeeman's unknotting theorem [Zee63] instead; the join structure also covers codimensions one and two.

**Corollary 1.3.** *Let  $d \geq 2$  and  $0 \leq i \leq d-2$ .*

- (a)  $\partial B(i, d)$  is a combinatorial triangulation of  $S^i \times S^{d-i-2}$ .
- (b)  $|B(i, d)|$  is PL homeomorphic to  $|ST(i+1, d-i-1)|$ , where  $ST(m, n)$  is the staircase triangulation of  $\partial\sigma^m \times \sigma^n$  of [CKP14], and  $B(i, d)$  can be obtained from  $ST(i+1, d-i-1)$  by elementary shellings, inverse shellings and a simplicial isomorphism.

Part (a) upgrades [KN12, Thm. 1.2(e)] to a PL statement. Part (b) confirms the conjecture of [CKP14] that the two polyhedra are homeomorphic. It also answers their Question 1.1, which asks whether  $B(k, d)$  can be obtained from  $ST(k+1, d-k-1)$  by bistellar moves, stellar exchanges, elementary shellings and their inverses, but only existentially: the sequence of moves comes from the theorem of Newman and Pachner, it is not constructed, and it includes a simplicial isomorphism.

**Sources.** Statement numbers for [KN12] are those of arXiv:1102.0542; they agree with the authors' revised version of July 2011, and we could not access the published version. Section and statement numbers for [CKP14] are those of the preprint dated August 18, 2011, and those for [WZ20] are those of arXiv:1811.08505v2. The book [RS72] was not available to us, so we cite it by chapter, without statement numbers, and state in Facts 2.1 and 2.2 what we use.

## 2 Preliminaries

Complexes are finite,  $|K|$  is the polyhedron of  $K$ , and  $\text{lk}_K \sigma$  and  $\text{st}_K \sigma = \{\tau \in K : \tau \cup \sigma \in K\}$  are the link and the closed star of a face  $\sigma$ . A *combinatorial  $n$ -manifold* is a pure  $n$ -dimensional complex in which the link of every nonempty face  $F$  is a PL ball or sphere of dimension  $n - |F|$ . Its boundary  $\partial K$  is generated by the  $(n-1)$ -faces that lie in exactly one facet, and  $|\partial K| = \partial|K|$ . A *shelling* of a pure complex is an order  $F_1, \dots, F_s$  of its facets such that the faces of  $F_j$  that are not faces of earlier facets form an interval  $[R(F_j), F_j] = \{G : R(F_j) \subseteq G \subseteq F_j\}$ ;  $R(F_j)$  is the *restriction* of  $F_j$ . An *elementary collapse* removes faces  $H \subset H'$  with  $|H'| = |H| + 1$  when  $H'$  is the only face properly containing  $H$ ;  $K \searrow L$  means that  $L$  arises from  $K$  by such steps. Collapses of polyhedra are as in [RS72, Ch. 3]; simplicial collapses and removals of a cell with a free facet are special cases.

**Fact 2.1** ([RS72, Ch. 2]). (a) *Collars. If  $M$  is a compact PL manifold, there is a PL embedding  $c : \partial M \times [0, 1] \rightarrow M$  with  $c(p, 0) = p$  whose image is a neighbourhood of  $\partial M$ . Then  $c(\partial M \times (0, 1))$  misses  $\partial M = c(\partial M \times \{0\})$ , as  $c$  is injective, and  $c(\partial M \times [0, 1))$  is open in  $M$ : near the points  $c(p, t)$  with  $0 < t < 1$  by invariance of domain, and near  $\partial M$  because the image of  $c$  is a neighbourhood of  $\partial M$  and  $c(\partial M \times \{1\})$  is compact.* (b) *Derived subdivisions. Let  $K$  be a polyhedral complex (a finite set of convex polytopes, closed under taking faces, any two of which meet in a common face). Choose a point  $\hat{P}$  in the relative interior of each cell  $P$ ; the simplices spanned by  $\hat{P}_0, \dots, \hat{P}_k$  for chains  $P_0 \subsetneq \dots \subsetneq P_k$  triangulate  $|K|$ . Hence an isomorphism of the face posets of two polyhedral complexes induces a PL homeomorphism of their polyhedra that maps each cell onto the corresponding cell.* (c) *Every triangulation of a PL manifold is a combinatorial manifold.*

**Fact 2.2** ([RS72, Ch. 3]). *Let  $M$  be a PL  $m$ -manifold without boundary and  $X \subset M$  a compact polyhedron. (a) Collapsing criterion. A compact polyhedron  $N \subset M$  that is a PL  $m$ -manifold, contains  $X$  in its interior and collapses onto  $X$  is a regular neighbourhood of  $X$  in  $M$ . (b) Uniqueness. Any two regular neighbourhoods of  $X$  in  $M$  are PL homeomorphic.*

**The complexes  $B(i, d)$ .** Let  $V_k = \{x_1, y_1, \dots, x_k, y_k\}$ . The faces of  $C_d^*$  are the subsets of  $V_d$  containing at most one of  $x_j, y_j$  for each  $j$ , and  $C_k^*$  is the subcomplex induced on  $V_k$ . We put  $x_j = e_j$  and  $y_j = -e_j$  in  $\mathbb{R}^d$ ; then  $|C_d^*| = \{z \in \mathbb{R}^d : \|z\|_1 = 1\}$ , a PL  $(d-1)$ -sphere. For a facet  $\tau$  let  $S(\tau)$  be the switch set of its word and  $\text{Sel}(\tau) = \tau \cap \bigcup_{j \in S(\tau)} \{x_j, y_j\}$ . For  $I, J \subseteq [d-1]$  let  $I \prec J$  if  $|I| < |J|$ , or if  $|I| = |J|$  and the least element of the symmetric difference of  $I$  and  $J$  lies in  $I$ .

**Lemma 2.3** (Klee–Novik). *Let  $0 \leq i \leq d-2$ . (a) [KN12, Lemma 3.5] Ordering the facets  $\tau$  of  $\text{st}_{B(i,d)} x_d$  by  $\prec$  on  $S(\tau)$  gives a shelling with restrictions  $\text{Sel}(\tau)$ . (b) [KN12, proof of Thm. 1.2(b)] Exchanging  $x_j$  and  $y_j$  for all  $j$  preserves switch sets, so it is an automorphism of  $B(i, d)$  exchanging  $x_d$  and  $y_d$ . (c) [KN12, Thm. 1.2(c)] Exchanging  $x_j$  and  $y_j$  for even  $j$  only maps  $B(d-i-2, d)$  isomorphically onto  $C(i, d)$ . (d) [KN12, Thm. 1.2(d)]  $B(i, d)$  is a combinatorial  $(d-1)$ -manifold with boundary.*

### 3 Proofs

**Lemma 3.1** ([KN12, Remark 3.7]). *Let  $0 \leq i \leq k-2$ . The faces of  $B(i, k)$  that avoid  $x_k$  and  $y_k$  are the faces of  $B(i, k-1)$ , and  $B(i, k) \searrow B(i, k-1)$ . Hence  $B(i, d) \searrow B(i, i+1) = C_{i+1}^*$  for  $0 \leq i \leq d-2$ .*

*Proof.* Klee and Novik state this without proof. Deleting or repeating the last letter of a word does not increase the number of switches; this gives the first claim. Let  $v = x_k$  and  $L = \text{lk}_{B(i,k)} v$ ; every face containing  $v$  is  $v \cup H$  with  $H \in L$ . Order the facets  $\tau_1, \dots, \tau_s$  of  $\text{st } v = v * L$  as in Lemma 2.3(a). Since  $S(\tau_j) \subseteq [k-1]$ , we have  $v \notin \text{Sel}(\tau_j)$ . A face  $H \subseteq G_j := \tau_j \setminus \{v\}$  is new at step  $j$  exactly when  $v \cup H$  is, so  $G_1, \dots, G_s$  is a shelling of  $L$  with  $R(G_j) = \text{Sel}(\tau_j)$ . Moreover  $|\text{Sel}(\tau_j)| = |S(\tau_j)| \leq i < k-1 = |G_j|$ .

Let  $L_j$  be the complex generated by  $G_1, \dots, G_j$ , and let  $j \geq 2$ . Only the first facet, with the word  $x \cdots x$ , has no switches, so  $R(G_j) = \text{Sel}(\tau_j) \neq \emptyset$ ; pick a vertex  $a_j \in G_j \setminus R(G_j)$ . The faces of  $L_j$  not in  $L_{j-1}$  form the interval  $[R(G_j), G_j]$ , which consists of nonempty faces, and every face of  $L_j$  containing one of them lies in this interval. So removing the pairs  $(H, H \cup \{a_j\})$  with  $H \in [R(G_j), G_j]$  and  $a_j \notin H$ , in order of decreasing dimension, collapses  $L_j$  onto  $L_{j-1}$ . The simplex  $L_1$  collapses onto a vertex  $p$ . Each step  $(H, H')$  of this collapse of  $L$  lifts to the elementary collapse  $(v \cup H, v \cup H')$ : if it is performed in the current subcomplex  $L' \subseteq L$ , and among the faces containing  $v$  only those of  $v * L'$  have been kept, then  $v \cup H$  and  $v \cup H'$  are the only faces containing  $v \cup H$ . Removing  $(v, v \cup \{p\})$  at the end collapses  $B(i, k)$  onto the faces not containing  $x_k$ . By Lemma 2.3(b) the same works for  $y_k$ , whose star is untouched because no face contains both  $x_k$  and  $y_k$ . What remains is  $B(i, k-1)$ . Iterating for  $k = d, \dots, i+2$  gives  $B(i, d) \searrow B(i, i+1)$ , which is  $C_{i+1}^*$  because every word of length  $i+1$  has at most  $i$  switches.  $\square$

**The standard neighbourhood of a join factor.** Fix  $0 \leq i \leq d-2$ , let  $\Sigma = C_{i+1}^*$ , and let  $\Sigma'$  be the boundary complex of the cross-polytope on  $V' = \{x_j, y_j : i+2 \leq j \leq d\}$ . A subset of  $V_d$  is a face of  $C_d^*$  if and only if its intersections with  $V_{i+1}$  and  $V'$  are faces of  $\Sigma$  and  $\Sigma'$ , so  $C_d^* = \Sigma * \Sigma'$ . Write  $\mathbb{R}^d = \mathbb{R}^{i+1} \times \mathbb{R}^{d-i-1}$  and  $z = (z', z'')$ ; then  $|\Sigma| = \{z \in |C_d^*| : z'' = 0\}$ .

**Lemma 3.2.** *Let  $N_0 = \{z \in |C_d^*| : \|z''\|_1 \leq \frac{1}{2}\}$ , and let  $D \subset \mathbb{R}^{d-i-1}$  be the  $\ell_1$ -ball of radius  $\frac{1}{2}$  about 0. There is a PL homeomorphism  $h: N_0 \rightarrow |\Sigma| \times D$  with  $h(z) = (z, 0)$  on  $|\Sigma|$ , and  $N_0$  is a regular neighbourhood of  $|\Sigma|$  in  $|C_d^*|$ . In particular  $N_0 \cong S^i \times B^{d-i-1}$ .*

*Proof.* Let  $\sigma \in \Sigma$  and  $\tau \in \Sigma'$ . A point of the simplex  $|\sigma \cup \tau|$  is  $\sum_{a \in \sigma} \lambda_a a + \sum_{b \in \tau} \mu_b b$  (vertices identified with the points  $\pm e_j$ ) with  $\lambda, \mu \geq 0$  and  $\sum \lambda_a + \sum \mu_b = 1$ , and there  $\|z''\|_1 = \sum \mu_b$ . So  $Q(\sigma, \tau) := N_0 \cap |\sigma \cup \tau|$  is the polytope given by these conditions and  $\sum \mu_b \leq \frac{1}{2}$ ; it is nonempty if and only if  $\sigma \neq \emptyset$ . For  $\sigma, \tau \neq \emptyset$  let  $Q'(\sigma, \tau)$  be the part of  $Q(\sigma, \tau)$  where  $\sum \mu_b = \frac{1}{2}$ . The faces of  $Q(\sigma, \tau)$  arise by turning some defining inequalities into equalities; they are the  $Q(\sigma_1, \tau_1)$  and  $Q'(\sigma_1, \tau_1)$  with  $\emptyset \neq \sigma_1 \subseteq \sigma$  and  $\tau_1 \subseteq \tau$  ( $\tau_1 \neq \emptyset$  for  $Q'$ ). They are pairwise distinct, since a face determines the coordinates that vanish identically on it and whether  $\sum \mu_b = \frac{1}{2}$  holds on it. As simplices of  $C_d^*$  meet in common faces, these polytopes form a polyhedral complex  $\mathcal{Q}$  with  $|\mathcal{Q}| = N_0$ . In its face poset, each of  $Q(\sigma_1, \tau_1) \leq Q(\sigma_2, \tau_2)$ ,  $Q'(\sigma_1, \tau_1) \leq Q(\sigma_2, \tau_2)$  and  $Q'(\sigma_1, \tau_1) \leq Q'(\sigma_2, \tau_2)$  holds if and only if  $\sigma_1 \subseteq \sigma_2$  and  $\tau_1 \subseteq \tau_2$ , and no  $Q(\cdot)$  lies in a  $Q'(\cdot)$ .

Next,  $D$  is the cone from 0 over  $\partial D = \frac{1}{2}|\Sigma'|$ . So  $|\Sigma| \times D$  is the union of the polytopes  $P(\sigma, \tau) = |\sigma| \times \text{conv}(\{0\} \cup \frac{1}{2}|\tau|)$  and  $P'(\sigma, \tau) = |\sigma| \times \frac{1}{2}|\tau|$  ( $\tau \neq \emptyset$ ). Computing faces and intersections factorwise, they form a polyhedral complex  $\mathcal{P}$ , and  $Q(\sigma, \tau) \mapsto P(\sigma, \tau)$ ,  $Q'(\sigma, \tau) \mapsto P'(\sigma, \tau)$  is an isomorphism of face posets. By Fact 2.1(b) it induces a PL homeomorphism  $h: N_0 \rightarrow |\Sigma| \times D$ . If the points  $\hat{P}$  of the cells  $Q(\sigma, \emptyset) = |\sigma|$  and  $P(\sigma, \emptyset) = |\sigma| \times \{0\}$  are chosen to correspond under  $z \mapsto (z, 0)$ , then  $h(z) = (z, 0)$  on  $|\Sigma|$ . As  $|\Sigma|$  is a PL  $i$ -sphere and  $D$  a PL  $(d-i-1)$ -ball,  $N_0$  is a PL  $(d-1)$ -manifold and  $N_0 \cong S^i \times B^{d-i-1}$ . The open subset of  $|C_d^*|$  where  $\|z''\|_1 < \frac{1}{2}$  contains  $|\Sigma|$  and lies in  $N_0$ . Finally,  $|\Sigma| \times D \searrow |\Sigma| \times \{0\}$ : remove the pairs  $(P'(\sigma, \tau), P(\sigma, \tau))$ ,  $\sigma, \tau \neq \emptyset$ , in order of decreasing  $\dim \sigma + \dim \tau$ . All other cells containing  $P'(\sigma, \tau)$  are then gone, so it is a free facet of  $P(\sigma, \tau)$ . Through  $h$ ,  $N_0 \searrow |\Sigma|$ ; now apply Fact 2.2(a).  $\square$

**Lemma 3.3.** *Let  $0 \leq i \leq d-2$ ,  $B = B(i, d)$  and  $C = C(i, d)$ . Then  $C$  is a combinatorial manifold with boundary,  $\partial B = \partial C$ ,  $|B| \cup |C| = |C_d^*|$  and  $|B| \cap |C| = |\partial B|$ .*

*Proof.* By Lemma 2.3(c),(d),  $C \cong B(d-i-2, d)$  is a combinatorial manifold with boundary, as  $0 \leq d-i-2 \leq d-2$ . Each facet of  $C_d^*$  lies in exactly one of  $B$  and  $C$ , and each ridge  $\rho$  of  $C_d^*$  lies in exactly two facets,  $\rho \cup \{x_j\}$  and  $\rho \cup \{y_j\}$ , where  $j$  is the index missed by  $\rho$ . This gives the second and third claims: a ridge lies in exactly one facet of  $B$  if and only if it lies in exactly one facet of  $C$ . Now let  $F \neq \emptyset$  be a face of both  $B$  and  $C$ . Then  $|F| \leq d-1$ , since no facet of  $C_d^*$  lies in both, so  $\text{lk}_{C_d^*} F$  is the boundary complex of a cross-polytope of dimension  $d-|F| \geq 1$ . Its facets split into the nonempty sets of facets of  $\text{lk}_B F$  and of  $\text{lk}_C F$ . The facet-adjacency graph of  $\text{lk}_{C_d^*} F$ , in which two facets are adjacent if they share a ridge, is the graph of a  $(d-|F|)$ -cube, hence connected. So some ridge  $\rho'$  of  $\text{lk}_{C_d^*} F$  lies in a facet of each. Then  $F \cup \rho'$  is a ridge of  $C_d^*$  in exactly one facet of  $B$ , and  $F \in \partial B$ . Conversely  $\partial B = \partial C$  lies in  $B \cap C$ . Thus  $B \cap C = \partial B$ , and  $|B| \cap |C| = |\partial B|$ .  $\square$

*Remark 3.4.* Every vertex of  $C_d^*$  lies in  $\partial B(i, d)$ . Indeed,  $x_j$  lies in the facet of  $B(i, d)$  with the word  $x \cdots x$ , and in the facet with the alternating word that has  $x$  in position  $j$ ; this word has  $d-1 > i$  switches, so that facet lies in  $C(i, d)$ . By Lemma 3.3,  $x_j \in B \cap C = \partial B$ , and similarly for  $y_j$ . In particular  $|C_{i+1}^*|$  meets  $\partial|B(i, d)|$ , so  $|B(i, d)|$  is not a neighbourhood of  $|C_{i+1}^*|$  in  $|C_d^*|$  and Fact 2.2(a) does not apply to it directly. This is why the proof below adds an outer collar.

*Proof of Theorem 1.2.* Let  $B$  and  $C$  be as in Lemma 3.3,  $\Sigma = C_{i+1}^* \subseteq B$  and  $M = |C_d^*|$ . By Fact 2.1(a) and Lemma 3.3 there are collars  $c: |\partial B| \times [0, 1] \rightarrow |C|$  and  $c': |\partial B| \times [-1, 0] \rightarrow |B|$ , with  $c(p, 0) = c'(p, 0) = p$ . Put  $N = |B| \cup c(|\partial B| \times [0, 1])$ . (i)  $N \cong |B|$ . The sets  $c'(|\partial B| \times [-1, 0]) \subseteq |B| \setminus |\partial B|$  and  $c(|\partial B| \times (0, 1]) \subseteq |C| \setminus |\partial B|$  are disjoint, so  $c'' = c' \cup c$  embeds  $|\partial B| \times [-1, 1]$  in  $N$ . The map that is the identity outside  $c''(|\partial B| \times (-1, 1])$  and sends  $c''(p, t)$  to  $c''(p, \frac{t-1}{2})$  is a PL homeomorphism  $N \rightarrow |B|$ . (ii)  $|B|$  lies in the interior of  $N$ . The set  $U = c(|\partial B| \times [0, 1])$  is open in  $|C|$ , so  $(M \setminus |C|) \cup U = M \setminus (|C| \setminus U)$  is open in  $M$ . It lies in  $N$  and contains  $|B|$ . (iii)  $N \searrow |\Sigma|$ . We have  $|\partial B| \times [0, 1] \searrow |\partial B| \times \{0\}$  (remove  $\sigma \times \{1\}$  and  $\sigma \times [0, 1]$ ,  $\sigma \in \partial B$ , in order of decreasing dimension), and  $c(|\partial B| \times [0, 1])$  meets  $|B|$  only in  $|\partial B|$ . So  $N \searrow |B| \searrow |\Sigma|$ , by Lemma 3.1. By Fact 2.2(a) and

Lemma 3.2,  $N$  and  $N_0$  are regular neighbourhoods of  $|\Sigma|$  in  $M$ , so  $|B| \cong N \cong N_0 \cong S^i \times B^{d-i-1}$  by Fact 2.2(b).  $\square$

*Proof of Corollary 1.3.* (a)  $\partial B(i, d)$  is a combinatorial manifold, and  $|\partial B(i, d)| = \partial|B(i, d)| \cong \partial(S^i \times B^{d-i-1}) = S^i \times S^{d-i-2}$ , as PL homeomorphisms preserve boundaries. (b)  $ST(m, n)$  triangulates  $|\partial\sigma^m| \times |\sigma^n| \cong S^{m-1} \times B^n$  [CKP14, Sect. 1.2], so it is a combinatorial manifold (Fact 2.1(c)), and  $|ST(i+1, d-i-1)| \cong |B(i, d)|$  by Theorem 1.2. For  $i = 0$  the two complexes are isomorphic [CKP14, Sect. 2]. For  $i \geq 1$  both are connected and have nonempty boundary, and the theorem of Newman and Pachner [Pac91], in the form [Lic99, Thm. 5.10], applies.  $\square$

**Verification.** The proofs do not depend on computation; the programs below are checks. They are included, with their recorded outputs, in the source files of this note (folder `reproducibility`), and they need only Python 3 and its standard library. (1) The program `verify_owr4798013.py`, written for this note with AI assistance (see the last paragraph), treats the 66 pairs with  $2 \leq d \leq 12$ ,  $0 \leq i \leq d-2$ , in about three minutes. It checks the filling criterion [KN12, Lemma 3.2], Lemma 2.3(a),(c), and  $\partial B = \partial C$  and  $B \cap C = \partial B$  (Lemma 3.3), and it executes the collapse of Lemma 3.1 step by step, checking that each step is an elementary collapse; the result is  $C_{i+1}^*$  in every case. For  $d \leq 8$  the mod-2 Betti numbers of  $B(i, d)$  and  $\partial B(i, d)$  are those of  $S^i$  and  $S^i \times S^{d-i-2}$ . (2) Two independent verification runs, both AI-assisted, checked the argument and ran their own code. The first confirmed for  $d \leq 10$  that the links of  $x_k$  and  $y_k$  in  $B(i, k)$  are collapsible and that the faces avoiding  $x_k$  and  $y_k$  form  $B(i, k-1)$ . It also ran a greedy collapse of all of  $B(i, d)$  onto  $C_{i+1}^*$ , which got stuck at  $(i, d) = (5, 9)$ . A greedy collapse can get stuck on a collapsible complex, so this does not contradict Lemma 3.1; it is why all checks follow the proof of Lemma 3.1. The second run reviewed an earlier draft of this note and ran its checks in memory. They are reproduced by the program `check_owr4798013_run2.py`: the collapse of Lemma 3.1, step by step, for all 45 pairs with  $d \leq 10$ , Lemma 2.3(a) for  $d \leq 8$ , Lemma 3.3 for  $d \leq 10$  and Remark 3.4 for  $d \leq 9$ . (3) An exploratory program used while the proof was found, `collapse_check.py`, runs a single greedy collapse; it succeeded for eight pairs with  $d \leq 7$ , and nothing here depends on it.

**Scope and priority.** Theorem 1.2 answers Question 4 of [Kle11] as posed, and the question of the record [ula26]. The collapse onto  $C_{i+1}^*$  and the disc-bundle remark are due to Klee and Novik [KN12, Remark 3.7]; our contribution is that  $|B(i, d)|$  is a product, via the join structure of  $C_d^*$ . The cases  $i = 0, 1$  were known [KN12, WZ20], and Cohen, Klee and Pannell treated  $i = d-2$  [CKP14]; their paper (read in its 2011 preprint version) states the general question as open. Machacek [Mac22] showed that the subspace of  $\mathbb{R}P^{d-1}$  of points whose nonzero coordinates have at most  $i$  sign changes is a PL manifold that collapses onto  $\mathbb{R}P^i$ . He does not mention Klee and Novik (we read arXiv:1909.04640v2). By the filling criterion [KN12, Lemma 3.2], this subspace is the antipodal quotient of  $|B(i, d)|$ ; Karp and Machacek [KM23] note the corresponding relation between face posets. We searched arXiv, Crossref, zbMATH, Semantic Scholar, Novik’s web page and the web, including the works citing [KN12] listed by Semantic Scholar and zbMATH, and found no answer to the question. The argument is short and uses only standard PL topology, so it may be known to experts. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the proof, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

## References

- [CKP14] Sara Cohen, Steven Klee, and Katherine Pannell. Bistellar equivalences of two families of simplicial complexes. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 89:65–85, 2014. zbMATH 1304.05162; preprint dated August 18, 2011.
- [Kle11] Steven Klee. Centrally symmetric manifolds with few vertices. *Oberwolfach Reports*, 8(1):370–373, 2011. Joint work with Isabella Novik. Abstract in Report No. 08/2011, *Topological and Geometric Combinatorics*, organised by A. Björner, G. Kalai, I. Novik and G. M. Ziegler, pp. 349–423. doi:10.4171/OWR/2011/08.
- [KM23] Steven N. Karp and John Machacek. Shelling the  $m = 1$  amplituhedron. *Combinatorial Theory*, 3(1), 2023. Preprint arXiv:2104.02786. doi:10.5070/C63160419.
- [KN12] Steven Klee and Isabella Novik. Centrally symmetric manifolds with few vertices. *Advances in Mathematics*, 229(1):487–500, 2012. Preprint arXiv:1102.0542. doi:10.1016/j.aim.2011.07.024.
- [Lic99] W. B. R. Lickorish. Simplicial moves on complexes and manifolds. In *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, volume 2 of *Geometry & Topology Monographs*, pages 299–320. Geometry & Topology Publications, 1999. doi:10.2140/gtm.1999.2.299.
- [Mac22] John Machacek. Boundary measurement and sign variation in real projective space. *Annales de l’Institut Henri Poincaré D, Combinatorics, Physics and their Interactions*, 9(3):543–565, 2022. Preprint arXiv:1909.04640. doi:10.4171/AIHPD/125.
- [Pac91] Udo Pachner. P.L. homeomorphic manifolds are equivalent by elementary shellings. *European Journal of Combinatorics*, 12(2):129–145, 1991. doi:10.1016/S0195-6698(13)80080-7.
- [RS72] Colin P. Rourke and Brian J. Sanderson. *Introduction to Piecewise-Linear Topology*, volume 69 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1972. doi:10.1007/978-3-642-81735-9.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record OWR-4798-013.
- [WZ20] Alexander Wang and Hailun Zheng. Centrally symmetric and balanced triangulations of  $\mathbb{S}^2 \times \mathbb{S}^{d-3}$  with few vertices. *European Journal of Combinatorics*, 85:Paper No. 103043, 2020. Preprint arXiv:1811.08505. doi:10.1016/j.ejc.2019.103043.
- [Zee63] E. C. Zeeman. Unknotting combinatorial balls. *Annals of Mathematics (2)*, 78(3):501–526, 1963. doi:10.2307/1970538.