

The Cubic Fermat Equation in Model Spaces: A Negative Answer to a Question of Dyakonov

Alper Ferudun*

October 17, 2026

Abstract

For an inner function θ on the unit disc and $1 \leq p \leq \infty$, let $K_\theta^p = H^p \cap \theta \overline{H_0^p}$ be the associated model (star-invariant) subspace of the Hardy space H^p . K. M. Dyakonov asked (Oberwolfach Reports 7 (2010); Integral Equations Operator Theory 78 (2014)) whether “Fermat’s last theorem” holds in these spaces: does $f^n + g^n = h^n$ with $f, g, h \in K_\theta^p$ and $n \geq 3$ force f, g, h to lie in a one-dimensional subspace? Dyakonov noted that the answer is yes when θ is a finite Blaschke product, and S. R. Garcia (2026) proved that it is yes for $n \geq 4$ when the boundary spectrum of θ has logarithmic capacity zero, and for $n = 3$ when the boundary spectrum has at most one point. We show that the answer is no for $n = 3$. We construct an explicit inner function $\theta_0 = zB\tilde{B}$, where B and $\tilde{B}$ are Blaschke products whose zeros accumulate on the unit circle exactly at -1 and 1 , and functions $F, G, H \in K_{\theta_0}^\infty$, hence in $K_{\theta_0}^p$ for every p , such that $F^3 + G^3 = H^3$ and F, G, H are linearly independent; moreover $F + iH = B$, and the inner factors of F, G, H are pairwise coprime. The construction composes Dixon’s elliptic functions sm and cm with a conformal map of the disc onto a period strip on whose two boundary lines both functions are real. Consequently the one-point hypothesis in Garcia’s cubic theorem cannot be relaxed to two points, and every closed subset of the unit circle with at least two points is the boundary spectrum of an inner function θ for which K_θ^p contains a nontrivial solution of the cubic Fermat equation. The case $n \geq 4$ and Dyakonov’s companion question on extreme points are not addressed. This is an unrefereed note.

1 Introduction

1.1 The problem

Let $\mathbb{D}$ be the open unit disc, $\mathbb{T} = \partial\mathbb{D}$, and m the normalized arc length on $\mathbb{T}$. For $1 \leq p \leq \infty$ we identify functions in the Hardy space H^p with their nontangential boundary functions in $L^p(\mathbb{T}, m)$ [Gar07], and we write $H_0^p = zH^p = \{f \in H^p : f(0) = 0\}$. An *inner function* is a function $\theta \in H^\infty$ with $|\theta| = 1$ a.e. on $\mathbb{T}$. Following Dyakonov [Dya10, Dya14], the *model* (or *star-invariant*) *subspace* associated with θ is

$$K_\theta^p = H^p \cap \theta \overline{H_0^p}, \quad 1 \leq p \leq \infty, \quad (1)$$

where the bar denotes complex conjugation of boundary functions.¹ The space K_θ^p is invariant under the backward shift; for $p = 2$ it is $H^2 \ominus \theta H^2$, and for every $p \in [1, \infty]$ it is the kernel in H^p of the Toeplitz operator with symbol $\bar{\theta}$ [Dya14, pp. 1–2].

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

¹In the statement of record OWR-4531-016 of the UnsolvedMath dataset [ula26] the bar over H_0^p has been lost: the record reads $K_\theta^p = H^p \cap \theta H_0^p$. Read literally, this space is θH_0^p , and the question becomes trivial: for every n the triple $(\theta z \cdot \frac{z}{2}, \theta z(1 - z^n/2^n)^{1/n}, \theta z)$, with the principal branch of the root, solves $f^n + g^n = h^n$ in θH_0^∞ and spans a space of dimension three. Both sources print the bar [Dya10, p. 2870], [Dya14, p. 1], and we use their definition (1).

Dyakonov posed two problems on these spaces at the Oberwolfach workshop “Operator Theory and Harmonic Analysis” of November 2010 [Dya10, p. 2870], and published them with more detail in [Dya14]. Problem 1 asks for the extreme points of the unit ball of K_θ^∞ ; it is not considered here. Problem 2 reads “Prove or disprove ‘Fermat’s last theorem’ for K_θ^p ” [Dya14, p. 3]: is it true that, whenever θ is inner, $p \in [1, \infty]$ and $n \in \{3, 4, 5, \dots\}$, no nontrivial triple of functions $f, g, h \in K_\theta^p$ satisfies

$$f^n + g^n = h^n ? \quad (2)$$

A triple is called *nontrivial* if the three functions do not lie in a single one-dimensional subspace of K_θ^p , that is, if $\dim \text{span}\{f, g, h\} \geq 2$ [Dya14, p. 3]; the problem in [Dya10] has the same content.

1.2 What was known

Dyakonov observed that the answer is positive when θ is a finite Blaschke product: then K_θ^p consists of rational functions, and Fermat’s last theorem for polynomials and rational functions, a consequence of the Mason–Stothers theorem, applies [Dya10], [Dya14, p. 3]. Garcia [Gar26a, Theorem 1.1] proved that the answer is positive for $n \geq 4$ when the boundary spectrum $\sigma(\theta) \cap \mathbb{T}$ has logarithmic capacity zero. In [Gar26b, Theorem 1.1] he extended this to equations $P(f, g, h) = 0$, where P is a homogeneous polynomial whose projective plane curve is smooth, and treated the cubic case: then $\dim \text{span}\{f, g, h\} \leq 1$ if either $\deg P = 3$ and $\#(\sigma(\theta) \cap \mathbb{T}) \leq 1$, or $\deg P \geq 4$ and $\text{cap}(\sigma(\theta) \cap \mathbb{T}) = 0$. Here, as in [Gar26a, Gar26b], the spectrum of an inner function $\theta = cB_\Lambda S_\mu$ (c unimodular, B_Λ the Blaschke product with zero sequence Λ , S_μ the singular inner function with singular measure μ) is $\sigma(\theta) = \overline{\Lambda} \cup \text{supp } \mu$ [GMR16, Def. 7.3.2], and the boundary spectrum $\sigma(\theta) \cap \mathbb{T}$ is the set of points of $\mathbb{T}$ across which θ cannot be continued analytically [GMR16, Thm. 7.3.4 and Prop. 7.3.8]. Garcia explains that the cubic case is different: the Fermat cubic has genus one, and $f^n + g^n = 1$ has non-constant meromorphic solutions in $\mathbb{C}$ for $n = 3$, coming from elliptic functions, but not for $n \geq 4$ (Gross, Baker; see [Gar26a, Remark 1.2] and [Gar26b, Remark 1.5]). Without the condition of the model space, (2) has nontrivial solutions in H^∞ for every n , for instance $(z/2, (1 - z^n/2^n)^{1/n}, 1)$.

1.3 Results

Our main result answers Problem 2 negatively.

Theorem 1.1. *There are an inner function θ_0 and functions $F, G, H \in K_{\theta_0}^\infty$ with the following properties.*

- (a) $\theta_0 = zB\tilde{B}$, where B and $\tilde{B}$ are Blaschke products with simple zeros, and the zeros of B and those of $\tilde{B}$ accumulate on $\mathbb{T}$ exactly at -1 and 1 . In particular $\sigma(\theta_0) \cap \mathbb{T} = \{-1, 1\}$.
- (b) $F^3 + G^3 = H^3$ on $\mathbb{D}$.
- (c) F, G and H are linearly independent; in particular the triple (F, G, H) is nontrivial.
- (d) $F + iH = B$. The inner factors of F, G and H are Blaschke products with pairwise disjoint zero sets, so that no nonconstant inner function divides two of the functions F, G, H .

Consequently $F, G, H \in K_{\theta_0}^p$ for every $p \in [1, \infty]$, and the answer to Problem 2 of [Dya10, Dya14] is negative for $n = 3$ and every $p \in [1, \infty]$.

The functions are explicit. Let sm and cm be Dixon’s elliptic functions, the solution of $\text{sm}' = \text{cm}^2$, $\text{cm}' = -\text{sm}^2$, $\text{sm}(0) = 0$, $\text{cm}(0) = 1$, which parametrizes the Fermat cubic $x^3 + y^3 = 1$ [Dix90, CF06];

let p_0 be the least positive real period of sm (classically $p_0 = \pi_3 = B(\frac{1}{3}, \frac{1}{3}) = 5.29991 \dots$ [CF06, (12)]), $h = \frac{\sqrt{3}}{2}p_0$, and

$$\psi(z) = \frac{h}{\pi} \text{Log} \frac{1+z}{1-z} + \frac{ih}{2},$$

a conformal map of $\mathbb{D}$ onto the strip $\{0 < \text{Im } w < h\}$. Then

$$F = B \frac{\text{sm} \circ \psi}{\text{sm} \circ \psi + i}, \quad G = B \frac{\text{cm} \circ \psi}{\text{sm} \circ \psi + i}, \quad H = \frac{B}{\text{sm} \circ \psi + i}, \quad (3)$$

where B and $\tilde{B}$ are the Blaschke products whose zeros are the points of $\mathbb{D}$ at which $\text{sm} \circ \psi$ takes the value $-i$ and $+i$, respectively. Both zero sets are invariant under the hyperbolic automorphism $z \mapsto (z + \tau)/(1 + \tau z)$, $\tau = \tanh(\pi/\sqrt{3})$; numerically each of them consists of three of its orbits (Section 7).

Corollary 1.2. *The hypothesis $\#(\sigma(\theta) \cap \mathbb{T}) \leq 1$ in [Gar26b, Theorem 1.1(a)] cannot be replaced by $\#(\sigma(\theta) \cap \mathbb{T}) \leq 2$, already for the Fermat cubic $P(x, y, z) = x^3 + y^3 - z^3$.*

Corollary 1.3. *Let $E \subset \mathbb{T}$ be a closed set with at least two points. Then there is a Blaschke product θ with $\sigma(\theta) \cap \mathbb{T} = E$ such that K_θ^∞ contains three linearly independent functions f, g, h with $f^3 + g^3 = h^3$.*

Together with [Gar26b, Theorem 1.1(a)], Corollary 1.3 shows that for a closed set $E \subset \mathbb{T}$ the following are equivalent: every inner function θ with $\sigma(\theta) \cap \mathbb{T} = E$ admits no nontrivial solution of the cubic Fermat equation in any K_θ^p ; and $\#E \leq 1$. Whether the cubic Fermat equation has nontrivial solutions for *every* inner function with two boundary spectral points is not addressed here, and neither is the case $n \geq 4$.

To our knowledge, Theorem 1.1 and its corollaries are new: no printed answer to Dyakonov's question was found in the sources searched, which are listed in the paragraph "Scope and priority" at the end. A search that finds nothing is not a proof of novelty, and no priority is claimed.

1.4 The idea

Dixon's functions are real on the real axis and have the period ωp_0 , $\omega = e^{2\pi i/3}$, whose imaginary part is h ; so they are real on both boundary lines of the strip. As ψ maps the two arcs of $\mathbb{T} \setminus \{\pm 1\}$ onto these lines, the conjugate $\theta_0 \bar{z} \bar{F}$ of F is a function of the same form with $\text{sm} -i$ in place of $\text{sm} +i$, and membership in $K_{\theta_0}^\infty$ follows. The poles in the strip, the zeros of $\text{sm} +i$, are finitely many modulo p_0 , and the translation by p_0 is a hyperbolic automorphism of $\mathbb{D}$ with fixed points ± 1 ; so the poles in $\mathbb{D}$ satisfy the Blaschke condition and are cancelled by B . The two boundary spectral points are the two ends of the strip (see also Remark 6.3). Section 7 describes numerical checks, which are not part of the proofs.

2 Preliminaries on Hardy spaces

We use the following standard facts [Gar07, Chapters I and II].

- (H1) Every $f \in H^p$ has nontangential limits a.e. on $\mathbb{T}$, and two functions in H^p with the same boundary values a.e. coincide. For $f \in H^\infty$, $|f| \leq \|f\|_\infty$ on $\mathbb{D}$, where $\|f\|_\infty$ is the essential supremum of $|f|$ on $\mathbb{T}$.
- (H2) If Λ is a sequence in $\mathbb{D}$ with $\sum_{a \in \Lambda} (1 - |a|) < \infty$ (the *Blaschke condition*), the Blaschke product $B_\Lambda = \prod_{a \in \Lambda} b_a$, $b_a(z) = \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}$ ($b_0(z) = z$), converges locally uniformly on $\mathbb{D}$, is inner, and

vanishes exactly on Λ , with the multiplicities of Λ . Moreover $|b_a(z)| = \rho(z, a)$, where $\rho(z, a) = |z - a|/|1 - \bar{a}z|$ is the pseudo-hyperbolic distance, which is invariant under the automorphisms of $\mathbb{D}$.

- (H3) Every $f \in H^p$, $f \neq 0$, has a unique factorization $f = cB_f S_\mu O_f$, where $|c| = 1$, B_f is the Blaschke product over the zeros of f , O_f is outer with $|O_f| = |f|$ a.e. on $\mathbb{T}$, and S_μ is the singular inner function of a positive singular measure μ on $\mathbb{T}$: $|S_\mu(z)| = e^{-P[\mu](z)}$, $P[\mu](z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2} d\mu(\zeta)$. The inner factor of f is $cB_f S_\mu$. If $f \in H^\infty$, then $|O_f| \leq \|f\|_\infty$ on $\mathbb{D}$. The function $1 - \bar{\zeta}_0 z$, $|\zeta_0| = 1$, is outer.
- (H4) For a positive finite measure μ on $\mathbb{T}$, the measures $P[\mu](r\zeta) dm(\zeta)$ converge weak-* to μ as $r \rightarrow 1^-$.

An inner function I divides $f \in H^p$ if $f/I \in H^p$; then, by the uniqueness in (H3), the inner factor of f divided by I is inner. So if the inner factor of f is a Blaschke product, every inner divisor of f is, up to a unimodular constant, a Blaschke product whose zeros are zeros of f (singular measures add under multiplication).

The next lemma is a direct consequence of the definition (1).

Lemma 2.1. *Let θ be inner, $1 \leq p \leq \infty$ and $f \in H^p$. Then $f \in K_\theta^p$ if and only if there is $g \in H^p$ with $\theta \bar{z} \bar{f} = g$ a.e. on $\mathbb{T}$. Consequently:*

- (a) $K_\theta^\infty \subset K_\theta^p$ for every $p \in [1, \infty]$;
- (b) if $f \in K_\theta^p$ and θ_1 is inner, then $f \in K_{\theta\theta_1}^p$;
- (c) the constant function 1 belongs to K_θ^p if and only if $\theta(0) = 0$.

Proof. We have $f \in \theta \overline{H_0^p}$ if and only if $f = \theta \bar{z} \bar{g}$ for some $g \in H^p$, that is, $\bar{f} = \bar{\theta} z g$. Since $|\theta| = |z| = 1$ a.e. on $\mathbb{T}$, this is equivalent to $\theta \bar{z} \bar{f} = g$. (a) The same $g \in H^\infty \subset H^p$ serves for every p . (b) $\theta \theta_1 \bar{z} \bar{f} = \theta_1 g \in H^p$. (c) For $f = 1$ the condition reads $\theta \bar{z} = g$, that is, $\theta = zg$ a.e. on $\mathbb{T}$. If $\theta(0) = 0$, take $g = \theta/z$. Conversely, $\theta = zg$ on $\mathbb{T}$ implies $\theta = zg$ on $\mathbb{D}$ by (H1), so $\theta(0) = 0$. $\square$

For $f \in K_\theta^p$ the function g of Lemma 2.1 is the conjugate $\tilde{f}$ of f in the sense of [Gar26b, Section 2.1].

Lemma 2.2. *Let Λ be a sequence in $\mathbb{D}$ satisfying the Blaschke condition, and let φ be an automorphism of $\mathbb{D}$. Then $\varphi^{-1}(\Lambda)$ satisfies the Blaschke condition, and $B_\Lambda \circ \varphi = c B_{\varphi^{-1}(\Lambda)}$ for a constant c with $|c| = 1$. In particular, if $\varphi^{-1}(\Lambda) = \Lambda$ (with multiplicities), then $|B_\Lambda \circ \varphi| = |B_\Lambda|$ on $\mathbb{D}$.*

Proof. For the automorphism φ^{-1} we have $1 - |\varphi^{-1}(a)|^2 = |(\varphi^{-1})'(a)| (1 - |a|^2)$, and $(\varphi^{-1})'$ is bounded on $\mathbb{D}$, since φ^{-1} is a Möbius map holomorphic on a neighbourhood of $\overline{\mathbb{D}}$; hence $\varphi^{-1}(\Lambda)$ satisfies the Blaschke condition. By (H2), for $z \in \mathbb{D}$,

$$|B_\Lambda(\varphi(z))| = \prod_{a \in \Lambda} \rho(\varphi(z), a) = \prod_{a \in \Lambda} \rho(z, \varphi^{-1}(a)) = |B_{\varphi^{-1}(\Lambda)}(z)|.$$

The two functions $B_\Lambda \circ \varphi$ and $B_{\varphi^{-1}(\Lambda)}$ have the same zeros with the same multiplicities, so their quotient is holomorphic and zero-free on $\mathbb{D}$, and it has modulus 1; hence it is a unimodular constant. $\square$

Lemma 2.3. *Let $f \in H^\infty$, $f \neq 0$, and let $J \subset \mathbb{T}$ be an open arc such that f extends holomorphically to a neighbourhood of each point of J . Then the singular measure μ of the inner factor of f satisfies $\mu(J) = 0$.*

Proof. Let $\zeta_0 \in J$. The extension of f has a zero of some finite order $k \geq 0$ at ζ_0 (it does not vanish identically near ζ_0 , since $f \not\equiv 0$ on $\mathbb{D}$). Hence $f_1 = f/(z - \zeta_0)^k$ is bounded on $\mathbb{D}$ and extends continuously to ζ_0 with $f_1(\zeta_0) \neq 0$. Since $z - \zeta_0$ is outer by (H3), f_1 and f have the same inner factor. Choose $\delta > 0$ and $\eta > 0$ with $|f_1| \geq \eta$ on $\mathbb{D} \cap D(\zeta_0, \delta)$. By (H3), $\eta \leq |f_1| \leq |S_\mu| \|f_1\|_\infty$ there, so $P[\mu] \leq M := \log(\|f_1\|_\infty/\eta)$ on $\mathbb{D} \cap D(\zeta_0, \delta)$. Let $J_0 = \mathbb{T} \cap D(\zeta_0, \delta/2)$. If $\chi \geq 0$ is continuous with support in J_0 , then $r\zeta \in D(\zeta_0, \delta)$ for $\zeta \in J_0$ and $r > 1 - \delta/2$, so by (H4)

$$\int \chi d\mu = \lim_{r \rightarrow 1^-} \int \chi(\zeta) P[\mu](r\zeta) dm(\zeta) \leq M \int \chi dm.$$

Hence $\mu(K) \leq M m(K)$ for every compact $K \subset J_0$, so the restriction of μ to J_0 is absolutely continuous; as μ is singular, $\mu(J_0) = 0$. Since J is covered by countably many such arcs J_0 , $\mu(J) = 0$. $\square$

3 Dixon's elliptic functions

Throughout, $\omega = e^{2\pi i/3}$.

Lemma 3.1. *There are meromorphic functions sm and cm on $\mathbb{C}$ with the following properties.*

(D1) $\text{sm}' = \text{cm}^2$, $\text{cm}' = -\text{sm}^2$, $\text{sm}(0) = 0$, $\text{cm}(0) = 1$, and $\text{sm}^3 + \text{cm}^3 = 1$.

(D2) $\text{sm}(\bar{u}) = \overline{\text{sm}(u)}$ and $\text{cm}(\bar{u}) = \overline{\text{cm}(u)}$.

(D3) *The set Λ of periods of sm is a lattice, that is, a discrete subgroup of $\mathbb{C}$ of rank two. Every element of Λ is a period of cm , $\bar{\Lambda} = \Lambda$ and $\omega\Lambda = \Lambda$; moreover $\text{sm}(\omega u) = \omega \text{sm}(u)$ and $\text{cm}(\omega u) = \text{cm}(u)$.*

(D4) $\Lambda \cap \mathbb{R} = p_0\mathbb{Z}$ for some $p_0 > 0$, and $P := \omega p_0 = -\frac{p_0}{2} + ih$ belongs to Λ , where $h = \frac{\sqrt{3}}{2}p_0$.

These are Dixon's functions [Dix90]; see [CF06, Section 1], where the real period $\pi_3 = B(\frac{1}{3}, \frac{1}{3})$ and the period lattice $\pi_3\mathbb{Z} \oplus \pi_3\omega\mathbb{Z}$ are recalled [CF06, (12)]. We do not use these values (they are confirmed numerically in Section 7), and for completeness we derive (D1)–(D4) from the differential system.

Proof. Let $E \subset \mathbb{CP}^2$ be the projective Fermat cubic $X^3 + Y^3 = W^3$ and $E_0 = \{(x, y) \in \mathbb{C}^2 : x^3 + y^3 = 1\}$ its affine part $W = 1$. The partial derivatives $3X^2$, $3Y^2$, $-3W^2$ have no common zero on E , so E is a smooth compact Riemann surface. The points of $E \setminus E_0$ are $Q_k = [1 : -\omega^k : 0]$, $k = 0, 1, 2$. Near them we use the chart $(S, C) = (W/X, Y/X)$, in which E is the curve $1 + C^3 = S^3$ and $Q_k = (0, -\omega^k)$; on $E_0 \cap \{x \neq 0\}$ we have $S = 1/x$ and $C = y/x$.

On E_0 consider the vector field $V = y^2\partial_x - x^2\partial_y$. It is tangent to E_0 , because $V(x^3 + y^3) = 3x^2y^2 - 3y^2x^2 = 0$, and it has no zeros on E_0 , because $(0, 0) \notin E_0$. Along an integral curve, $x' = y^2$ and $y' = -x^2$, hence

$$S' = -\frac{x'}{x^2} = -C^2, \quad C' = \frac{y'x - yx'}{x^2} = -\frac{x^3 + y^3}{x^2} = -S^2.$$

So in the chart (S, C) the field is $-C^2\partial_S - S^2\partial_C$, which is holomorphic and equal to $(-\omega^{2k}, 0) \neq 0$ at Q_k . Thus V extends to a holomorphic vector field on E without zeros.

Since E is compact, the flow of V is complete: there is a holomorphic map $(t, Q) \mapsto \Phi_t(Q)$ from $\mathbb{C} \times E$ to E with $\Phi_0(Q) = Q$, $\partial_t \Phi_t(Q) = V(\Phi_t(Q))$ and $\Phi_{s+t} = \Phi_s \circ \Phi_t$. (By compactness there is $\varepsilon > 0$ such that every solution of $u' = V(u)$ exists on a disc $|t - t_1| < \varepsilon$ around its initial time t_1 ; a solution on $|t| < r$ extends to $|t| < r + \varepsilon/2$ by the solutions starting at the points with $r - \varepsilon/2 < |t_1| < r$,

which agree with each other and with the given solution on the convex overlaps, by uniqueness.) Let $O = (0, 1) \in E_0$ and $\phi(t) = \Phi_t(O)$. Then $\phi : \mathbb{C} \rightarrow E$ is holomorphic, $\phi' = V(\phi) \neq 0$, so ϕ is a local biholomorphism, and $\phi(t+s) = \Phi_s(\phi(t))$. Put $\text{sm} = x \circ \phi$ and $\text{cm} = y \circ \phi$; these are meromorphic on $\mathbb{C}$, since x and y are meromorphic on E , and (D1) holds.

Let Λ' be the group of periods of ϕ ; it is discrete, since ϕ is injective near 0. If λ is a period of sm , then $\text{cm}(t+\lambda)^2 = \text{cm}(t)^2$ by differentiation, so $\text{cm}(t+\lambda) = \epsilon \text{cm}(t)$ with a constant $\epsilon \in \{\pm 1\}$, and differentiating again, $\text{sm}(t)^2 = \text{sm}(t+\lambda)^2 = \epsilon \text{sm}(t)^2$, which forces $\epsilon = 1$; hence $\phi(t+\lambda) = \phi(t)$ where $\text{sm}(t)$ is finite, and by continuity everywhere, that is, $\lambda \in \Lambda'$. So $\Lambda = \Lambda'$, and every element of Λ is a period of cm . If $\phi(t) = \phi(s)$, then $\phi(t+u) = \Phi_u(\phi(t)) = \Phi_u(\phi(s)) = \phi(s+u)$ for all u , so $t-s \in \Lambda$. Hence ϕ induces an injective local biholomorphism $\mathbb{C}/\Lambda \rightarrow E$, that is, a biholomorphism onto the open set $\phi(\mathbb{C})$. The orbits $\{\Phi_t(Q) : t \in \mathbb{C}\}$, $Q \in E$, are open and pairwise disjoint or equal, so each of them is also closed; thus $\phi(\mathbb{C})$ is compact, and so is $\mathbb{C}/\Lambda$. A discrete subgroup of $\mathbb{C}$ with compact quotient has rank two.

The pair $(\overline{\text{sm}(\bar{u})}, \overline{\text{cm}(\bar{u})})$ satisfies the same system with the same initial values, since the system has real coefficients; by uniqueness it coincides with (sm, cm) , which gives (D2) and $\bar{\Lambda} = \Lambda$. The pair $(\omega^{-1} \text{sm}(\omega u), \text{cm}(\omega u))$ also satisfies the system with the same initial values: $\frac{d}{du}[\omega^{-1} \text{sm}(\omega u)] = \text{cm}(\omega u)^2$ and $\frac{d}{du} \text{cm}(\omega u) = -\omega \text{sm}(\omega u)^2 = -(\omega^{-1} \text{sm}(\omega u))^2$, as $\omega^{-2} = \omega$. Hence $\text{sm}(\omega u) = \omega \text{sm}(u)$, $\text{cm}(\omega u) = \text{cm}(u)$ and $\omega \Lambda = \Lambda$. Finally, Λ is not contained in $i\mathbb{R}$, and for $\lambda \in \Lambda$ with $\text{Re } \lambda \neq 0$ we get $0 \neq \lambda + \bar{\lambda} \in \Lambda \cap \mathbb{R}$. So $\Lambda \cap \mathbb{R}$ is a nonzero discrete subgroup of $\mathbb{R}$, that is, $p_0\mathbb{Z}$ with $p_0 > 0$, and $P = \omega p_0 \in \Lambda$. $\square$

Let $L_0 = \mathbb{R}$ and $L_h = \mathbb{R} + ih$ be the two boundary lines of the strip $S = \{w : 0 < \text{Im } w < h\}$, and define the meromorphic functions

$$q = \frac{\text{sm}}{\text{sm}+i}, \quad q_G = \frac{\text{cm}}{\text{sm}+i}, \quad q_H = \frac{1}{\text{sm}+i}, \quad \tilde{q} = \frac{\text{sm}}{\text{sm}-i}, \quad \tilde{q}_G = \frac{\text{cm}}{\text{sm}-i}, \quad \tilde{q}_H = \frac{1}{\text{sm}-i}. \quad (4)$$

Corollary 3.2. (a) sm and cm are p_0 -periodic, and their values on $L_0 \cup L_h$ are real or ∞ .

(b) If $\text{sm}(w) = \epsilon i$ with $\epsilon \in \{-1, 1\}$, then $\text{cm}(w)^3 = 1 + \epsilon i \neq 0$; hence w is a simple zero of $\text{sm} - \epsilon i$.

(c) The poles of q , q_G , q_H are exactly the zeros of $\text{sm} + i$, and they are simple; likewise for $\tilde{q}$, $\tilde{q}_G$, $\tilde{q}_H$ and the zeros of $\text{sm} - i$. Moreover $q + iq_H = 1$.

(d) The six functions (4) are holomorphic in a neighbourhood of $L_0 \cup L_h$, and $\bar{q} = \tilde{q}$, $\overline{q_G} = \tilde{q}_G$, $\overline{q_H} = \tilde{q}_H$ on $L_0 \cup L_h$.

(e) The zero sets of q , q_G and q_H are the zero set of sm , the zero set of cm and the set of poles of sm , respectively, and these three sets are pairwise disjoint.

Proof. (a) Periodicity holds since $p_0 \in \Lambda$. On L_0 use (D2); for $u \in L_h$ the point $u - P = \text{Re } u + \frac{p_0}{2}$ is real, and $\text{sm}(u) = \text{sm}(u - P)$, $\text{cm}(u) = \text{cm}(u - P)$. (b) $\text{cm}(w)^3 = 1 - (\epsilon i)^3 = 1 + \epsilon i$, so $\text{sm}'(w) = \text{cm}(w)^2 \neq 0$. (c) Near a pole of sm , $q_H = 1/(\text{sm}+i) \rightarrow 0$ and $q = 1 - iq_H$, so both are holomorphic there; moreover $(\text{cm}/\text{sm})^3 = \text{sm}^{-3} - 1 \rightarrow -1$, so cm/sm is bounded, hence holomorphic, near the pole, with a cube root of -1 as its value, and $q_G = (\text{cm}/\text{sm})/(1 + i/\text{sm})$ is holomorphic there. Elsewhere the only singularities are the zeros of $\text{sm} + i$; they are simple by (b), and the numerators $\text{sm} = -i$, $\text{cm} \neq 0$ and 1 do not vanish there. The tilde functions are treated in the same way, and $q + iq_H = (\text{sm}+i)/(\text{sm}+i) = 1$. (d) On $L_0 \cup L_h$ the values of sm are real or ∞ , so $\text{sm} \pm i \neq 0$ there, and (c) gives holomorphy near these lines. For real s and c we have $\overline{s/(s+i)} = s/(s-i)$, $\overline{c/(s+i)} = c/(s-i)$ and $\overline{1/(s+i)} = 1/(s-i)$; this gives the identities away from the poles of sm , and at the poles by continuity. (e) By (c), q is 1 at the poles of sm , so it vanishes exactly where $\text{sm} = 0$; q_H vanishes exactly at the poles of sm ; q_G vanishes where $\text{cm} = 0$ (there $\text{sm}^3 = 1$, so sm is finite and

$\text{sm} \neq -i$) and nowhere else, since at a pole of sm its value is a cube root of -1 . The three sets are disjoint because $\text{sm} = \text{cm} = 0$ contradicts $\text{sm}^3 + \text{cm}^3 = 1$ and a pole of sm is a pole of cm . $\square$

4 The strip, the zero sets and the inner function

Recall $S = \{0 < \text{Im } w < h\}$, $L_0 = \mathbb{R}$, $L_h = \mathbb{R} + ih$ and $\psi(z) = \frac{h}{\pi} \text{Log } \frac{1+z}{1-z} + \frac{ih}{2}$, with the principal branch of the logarithm.

Lemma 4.1. *The map ψ is holomorphic on $\Omega = \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ and maps $\mathbb{D}$ biholomorphically onto S , with inverse $\psi^{-1}(w) = \tanh \frac{\pi(w-ih/2)}{2h}$. It maps the arc $\{e^{it} : 0 < t < \pi\}$ onto L_h and the arc $\{e^{it} : \pi < t < 2\pi\}$ onto L_0 . The map $\gamma = \psi^{-1} \circ (\psi + p_0)$ is the automorphism*

$$\gamma(z) = \frac{z + \tau}{1 + \tau z}, \quad \tau = \tanh \frac{\pi}{\sqrt{3}} \in (0, 1),$$

with fixed points ± 1 ; it maps $(-1, 1)$ onto itself, and $\gamma^k(x) \rightarrow \pm 1$ as $k \rightarrow \pm\infty$ for every $x \in (-1, 1)$.

Proof. Since $(1+z)/(1-z) \in (-\infty, 0]$ exactly when $z \in (-\infty, -1] \cup (1, \infty)$, ψ is holomorphic on Ω . The map $z \mapsto (1+z)/(1-z)$ sends $\mathbb{D}$ onto the right half-plane, which Log maps onto $\{|\text{Im } s| < \pi/2\}$; hence $\psi(\mathbb{D}) = S$, and solving for z gives the inverse. For $\zeta = e^{it}$,

$$\frac{1 + e^{it}}{1 - e^{it}} = \frac{e^{-it/2} + e^{it/2}}{e^{-it/2} - e^{it/2}} = i \cot \frac{t}{2},$$

whose argument is $\pi/2$ for $0 < t < \pi$ and $-\pi/2$ for $\pi < t < 2\pi$; so $\text{Im } \psi(\zeta)$ is h , respectively 0 , and $\text{Re } \psi(\zeta) = \frac{h}{\pi} \log |\cot \frac{t}{2}|$ runs through $\mathbb{R}$ on each arc. In the variable $s = \pi(w - ih/2)/h$ we have $\psi^{-1}(w) = \tanh(s/2)$, and $w \mapsto w + p_0$ becomes $s \mapsto s + \pi p_0/h = s + 2\pi/\sqrt{3}$; the addition theorem for $\tanh$ gives the formula for γ . The points of $(-1, 1)$ correspond to real s , and $\gamma^k(x) = \tanh(s/2 + k\pi/\sqrt{3}) \rightarrow \pm 1$ as $k \rightarrow \pm\infty$. $\square$

Let

$$Z_S = \{w \in S : \text{sm}(w) = -i\}, \quad \tilde{Z}_S = \{w \in S : \text{sm}(w) = i\}, \quad Z = \psi^{-1}(Z_S), \quad \tilde{Z} = \psi^{-1}(\tilde{Z}_S).$$

Lemma 4.2. (a) Z_S and $\tilde{Z}_S$ are invariant under $w \mapsto w \pm p_0$, consist of finitely many points modulo p_0 , are nonempty, and have positive distance from $L_0 \cup L_h$. Their points are simple zeros of $\text{sm} + i$ and of $\text{sm} - i$, respectively.

(b) Z and $\tilde{Z}$ are invariant under γ and γ^{-1} , satisfy the Blaschke condition, and the set of their accumulation points on $\mathbb{T}$ is $\{-1, 1\}$ in both cases.

Proof. (a) Invariance follows from Corollary 3.2(a). In the compact rectangle $R = [0, p_0] \times [0, h]$ the function $\text{sm} + i$ has finitely many zeros, since its zeros are isolated and cannot accumulate at poles of sm , and it has none on $L_0 \cup L_h$, where sm is real or ∞ . Since $Z_S = (Z_S \cap R) + p_0\mathbb{Z}$, the set Z_S is finite modulo p_0 and has positive distance from the two lines. The function sm induces a nonconstant holomorphic map from the compact Riemann surface $\mathbb{C}/\Lambda$ to the Riemann sphere, which is therefore surjective; take $u \in \mathbb{C}$ with $\text{sm}(u) = -i$. As $P \in \Lambda$ and $\text{Im } P = h$, there is $k \in \mathbb{Z}$ with $0 \leq \text{Im}(u - kP) < h$, and $\text{Im}(u - kP) \neq 0$ because $\text{sm}(u - kP) = -i$ is not real; so $u - kP \in Z_S$. Simplicity is Corollary 3.2(b). The same arguments apply to $\tilde{Z}_S$.

(b) The invariance follows from (a) and Lemma 4.1. For $w \in S$ write $s = \pi(w - ih/2)/h = \sigma + i\beta$ with $|\beta| < \pi/2$, so that $z = \psi^{-1}(w) = \tanh(s/2) = (e^s - 1)/(e^s + 1)$ and

$$1 - |z|^2 = \frac{|e^s + 1|^2 - |e^s - 1|^2}{|e^s + 1|^2} = \frac{4e^\sigma \cos \beta}{|e^s + 1|^2}.$$

If $\sigma \geq 1$, then $|e^\sigma + 1| \geq e^\sigma - 1 \geq e^\sigma(1 - e^{-1})$; if $\sigma \leq -1$, then $|e^\sigma + 1| \geq 1 - e^\sigma \geq 1 - e^{-1}$; and if $|\sigma| < 1$, then $1 - |z|^2 \leq 1$. In all cases

$$1 - |z|^2 \leq C_0 e^{-|\sigma|}, \quad C_0 = \frac{4}{(1 - e^{-1})^2} > e. \quad (5)$$

Along a γ -orbit $w_k = w_0 + kp_0$ we have $\sigma_k = \sigma_0 + 2\pi k/\sqrt{3}$, hence

$$\sum_{k \in \mathbb{Z}} (1 - |\psi^{-1}(w_k)|) \leq \sum_{k \in \mathbb{Z}} (1 - |\psi^{-1}(w_k)|^2) \leq C_0 \sum_{k \in \mathbb{Z}} e^{-|\sigma_0 + 2\pi k/\sqrt{3}|} < \infty.$$

By (a), Z is a finite union of such orbits, so it satisfies the Blaschke condition. Since $|\tanh(s/2) \mp 1| = 2/|e^{\pm s} + 1| \leq 2/(e^{\pm \sigma} - 1)$ for $\pm \sigma > 0$, the points of an orbit tend to 1 as $k \rightarrow \infty$ and to -1 as $k \rightarrow -\infty$, and only finitely many points of Z have $|\sigma|$ below any given bound. Hence the accumulation points of Z on $\mathbb{T}$ are exactly -1 and 1 . The same holds for $\tilde{Z}$. $\square$

Let B and $\tilde{B}$ be the Blaschke products with the simple zeros Z and $\tilde{Z}$, respectively, and put

$$\theta_0 = zB\tilde{B}. \quad (6)$$

Proposition 4.3. θ_0 is a Blaschke product with $\theta_0(0) = 0$, the zeros of B and those of $\tilde{B}$ accumulate on $\mathbb{T}$ exactly at -1 and 1 , and $\sigma(\theta_0) \cap \mathbb{T} = \{-1, 1\}$. Moreover $|B \circ \gamma| = |B|$ and $|\tilde{B} \circ \gamma| = |\tilde{B}|$ on $\mathbb{D}$.

Proof. By Lemma 4.2 and (H2), B and $\tilde{B}$ exist, and θ_0 is a Blaschke product with zero set $\{0\} \cup Z \cup \tilde{Z}$. Its singular measure is zero, and the closure of its zero set meets $\mathbb{T}$ in $\{-1, 1\}$; so $\sigma(\theta_0) \cap \mathbb{T} = \{-1, 1\}$ by the definition recalled in Section 1. The last statement is Lemma 2.2 with $\varphi = \gamma$, since $\gamma^{-1}(Z) = Z$ and $\gamma^{-1}(\tilde{Z}) = \tilde{Z}$. $\square$

5 Proof of Theorem 1.1

With B , $\tilde{B}$ and the functions (4) as above, define on $\mathbb{D}$

$$\begin{aligned} F &= B(q \circ \psi), & G &= B(q_G \circ \psi), & H &= B(q_H \circ \psi), \\ \tilde{F} &= \tilde{B}(\tilde{q} \circ \psi), & \tilde{G} &= \tilde{B}(\tilde{q}_G \circ \psi), & \tilde{H} &= \tilde{B}(\tilde{q}_H \circ \psi). \end{aligned} \quad (7)$$

This agrees with (3). Proposition 4.3 gives part (a) of Theorem 1.1.

5.1 Holomorphy and the identities

By Corollary 3.2(c) and Lemma 4.1, the functions $q \circ \psi$, $q_G \circ \psi$ and $q_H \circ \psi$ are meromorphic on $\mathbb{D}$, and their poles are exactly the points of Z , all simple; B has a simple zero at each of these points. Hence F , G , H are holomorphic on $\mathbb{D}$, and they do not vanish on Z . The same holds for $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ and $\tilde{Z}$. Away from Z and from the poles of $\text{sm} \circ \psi$, (D1) and Corollary 3.2(c) give

$$F^3 + G^3 - H^3 = B^3 \frac{(\text{sm}^3 + \text{cm}^3 - 1) \circ \psi}{(\text{sm} \circ \psi + i)^3} = 0, \quad F + iH = B(q + iq_H) \circ \psi = B,$$

and by continuity both identities hold on $\mathbb{D}$. This proves (b) and the first statement of (d).

5.2 Boundedness

Lemma 5.1. *The six functions (7) belong to H^∞ .*

Proof. Consider F . By Lemma 4.2(a) we can choose $\rho > 0$ smaller than half the distance between any two distinct points of Z_S and smaller than the distance from Z_S to $L_0 \cup L_h$. The closed discs $\overline{D(w, \rho)}$, $w \in Z_S$, are then pairwise disjoint and contained in S . By Corollary 3.2(c),(d), q is continuous on the compact set $K = R \setminus \bigcup_{w \in Z_S} D(w, \rho)$, $R = [0, p_0] \times [0, h]$, so $|q| \leq M$ on K for some M , and by periodicity $|q| \leq M$ on $\bar{S} \setminus \bigcup_{w \in Z_S} D(w, \rho)$. Since $|B| \leq 1$, we get $|F| \leq M$ on $\mathbb{D} \setminus \bigcup_w U_w$, where $U_w = \psi^{-1}(D(w, \rho))$. The closure of U_w is $\psi^{-1}(\overline{D(w, \rho)})$, a compact subset of $\mathbb{D}$, and on $\partial U_w = \psi^{-1}(\partial D(w, \rho))$, which lies outside all the discs, $|F| \leq M$. By the maximum principle $|F| \leq M$ on U_w . Hence $|F| \leq M$ on $\mathbb{D}$. The other five functions are treated in the same way, with $\tilde{Z}_S$ in place of Z_S for $\tilde{F}, \tilde{G}, \tilde{H}$. $\square$

5.3 Membership in the model space

Lemma 5.2. *A.e. on $\mathbb{T}$ we have $\theta_0 \bar{z} \tilde{F} = \tilde{F}$, $\theta_0 \bar{z} \tilde{G} = \tilde{G}$ and $\theta_0 \bar{z} \tilde{H} = \tilde{H}$. Consequently $F, G, H \in K_{\theta_0}^\infty$, and $F, G, H \in K_{\theta_0^p}^p$ for every $p \in [1, \infty]$.*

Proof. Let $\zeta \in \mathbb{T} \setminus \{\pm 1\}$. By Lemma 4.1 and Corollary 3.2(d), $q \circ \psi$ and $\tilde{q} \circ \psi$ are holomorphic in a neighbourhood of ζ in $\mathbb{C}$, and $\tilde{q}(\psi(\zeta)) = \overline{q(\psi(\zeta))}$, because $\psi(\zeta) \in L_0 \cup L_h$. By (H1) and (H2), B and $\tilde{B}$ have nontangential limits of modulus one at almost every $\zeta \in \mathbb{T}$. At such a point $\zeta \neq \pm 1$ the boundary values are $F(\zeta) = B(\zeta)q(\psi(\zeta))$ and $\tilde{F}(\zeta) = \tilde{B}(\zeta)\tilde{q}(\psi(\zeta))$, and

$$\theta_0(\zeta) \bar{\zeta} \overline{F(\zeta)} = \zeta B(\zeta) \tilde{B}(\zeta) \bar{\zeta} \overline{B(\zeta) q(\psi(\zeta))} = |B(\zeta)|^2 \tilde{B}(\zeta) \tilde{q}(\psi(\zeta)) = \tilde{F}(\zeta).$$

The same computation with $q_G, \tilde{q}_G$ and $q_H, \tilde{q}_H$ applies to G and H . Since $\tilde{F}, \tilde{G}, \tilde{H} \in H^\infty$ by Lemma 5.1, Lemma 2.1 shows that $F, G, H \in K_{\theta_0}^\infty$, and Lemma 2.1(a) gives the last statement. $\square$

Thus $\tilde{F}, \tilde{G}, \tilde{H}$ are the conjugates of F, G, H in the sense of [Gar26b, Section 2.1]. The inner function $B = F + iH$ lies in $K_{\theta_0}^\infty$; in agreement with Dyakonov's remark that K_{θ}^∞ contains inner functions only if $\theta(0) = 0$, and that these are then the divisors of θ/z [Dya14, p. 2], B divides $\theta_0/z = B\tilde{B}$. The factor z in θ_0 cannot be omitted for this triple: if F and H belonged to some $K_{B\tilde{B}}^p$, then so would B , and by Lemma 2.1 the function $\bar{z}\tilde{B}$ would be the boundary function of an H^p function, that is, $\tilde{B}(0) = 0$, which means $\psi(0) = ih/2 \in \tilde{Z}_S$; but $ih/2$ is not among the points of Table 1 (numerically, $\text{sm}(ih/2) = -0.629960 \dots + 0.429160 \dots i$).

5.4 Linear independence

Suppose that $aF + bG + cH = 0$ on $\mathbb{D}$ with $(a, b, c) \in \mathbb{C}^3 \setminus \{0\}$. Dividing by B and composing with ψ^{-1} , we get $(a \text{sm} + b \text{cm} + c)/(\text{sm} + i) = 0$ on S outside a discrete set, so $a \text{sm} + b \text{cm} + c = 0$ on $\mathbb{C}$ by the identity theorem. If $b \neq 0$, then $\text{cm} = -(a \text{sm} + c)/b$, and (D1) gives $\text{sm}^3 - (a \text{sm} + c)^3/b^3 - 1 = 0$. Since sm takes infinitely many values, the polynomial $X^3 - (aX + c)^3/b^3 - 1$ vanishes identically, that is,

$$1 - \frac{a^3}{b^3} = 0, \quad \frac{3a^2c}{b^3} = 0, \quad \frac{3ac^2}{b^3} = 0, \quad \frac{c^3}{b^3} = -1.$$

The first equation gives $a \neq 0$, the second then gives $c = 0$, and this contradicts the fourth. If $b = 0$, then $a \text{sm} + c = 0$ with sm nonconstant forces $a = c = 0$. Hence F, G, H are linearly independent, which is (c).

5.5 Inner factors

Lemma 5.3. *The inner factors of F , G and H are, up to unimodular constants, the Blaschke products whose zeros are the points of $\mathbb{D}$ at which $\text{sm} \circ \psi$ vanishes, at which $\text{cm} \circ \psi$ vanishes, and at which $\text{sm} \circ \psi$ has a pole, respectively, with multiplicities. These three zero sets are pairwise disjoint.*

Proof. Let $f \in \{F, G, H\}$ and write $f = B(r \circ \psi)$ with $r \in \{q, q_G, q_H\}$. The function f has no zeros on Z , and off Z its zeros are those of $r \circ \psi$; by Corollary 3.2(e) these form the three sets of the statement, which are pairwise disjoint. It remains to show that the singular measure μ of the inner factor of f is zero.

The function f extends holomorphically across $\mathbb{T} \setminus \{\pm 1\}$: the factor $r \circ \psi$ does by Lemma 4.1 and Corollary 3.2(d), and the Blaschke product B does because the accumulation points of its zeros on $\mathbb{T}$ are ± 1 [GMR16, Thm. 7.3.4 and Prop. 7.3.8]. By Lemma 2.3, $\mu(\mathbb{T} \setminus \{\pm 1\}) = 0$.

By Proposition 4.3 and the periodicity of r , $|f \circ \gamma| = |B \circ \gamma| |r \circ (\psi + p_0)| = |B| |r \circ \psi| = |f|$ on $\mathbb{D}$. Choose $x \in (-1, 1)$ with $f(x) \neq 0$. By Lemma 4.1, $x_k = \gamma^k(x) \in (-1, 1)$ tends to 1 as $k \rightarrow \infty$, and $|f(x_k)| = |f(x)| > 0$ for all k . If μ had an atom of mass $m > 0$ at 1, then by (H3), for $0 < r < 1$,

$$|f(r)| \leq \|f\|_\infty |S_\mu(r)| \leq \|f\|_\infty \exp\left(-m \frac{1-r^2}{(1-r)^2}\right) = \|f\|_\infty \exp\left(-m \frac{1+r}{1-r}\right) \rightarrow 0$$

as $r \rightarrow 1^-$, which is impossible. In the same way, $x_k \rightarrow -1$ as $k \rightarrow -\infty$ excludes an atom at -1 , since $(1-r^2)/(1+r)^2 \rightarrow \infty$ as $r \rightarrow -1^+$. Hence $\mu = 0$. $\square$

If a nonconstant inner function divided two of the functions F , G , H , then by Lemma 5.3 and the remark after (H4) it would be a Blaschke product whose zeros are common zeros of the two functions; there are none. This completes the proof of (d), and of Theorem 1.1.

Remark 5.4. $\|F\|_\infty = \|G\|_\infty = \|H\|_\infty = 1$. Indeed, as $|B| = 1$ a.e. on $\mathbb{T}$, $\|f\|_\infty$ is the supremum of $|r|$ on $L_0 \cup L_h$, where $s = \text{sm}$ and $c = \text{cm}$ are real; there $|q| = |s|/\sqrt{1+s^2}$, $|q_H| = 1/\sqrt{1+s^2}$ and $|q_G| = |c|/\sqrt{1+s^2} \leq 1$, as $(1+s^2)^3 - (1-s^2)^2 = s^2(3s^2+2s+3) \geq 0$. The value 1 is attained by $|q_G|$ and $|q_H|$ at $w = 0$ and by $|q|$ at the real poles of sm , which exist since sm is real, p_0 -periodic and nonconstant on $\mathbb{R}$ with $\text{sm}' = \text{cm}^2 \geq 0$.

6 Proofs of the corollaries, and remarks

Proof of Corollary 1.2. The projective Fermat cubic is smooth, so [Gar26b, Theorem 1.1(a)] applies to $P = x^3 + y^3 - z^3$. By Theorem 1.1, $\#(\sigma(\theta_0) \cap \mathbb{T}) = 2$ and, for every $p \in [1, \infty]$, $K_{\theta_0}^p$ contains the triple (F, G, H) with $P(F, G, H) = 0$ and $\dim \text{span}\{F, G, H\} = 3$. $\square$

Proposition 6.1. *Let φ be an automorphism of $\mathbb{D}$ and $\theta_\varphi = z(B \circ \varphi)(\tilde{B} \circ \varphi)$. Then θ_φ is a Blaschke product with $\sigma(\theta_\varphi) \cap \mathbb{T} = \{\varphi^{-1}(-1), \varphi^{-1}(1)\}$, and the functions $F \circ \varphi$, $G \circ \varphi$, $H \circ \varphi$ belong to $K_{\theta_\varphi}^\infty$, are linearly independent, and satisfy $(F \circ \varphi)^3 + (G \circ \varphi)^3 = (H \circ \varphi)^3$.*

Proof. By Lemma 2.2, $B \circ \varphi$ and $\tilde{B} \circ \varphi$ are, up to unimodular constants, the Blaschke products with the zeros $\varphi^{-1}(Z)$ and $\varphi^{-1}(\tilde{Z})$. Since φ^{-1} is a homeomorphism of $\mathbb{D}$, Lemma 4.2(b) shows that these zeros accumulate on $\mathbb{T}$ exactly at $\varphi^{-1}(\pm 1)$; hence θ_φ is a Blaschke product with $\sigma(\theta_\varphi) \cap \mathbb{T} = \{\varphi^{-1}(-1), \varphi^{-1}(1)\}$, and $B \circ \varphi$, $\tilde{B} \circ \varphi$ have nontangential limits of modulus one a.e. on $\mathbb{T}$. The Fermat identity and the linear independence are preserved under composition with the bijection φ . For $\zeta \in \mathbb{T}$ with $\varphi(\zeta) \neq \pm 1$, the functions $q \circ \psi \circ \varphi$ and $\tilde{q} \circ \psi \circ \varphi$ are holomorphic near ζ , and $\tilde{q}(\psi(\varphi(\zeta))) = \overline{q(\psi(\varphi(\zeta)))}$. Hence, as in Lemma 5.2, a.e. on $\mathbb{T}$

$$\theta_\varphi \overline{z F \circ \varphi} = z(B \circ \varphi)(\tilde{B} \circ \varphi) \overline{z B \circ \varphi} \overline{q \circ \psi \circ \varphi} = (\tilde{B} \circ \varphi)(\tilde{q} \circ \psi \circ \varphi) = \tilde{F} \circ \varphi \in H^\infty,$$

and Lemma 2.1 gives $F \circ \varphi \in K_{\theta_\varphi}^\infty$; the same argument applies to $G \circ \varphi$ and $H \circ \varphi$. $\square$

Proof of Corollary 1.3. Let $a \neq b$ be points of E . The automorphism group of $\mathbb{D}$ acts transitively on ordered pairs of distinct points of $\mathbb{T}$ (map b to 1 by a rotation, pass to the upper half-plane by a Cayley transform sending 1 to ∞ , and translate), so there is an automorphism φ with $\varphi(a) = -1$ and $\varphi(b) = 1$. Let $(e_j)_{j \geq 1}$ be a sequence whose set of values is dense in E , and let θ_1 be the Blaschke product with the zeros $(1 - 2^{-j-k})e_j$, $j, k \geq 1$; the Blaschke condition holds because $\sum_{j,k} 2^{-j-k} = 1$. Every e_j is an accumulation point of these zeros, and the set of accumulation points is closed, so it contains E ; conversely, if $(1 - 2^{-j_n-k_n})e_{j_n} \rightarrow \zeta \in \mathbb{T}$, then $j_n + k_n \rightarrow \infty$, hence $e_{j_n} \rightarrow \zeta$ and $\zeta \in E$. The Blaschke product $\theta = \theta_\varphi \theta_1$ therefore has $\sigma(\theta) \cap \mathbb{T} = \{a, b\} \cup E = E$, and by Proposition 6.1 and Lemma 2.1(b) the functions $F \circ \varphi$, $G \circ \varphi$, $H \circ \varphi$ lie in K_θ^∞ . $\square$

Remark 6.2. The factor z in θ_φ matters. For a rotation φ the function $\theta_0 \circ \varphi$ also works, but its boundary spectrum is an antipodal pair. For a general automorphism, $\theta_0 \circ \varphi = \varphi(B \circ \varphi)(\tilde{B} \circ \varphi)$, and the computation in the proof of Proposition 6.1 gives $(\theta_0 \circ \varphi) \bar{z} F \circ \varphi = \bar{z} \varphi(F \circ \varphi)$ a.e. on $\mathbb{T}$. For $g \in H^\infty$, the function $\bar{z}g$ is the boundary function of an H^1 function if and only if $g(0) = 0$; so by Lemma 2.1, $F \circ \varphi$ lies in some $K_{\theta_0 \circ \varphi}^p$ if and only if $\varphi(0)\tilde{F}(\varphi(0)) = 0$, which fails for a general automorphism. On the other hand $z(\theta_0 \circ \varphi) = \varphi \theta_\varphi$ is a multiple of θ_φ , so it works by Lemma 2.1(b).

Remark 6.3. In Garcia's proof of the one-point case [Gar26b, Lemmas 3.4 and 3.7], a nontrivial triple and its pseudocontinuation glue to a holomorphic map from the sphere minus one point into the cubic, which is lifted through the universal covering $\mathbb{C} \rightarrow \mathbb{C}/\Lambda$ and shown to be constant. For our triple the glued map is $z \mapsto [\text{sm} : \text{cm} : 1](\psi(z))$ on $\hat{\mathbb{C}} \setminus \{\pm 1\}$; it is single-valued, because a loop around ± 1 changes ψ by $\pm 2ih = \pm p_0(1 + 2\omega) \in \Lambda$, and for $|z| > 1$ one checks that $\psi(1/\bar{z}) - \overline{\psi(z)} \in \{0, 2ih\}$. Since $\hat{\mathbb{C}} \setminus \{\pm 1\}$ is conformally equivalent to $\mathbb{C} \setminus \{0\}$, which is not simply connected, the lift is not available, and there is no contradiction.

7 Numerical checks and reproducibility

All statements of this note are proved without computation; the computations are checks. The reproducibility package contains the program of this note, `check_note.py` (Python 3 with `mpmath`, `numpy` and `sympy`; about ten seconds; it reads no files and writes only to standard output), with its recorded output, and the programs and outputs of the first computation (`original/`), of the two verification runs (`verification_run_A/`, `verification_run_B/`), which compute Dixon's functions by other methods and test further wrong inputs, and of the verification run on the final text (`independent_run_2/`), described at the end of this section. Its `README.md` gives command lines, running times and what each program checks. All recorded outputs were reproduced by re-runs from the package.

The program of this note checks in exact arithmetic the algebraic identities used in the proofs, among them those in the proofs of Lemma 3.1 and Corollary 3.2, $2ih = p_0(1 + 2\omega)$, and the coefficient system of Section 5, whose only solution is $a = b = c = 0$. It computes $\text{sm} = 6\wp/(1 - 3\wp')$ and $\text{cm} = (3\wp' + 1)/(3\wp' - 1)$, with $\wp$ the Weierstrass function for $g_2 = 0$, $g_3 = 1/27$, to 40 digits from the Laurent series of $\wp$ and a reduction modulo $p_0\mathbb{Z}[\omega]$; the lattice is tested by comparing the series at points that differ by a period, and the formulas are tested against (D1), which identifies them with Dixon's functions. It finds $p_0 = 5.2999162508563498719\dots = B(\frac{1}{3}, \frac{1}{3})$, $h = 4.5898621111715786247\dots$, and exactly three solutions of each of the equations $\text{sm} = \mp i$ in $[0, p_0) \times (0, h)$ (Table 1); so Z and $\tilde{Z}$ consist of three γ -orbits each. The points of Z_S form an equilateral triangle with centroid $p_0/3 + 2ih/3$, those of $\tilde{Z}_S$ are the points of $\overline{Z_S} + P$ modulo p_0 , and sm has three poles in $[0, p_0) \times [0, h)$, one of them the

Z_S : solutions of $\text{sm}(w) = -i$	$\tilde{Z}_S$: solutions of $\text{sm}(w) = i$
$0.749770454882856 + 3.646997258195469i$	$0.133548920259869 + 0.942864852976109i$
$1.766638750285450 + 1.885729705952219i$	$3.399728580311031 + 0.942864852976109i$
$2.783507045688044 + 3.646997258195469i$	$4.416596875713625 + 2.704132405219360i$

Table 1: The zeros of $\text{sm} \pm i$ in $[0, p_0) \times (0, h)$, rounded to 15 decimals (computed to 40 digits).

real pole $2p_0/3$. The zeros agree to $7 \cdot 10^{-25}$ with those of the first computation, which integrated (D1) by Taylor series.

In double precision, with the Blaschke factors written in the strip variable s of Lemma 4.1, the program tests membership from the definition: for $f \in K_\theta^2 = H^2 \ominus \theta H^2$ the inner products $\langle f, \theta z^k \rangle = \int_{\mathbb{T}} f \bar{\theta} \bar{z}^k dm$ vanish for $k \geq 0$. Computed by quadrature on the two boundary lines, where $dm = \text{sech}(\pi x/h) dx/(2h)$, $x = \text{Re } w$, they are below $2.2 \cdot 10^{-16}$ in absolute value for $f = F, G, H$, $\theta = \theta_0$ and $k = 0, \dots, 8$, while wrong inputs give values up to 0.10 ($\theta = B\tilde{B}$, factor z omitted), 0.28 ($\theta = zB$) and $7.9 \cdot 10^{-3}$ (a strip of height $0.97h$, on whose upper line sm is not real). For $\varphi(z) = e^{0.7i}(z-c)/(1-\bar{c}z)$, $c = 0.3 + 0.2i$, whose boundary spectral points are not antipodal, the inner products for $F \circ \varphi$, $G \circ \varphi$, $H \circ \varphi$ and θ_φ are below $2.6 \cdot 10^{-16}$, while $|\langle F \circ \varphi, \theta_0 \circ \varphi \rangle| = 0.0572858 = |\varphi(0)\tilde{F}(\varphi(0))|$, as Remark 6.2 predicts. The identities $F^3 + G^3 = H^3$ and $F + iH = B$ hold to $6.4 \cdot 10^{-16}$ at 4000 random points, and the suprema of $|F|$, $|G|$, $|H|$ on a grid of the closed period rectangle are 0.999998, 0.999998 and 0.999978 (Remark 5.4).

The programs of the verification run on the final text (`independent_run_2/`; `mpmath` with 40 digits, about two minutes) use other methods: Dixon's functions by Taylor integration of (D1), by Abel integrals on the cubic, and through the Weierstrass function written with Jacobi's function `cn`; the zeros of $\text{sm} \mp i$ from the integrals $\int_0^{\mp i} dt/(1-t^3)^{2/3}$ and the automorphism $u \mapsto \omega(u - p_0/3) + p_0/3$ of the curve; Blaschke products as products of the factors b_a in the variable z ; and integrals over $\mathbb{T}$ in the angle, with nodes graded towards ± 1 . They reproduce p_0 , Table 1 (to $4.6 \cdot 10^{-16}$, the rounding), the poles and the value 0.0572858; the inner products $\langle f, \theta_0 z^k \rangle$ are below $5 \cdot 10^{-19}$ (below $8 \cdot 10^{-19}$ after transport by another automorphism), and for $\theta = B\tilde{B}$ they are $\tilde{F}(0)$, $\tilde{G}(0)$, $\tilde{H}(0)$ (of absolute values 0.0774..., 0.1015..., 0.1015...) for $k = 0$ and vanish for $k \geq 1$, as Lemma 5.2 predicts.

Verification. The result was first obtained, with an earlier written form of the proofs of Sections 3 to 5, with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. Both re-derived every step of the proof of Theorem 1.1 and confirmed it; the first also proved $F + iH = B$, the coprimality of the inner factors and Remark 5.4. Both found that the earlier justification of Corollary 1.3 was wrong, since it used rotations, which only reach antipodal pairs; both proved the corrected form (Proposition 6.1) and checked it numerically. Their other required corrections concern notation and wording, for instance the constant term $c^3/b^3 = -1$ in Section 5 and the bibliographic data of [Dya23a]; all were made. Lemmas 2.2 and 2.3, the present proof of Lemma 5.3 and the program `check_note.py` were written after the two runs, together with this text; the first run had proved Lemma 5.3 with a different argument for the singular measure on $\mathbb{T} \setminus \{\pm 1\}$. The program `check_note.py` is the exact program of this note.

A further AI-assisted verification run, called run 2 in the package, then examined the final text: first the parts that the two runs had not seen (Lemmas 2.2 and 2.3, the proof of Lemma 5.3, Remark 6.2 and `check_note.py`), then all other proofs, line by line. It read the sources again (the problem page with the bar over H_0^p , [Dya14], [Gar26a, Gar26b], [Dya23a, Dya23b] and Section 1 of [CF06], all but the problem page in their arXiv versions), wrote its own programs, ran all programs of the package again from the source archive, with identical outputs, and repeated the search of the literature. It

found no error in a proof and no earlier answer to the question. Its required corrections, all made: the page numbers of [Dya14] are now said to be those of the arXiv version; the reason why the factor z cannot be omitted in Section 5.3 ($\tilde{B}(0) \neq 0$) is now given; a symbol used twice in Section 4 was renamed; the digits of p_0 and h in this section were corrected; the papers [Dya23a, Dya23b] are described more precisely; and the searches of this run are recorded below.

Scope and priority. The note answers Problem 2 of [Dya10, Dya14], the Fermat part of record OWR-4531-016 [ula26], negatively for $n = 3$ and every $p \in [1, \infty]$. The case $n \geq 4$ is not addressed here, and neither is Problem 1 on extreme points; so the record is answered only in part. The question and the positive answer for finite Blaschke products are Dyakonov's [Dya10, Dya14]; the positive answers for $n \geq 4$ with boundary spectrum of capacity zero and for $n = 3$ with at most one boundary spectral point are Garcia's [Gar26a, Gar26b]; Dixon's functions are classical [Dix90, CF06], and so is the solution of the cubic Fermat equation by elliptic functions (Gross, Baker; see [Gar26a, Remark 1.2]). The contribution of this note is the explicit counterexample of Theorem 1.1 and Corollaries 1.2 and 1.3; to our knowledge they are new. No printed answer to the question was found in the sources searched: [Dya10] (the problem page), [Dya14], [Gar26a] and [Gar26b], read in full; Garcia's arXiv listing and the submissions in math.CV and math.FA announced up to 16 October 2026, checked twice on 17 October 2026, the second time by run 2; the works citing [Dya14] listed by Semantic Scholar (titles: besides [Gar26a, Gar26b, GMR16], two papers of Dyakonov on extreme points [Dya23a, Dya23b], which mention Problem 1 as open and do not mention Problem 2, and an unrelated paper); queries to zbMATH Open, Crossref and Semantic Scholar; and six web searches. Keyword queries to the arXiv interface and the citation lists of OpenAlex were refused (rate limits). A search that finds nothing is not a proof of novelty, and no priority is claimed. We did not read [Dix90]; what we say about it is taken from [CF06], of which we read Section 1 (arXiv version). The books [GMR16, Gar07] are cited for standard facts and were not consulted; the facts on the boundary spectrum are quoted from [GMR16] as stated in [Gar26a, Gar26b].

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [CF06] Eric van Fossen Conrad and Philippe Flajolet. The Fermat cubic, elliptic functions, continued fractions, and a combinatorial excursion. *Séminaire Lotharingien de Combinatoire*, 54:Art. B54g, 44 pp., 2006. [arXiv:math/0507268](#).
- [Dix90] Alfred Cardew Dixon. On the doubly periodic functions arising out of the curve $x^3 + y^3 - 3\alpha xy = 1$. *The Quarterly Journal of Pure and Applied Mathematics*, 24:167–233, 1890. Not consulted; cited as in [CF06].
- [Dya10] Konstantin M. Dyakonov. Two problems on star-invariant subspaces. *Oberwolfach Reports*, 7(4):2813–2875, 2010. Report No. 49/2010, Operator Theory and Harmonic Analysis, organised by A. Borichev, R. Mortini, N. Nikolski and K. Seip; list of open problems, item 3, p. 2870. [doi:10.4171/OWR/2010/49](#).
- [Dya14] Konstantin M. Dyakonov. Two problems on coinvariant subspaces of the shift operator. *Integral Equations and Operator Theory*, 78(2):151–154, 2014. Page numbers in our

citations refer to the arXiv version (v1, 4 pages). `arXiv:1401.8031`,
`doi:10.1007/s00020-013-2110-0`.

- [Dya23a] Konstantin M. Dyakonov. Functions with small and large spectra as (non)extreme points in subspaces of H^∞ . *St. Petersburg Mathematical Journal*, 34(3):453–462, 2023. `doi:10.1090/spmj/1763`.
- [Dya23b] Konstantin M. Dyakonov. Questions about extreme points. *Integral Equations and Operator Theory*, 95(2):Paper No. 14, 2023. `doi:10.1007/s00020-023-02733-8`.
- [Gar07] John B. Garnett. *Bounded Analytic Functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, revised first edition, 2007. `doi:10.1007/0-387-49763-3`.
- [Gar26a] Stephan Ramon Garcia. Fermat’s Last Theorem in star-invariant subspaces: capacity-zero boundary spectrum, 2026. Preprint, version 1 of 12 August 2026; cited in [Gar26b] as to appear in Proc. Amer. Math. Soc. `arXiv:2608.11653`.
- [Gar26b] Stephan Ramon Garcia. Fermat’s Last Theorem in star-invariant subspaces II: small boundary spectrum, projective plane curves, and the cubic Fermat equation, 2026. Preprint, version 1 of 12 October 2026. `arXiv:2610.17180`.
- [GMR16] Stephan Ramon Garcia, Javad Mashreghi, and William T. Ross. *Introduction to Model Spaces and their Operators*, volume 148 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016. `doi:10.1017/CB09781316258231`.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record OWR-4531-016.