

Positive Recurrence of the Two-Chunk Deterministic First Chunk System: A Stable Population Process with an Unstable Large System Limit

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Abstract

In Oberwolfach Report 48/2010, I. Norros (joint work with H. Reittu) considered the two-chunk Deterministic First Chunk (DFC) file-sharing system of Norros, Reittu and Eirola, in which one permanent seed holds both chunks and arriving peers choose by a fair coin the chunk they download first. The large system limit of this population process is not stable, but the stochastic system was conjectured, on simulation evidence, to be stable; if so, instability of a population process cannot be inferred from instability of its large system limit. We prove the conjecture for the chain defined in Fig. 2 of arXiv:0910.5577v1, whose contact population consists of the seed and the peers holding one chunk: for every arrival rate $\lambda > 0$ this chain on $\mathbb{N}^4$ is irreducible, non-explosive and positive recurrent. Hence DFC is a stable population process whose large system limit is not stable; the theorem also refutes Conjecture 3.4 of arXiv:0910.5577v1 (instability for large λ). The proof is an explicit Foster–Lyapunov function with drift at most $-\frac{1}{2}$ outside an explicit finite set; the constants are crude. The mechanism is an exact identity: the difference Q between the two classes of peers has drift $(X - Y)/(X + Y + 1)$, a term of order one that comes from the seed alone; it vanishes in the large system scaling, where Q becomes a first integral. The model, its large system limit and the instability of the limit are due to Norros, Reittu and Eirola, the question to Norros and Reittu. The variant in which empty peers are contacted as well (contact population $A + B + X + Y + 1$) is not covered and remains open. This is an unrefereed note.

1 Introduction

1.1 The question

In their abstract in Oberwolfach Report 48/2010 [Nor10, pp. 2782–2783], Norros and Reittu consider the following file-sharing system. A file has two chunks, and one permanent seed holds both. Peers arrive at the times of a Poisson process of rate $\lambda > 0$; every peer other than the seed makes contacts at rate 1, and a peer leaves as soon as it holds both chunks. In the *Deterministic First Chunk* (DFC) system an arriving peer decides by a fair coin which chunk it downloads first and downloads nothing until it meets a holder of that chunk; then it takes the other chunk from the first holder it meets. The system was introduced and analysed by Norros, Reittu and Eirola [NRE09] (published as [NRE11]). Its state is $(A, B, X, Y) \in \mathbb{N}^4$, the numbers of empty peers that want chunk 0 (resp. 1) first and of peers holding only chunk 0 (resp. only chunk 1), and its rates are those of [NRE09, Fig. 2] (Table 1). Its large system limit (2) is not stable [NRE09, Prop. 3.3]. The stochastic system, however, “is conjectured, by simulation evidence, to be stable” [Nor10, p. 2782]; if so, instability of a

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population process cannot be inferred from instability of its large system limit. The earlier preprint arXiv:0910.5577v1 had conjectured the opposite [NRE09, Conjecture 3.4]: instability when λ is large enough. The corpus record OWR-4530-006 asks to prove or disprove the stability conjecture, and hence to decide whether instability of a population process can fail to follow from instability of its large system limit.

1.2 Result

Following [NRE09, Sec. 2], a continuous-time Markov chain is *stable* if it is irreducible and positive recurrent.

Theorem 1.1. *For every $\lambda > 0$ the Deterministic First Chunk chain of [NRE09, Fig. 2] (Table 1; contact population $X + Y + 1$) is irreducible, non-explosive and positive recurrent. Thus it is stable in the sense of [NRE09, Sec. 2].*

Corollary 1.2. *The DFC chain is a population process whose large system limit (2) is not stable while the process itself is stable for every $\lambda > 0$. Hence instability of a population process does not follow from instability of its large system limit.*

That (2) is not stable is [NRE09, Prop. 3.3]; it also follows from its unbounded curve of equilibria (Lemma 2.2). Theorem 1.1 confirms the conjecture of [Nor10, p. 2782] and refutes Conjecture 3.4 of arXiv:0910.5577v1 [NRE09]. We did not read the full text of the published version [NRE11]. Its abstract says that proofs for the stochastic systems are given only in the two simplest cases, which in the arXiv version are the Plain Random Contact and the Friedman systems [NRE09, Props. 3.2 and 3.6]; this suggests, but does not prove, that the conjecture was kept there.

1.3 The mechanism

Put $D = X + Y + 1$, $N = A + B + X + Y$ and $Q = (B + Y) - (A + X)$, the difference between the two classes of peers (by the chunk they want first). The generator $\mathcal{L}$ of the chain gives exactly (Lemma 2.1)

$$\mathcal{L}Q = \frac{X - Y}{X + Y + 1}, \quad \mathcal{L}N = \lambda - T, \quad T = \frac{2XY + X + Y}{X + Y + 1}. \quad (1)$$

The products XY cancel in $\mathcal{L}Q$; what is left comes from the seed alone (the terms “+1” in the rates). This drift is of order one while the rates are of order N , so it disappears in the large system scaling, and $q = (b + y) - (a + x)$ is a first integral of (2). All equilibria of (2) lie in the level set $\{q = 0\}$ (Lemma 2.2), so no trajectory with $q \neq 0$ can converge to an equilibrium. In the chain the drift remains. On the escaping trajectories of [NRE09, Prop. 3.3] (x and b grow while a and y stay bounded) the larger class holds most chunks, and in the chain $|Q|$ decreases at rate about one; once $|Q|$ is small, Y grows (again by a seed term), the departure rate $T \approx 2Y + 1$ grows with it, and N falls. The Lyapunov function of Section 4 combines a Huber function of Q with bounded increasing functions of Y and of X .

1.4 What is known, what is new, what stays open

The model, the limit (2) and its instability, the notions of stability and the remark that λ scales the limit but not the chain are due to Norros, Reittu and Eirola [NRE09], the question and the stability conjecture to Norros and Reittu [Nor10]; the Foster–Lyapunov method is standard (Lemma 3.2, [MT93]). New, as far as we know, are the identity (1) with the first integral q , the Lyapunov function and Theorem 1.1; no earlier proof or disproof was found in the sources searched (Section 7), and a search that finds nothing is not a proof of novelty. The theorem concerns only the chain of [NRE09,

Fig. 2]. The wording of [Nor10] (contact with “a randomly selected peer (seed included)”) also fits a chain in which empty peers are contacted as well (contact population $A + B + X + Y + 1$); that chain is not covered (Section 6). The constants are crude, and the finite set outside which the drift is negative is far from sharp.

2 The chain and its large system limit

The DFC chain $S(t) = (A, B, X, Y)$ is the continuous-time Markov chain on $\mathbb{N}^4$ with the six transitions of Table 1, read off from [NRE09, Fig. 2]: the seed is counted in the contact population $X + Y + 1$, and empty peers are not contacted.

	event	transition	rate
r_1	arrival that wants chunk 0 first	$A \rightarrow A + 1$	$\lambda/2$
r_2	arrival that wants chunk 1 first	$B \rightarrow B + 1$	$\lambda/2$
r_3	empty peer gets chunk 0 (from X -peer or seed)	$(A, X) \rightarrow (A - 1, X + 1)$	$A(X + 1)/D$
r_4	empty peer gets chunk 1 (from Y -peer or seed)	$(B, Y) \rightarrow (B - 1, Y + 1)$	$B(Y + 1)/D$
r_5	X -peer gets chunk 1 and leaves	$X \rightarrow X - 1$	$X(Y + 1)/D$
r_6	Y -peer gets chunk 0 and leaves	$Y \rightarrow Y - 1$	$(X + 1)Y/D$

Table 1: The DFC chain of [NRE09, Fig. 2]; $D = X + Y + 1$.

The large system limit is defined in [NRE09, Sec. 2] (arrival rates and states are multiplied by a parameter tending to infinity, and the drift is divided by it); for the DFC chain it is [NRE09, eq. (5)]

$$\dot{a} = \frac{\lambda}{2} - \frac{ax}{x+y}, \quad \dot{b} = \frac{\lambda}{2} - \frac{by}{x+y}, \quad \dot{x} = \frac{(a-y)x}{x+y}, \quad \dot{y} = \frac{(b-x)y}{x+y} \quad (2)$$

on $(0, \infty)^4$. Such a system is *stable* [NRE09, Sec. 2] if there is a compact set $K \subset (0, \infty)^4$ which every trajectory reaches and in which it eventually stays. The finite-set criterion for the stability of chains recalled in [NRE09, Sec. 2] is proved, in the form we use, in Lemma 3.2.

Lemma 2.1. *For every state, with $T = r_5 + r_6 = (2XY + X + Y)/D$,*

$$\mathcal{L}Q = r_5 - r_6 = \frac{X - Y}{D}, \quad \mathcal{L}N = \lambda - T.$$

Proof. The transitions r_3 and r_4 leave Q and N unchanged. The arrivals change Q by -1 and $+1$ at the same rate $\lambda/2$, and N by $+1$. The departures r_5 and r_6 change Q by $+1$ and -1 and N by -1 . Hence $\mathcal{L}Q = r_5 - r_6 = (X(Y + 1) - (X + 1)Y)/D = (X - Y)/D$ and $\mathcal{L}N = \lambda - r_5 - r_6$. $\square$

Lemma 2.2. *The quantity $q = (b + y) - (a + x)$ is a first integral of (2). The equilibria of (2) in $(0, \infty)^4$ are the points*

$$(a, b, x, y) = \left(\theta, \frac{\lambda\theta}{2\theta - \lambda}, \frac{\lambda\theta}{2\theta - \lambda}, \theta \right), \quad \theta > \frac{\lambda}{2};$$

they satisfy $q = 0$, and (2) is not stable.

Proof. Adding the equations for a and x gives $(a + x)' = \frac{\lambda}{2} - \frac{xy}{x+y}$, and adding those for b and y gives the same expression; hence $q' = 0$. At an equilibrium, $\dot{x} = 0$ and $\dot{y} = 0$ give $a = y$ and $b = x$, and then $\dot{a} = 0$ and $\dot{b} = 0$ both reduce to $xy = \frac{\lambda}{2}(x + y)$; with $a = y = \theta$ this is $x(2\theta - \lambda) = \lambda\theta$, so $\theta > \lambda/2$ and $x = b = \lambda\theta/(2\theta - \lambda)$; since $a = y$ and $b = x$, $q = 0$. This is the curve of [NRE09, Prop. 3.3]. An equilibrium is a constant trajectory, so it lies in every compact set that every trajectory eventually stays in; since the curve is unbounded in $(0, \infty)^4$, there is no such set. $\square$

Proof of Corollary 1.2, given Theorem 1.1. The chain is of the kind considered in [NRE09, Sec. 2], with large system limit (2), which is not stable by Lemma 2.2. $\square$

Remark 2.3. The same computation is consistent with the two stochastic results of [NRE09]. In the Plain Random Contact system [NRE09, Fig. 1] (arrivals with chunk 0 and 1 at rates $\lambda(X + \frac{1}{2})/D$ and $\lambda(Y + \frac{1}{2})/D$, departures at rates r_5, r_6) one finds $\mathcal{L}(Y - X) = (1 - \lambda)(X - Y)/D$: the arrivals copy the majority chunk and beat the seed term exactly when $\lambda > 1$, the threshold of the instability theorem [NRE09, Prop. 3.2]. In the Friedman system [NRE09, Fig. 3] the arrival rates are exchanged and $\mathcal{L}(Y - X) = (1 + \lambda)(X - Y)/D$, restoring for every λ , as in [NRE09, Prop. 3.6]. In DFC the arrivals are split evenly and add nothing to $\mathcal{L}Q$. These are our consistency checks.

3 Preliminaries

Lemma 3.1. (a) *The DFC chain is irreducible.*

- (b) *On the set $\{N \leq n\}$ the total jump rate is at most $\lambda + n$. If σ_n denotes the exit time from $\{N \leq n\}$, then $\sigma_n \rightarrow \infty$ almost surely; in particular the chain is non-explosive.*
- (c) *If $X \geq Y$, then $Y \leq T = 2Y + 1 - \frac{2Y^2 + 2Y + 1}{D} < 2Y + 1$. In general $\min(X, Y) \leq T < 2\min(X, Y) + 1$.*
- (d) *The map $\sigma(A, B, X, Y) = (B, A, Y, X)$ maps the chain to itself (it exchanges r_1 and r_2 , r_3 and r_4 , r_5 and r_6), and $Q \circ \sigma = -Q$.*

Proof. (a) From any state the chain reaches 0: the empty peers move to X or Y ($r_3 > 0$ if $A \geq 1$, $r_4 > 0$ if $B \geq 1$, thanks to the seed), then the other peers leave. From 0 it reaches (a, b, x, y) by $a + x$ arrivals to A , x moves to X , $b + y$ arrivals to B and y moves to Y .

(b) Since $X + 1 \leq D$ and $Y + 1 \leq D$, we have $r_3 \leq A$, $r_4 \leq B$, $r_5 \leq X$ and $r_6 \leq Y$, so the total rate is at most $\lambda + N$. The process stopped at σ_n is a Markov chain with bounded rates on the finite set $\{N \leq n + 1\}$, since every jump changes N by at most one. For a chain on a finite set Dynkin's formula holds for every function; applied to N , with Lemma 2.1, it gives $\mathbb{E}_s N(t \wedge \sigma_n) = N(s) + \mathbb{E}_s \int_0^{t \wedge \sigma_n} (\lambda - T) du \leq N(s) + \lambda t$. As $N(\sigma_n) = n + 1$ on $\{\sigma_n \leq t\}$, we get $(n + 1) \mathbb{P}_s(\sigma_n \leq t) \leq N(s) + \lambda t$. Hence $\sigma_n \rightarrow \infty$ almost surely, and the chain makes only finitely many jumps in every bounded time interval.

(c) Let $X \geq Y$. Expanding, $(2Y + 1)D - (2XY + X + Y) = 2Y^2 + 2Y + 1 > 0$, which gives the formula for T and $T < 2Y + 1$. Moreover $2XY + X + Y - Y(2X + 1) = X \geq 0$, so $T \geq Y(2X + 1)/D \geq Y$; the last step uses $2X + 1 \geq X + Y + 1 = D$, which holds because $X \geq Y$. Since T is symmetric in X and Y , the general statement follows.

(d) is read off from Table 1. $\square$

We use the Foster–Lyapunov criterion in the following form. Criteria of this kind for continuous-time processes, proved by Dynkin's formula, are developed in [MT93]; the finite-set criterion is recalled in [NRE09, Sec. 2]. For completeness we give the short proof of the form we need. For a function V on the state space we write $\mathcal{L}V(s) = \sum_{s'} q(s, s') (V(s') - V(s))$, where $q(s, s')$ are the jump rates.

Lemma 3.2. *Let S be an irreducible, non-explosive continuous-time Markov chain on a countable set E such that every state s has finitely many possible jumps and a finite total rate $q(s) > 0$. Let $F \subset E$ be finite, $\varepsilon > 0$, and let $V : E \rightarrow \mathbb{R}$ be bounded below with $\mathcal{L}V(s) \leq -\varepsilon$ for all $s \notin F$. Then the hitting time τ_F of F satisfies $\mathbb{E}_s \tau_F \leq (V(s) - \inf V)/\varepsilon$ for $s \notin F$, and every state is positive recurrent.*

Proof. Choose finite sets $E_1 \subset E_2 \subset \dots$ with union E , and let σ_n be the exit time from E_n . As the chain is non-explosive, it visits only finitely many states in a bounded time interval, so $\sigma_n \rightarrow \infty$

almost surely. Let $s \notin F$. The process stopped at $t \wedge \tau_F \wedge \sigma_n$ moves in the finite set formed by E_n and its neighbours, and before this time it is outside F . Dynkin's formula for this finite chain gives

$$\inf V \leq \mathbb{E}_s V(S(t \wedge \tau_F \wedge \sigma_n)) = V(s) + \mathbb{E}_s \int_0^{t \wedge \tau_F \wedge \sigma_n} \mathcal{L}V(S(u)) du \leq V(s) - \varepsilon \mathbb{E}_s[t \wedge \tau_F \wedge \sigma_n].$$

Letting $n \rightarrow \infty$ and then $t \rightarrow \infty$ (monotone convergence) gives the bound on $\mathbb{E}_s \tau_F$.

Now let $F' \supseteq F$ be finite; the first part applies to F' . For $s \in F'$ let $\tau_{F'}^+$ be the first time after the first jump at which the chain is in F' . By the Markov property at the first jump, $\mathbb{E}_s \tau_{F'}^+ = 1/q(s) + \sum_{s'} \frac{q(s, s')}{q(s)} \mathbb{E}_{s'} \tau_{F'}^+ < \infty$, a finite sum. Let $m = \max_{s \in F'} \mathbb{E}_s \tau_{F'}^+$ and fix $f \in F'$. Starting from f , let $T_0 = 0$, let T_{k+1} be the first time after the first jump following T_k at which the chain is in F' (these are the jump times into F'), and $Z_k = S(T_k)$. By the strong Markov property (Z_k) is a Markov chain on F' with $\mathbb{E}[T_{k+1} - T_k \mid \mathcal{F}_{T_k}] = \mathbb{E}_{Z_k} \tau_{F'}^+ \leq m$. It is irreducible, because a path of positive rates from f to f' in E induces a path of positive probability of (Z_k) . A finite irreducible chain is positive recurrent, so $K = \min\{k \geq 1 : Z_k = f\}$ has $\mathbb{E}_f K < \infty$. The return time of S to f is T_K , and since $\{K > k\} \in \mathcal{F}_{T_k}$,

$$\mathbb{E}_f T_K = \sum_{k \geq 0} \mathbb{E}_f [\mathbf{1}_{\{K > k\}} \mathbb{E}[T_{k+1} - T_k \mid \mathcal{F}_{T_k}]] \leq m \mathbb{E}_f K < \infty.$$

So every state of F' is positive recurrent; taking $F' = F \cup \{s\}$ for an arbitrary state s proves the last claim. $\square$

4 The Lyapunov function

4.1 Parameters and definition

Let $\Lambda = \max(\lambda, 1)$ and fix

$$\begin{aligned} \delta = \eta = \frac{1}{8\Lambda}, \quad M = 32\Lambda, \quad \kappa = 16, \quad L' = \lceil 32\Lambda \rceil, \\ L = \lceil 32768\Lambda^2 \rceil, \quad M_0 = \lceil 65536\Lambda^2 \rceil, \quad X_1 = \lceil 2^{50}\Lambda^4 \rceil. \end{aligned} \tag{3}$$

Since $\lambda \leq \Lambda$, $L' \geq 32\Lambda = 4/\delta$, $L \geq 32768\Lambda^2 = 8\kappa M/\delta$ and $M_0 \geq 65536\Lambda^2 = 4\kappa M^2$, these satisfy

$$\begin{aligned} (\delta + \eta)\lambda \leq \frac{1}{4}, \quad \frac{1}{L'} \leq \frac{\delta}{4}, \quad \frac{\lambda}{2L'} \leq \frac{1}{64}, \quad \frac{2\kappa M}{L} \leq \frac{\delta}{4}, \quad \frac{\kappa M(\lambda+1)}{L} \leq \frac{1}{32}, \\ \delta M = 4, \quad M \geq 32, \quad \frac{2\kappa M^2}{M_0} \leq \frac{1}{2}, \quad L' \leq 33\Lambda, \quad L \leq 32769\Lambda^2, \quad M_0 \leq 65537\Lambda^2. \end{aligned} \tag{4}$$

(For instance $\kappa M(\lambda+1)/L \leq \frac{\delta}{8} \cdot 2\Lambda = \frac{1}{32}$.) Let h be the Huber function $h(Q) = \frac{Q^2}{2L'} + \frac{L'}{2}$ for $|Q| \leq L'$ and $h(Q) = |Q|$ otherwise; let $g(z) = \frac{Mz}{M+z}$ for $z \geq 0$; let $w(Q) = \min(1, \max(0, (Q + L' + L)/L))$ and $\bar{w}(Q) = w(-Q)$. Define

$$V(A, B, X, Y) = h(Q) + \delta N + \eta(A + B) - \kappa[g(Y)w(Q) + g(X)\bar{w}(Q)]. \tag{5}$$

The function h is even, convex and C^1 , with $|h'| \leq 1$ and h' $(1/L')$ -Lipschitz. Hence the increments $\varphi_{\pm}(Q) = h(Q \pm 1) - h(Q)$ satisfy $|\varphi_{\pm}| \leq 1$ and $0 \leq \varphi_+ + \varphi_- = \int_0^1 (h'(Q+u) - h'(Q-1+u)) du \leq 1/L'$; for $Q \leq -L' - 1$ one has $\varphi_+(Q) = -1$ and $\varphi_-(Q) = 1$. The function g is increasing with $0 \leq g < M$, and

$$\Delta^+ g(z) := g(z+1) - g(z) = \frac{M^2}{(M+z)(M+z+1)} \in (0, 1), \quad \Delta^- g(z) := \Delta^+ g(z-1) \quad (z \geq 1),$$

so $\Delta^-g(z) = g(z) - g(z-1) = M^2/((M+z-1)(M+z))$ for $z \geq 1$; we put $\Delta^-g(0) := 0$. The function w is nondecreasing, its increments are at most $1/L$, $w(Q) = 1$ if and only if $Q \geq -L'$, and $w(Q) = 0$ if and only if $Q \leq -L' - L$. We put

$$G_Y = r_4 \Delta^+g(Y) - r_6 \Delta^-g(Y), \quad G_X = r_3 \Delta^+g(X) - r_5 \Delta^-g(X), \quad \gamma(Y) = (Y+1) \Delta^+g(Y) - Y \Delta^-g(Y).$$

Lemma 4.1. (a) $V \circ \sigma = V$, hence $(\mathcal{L}V) \circ \sigma = \mathcal{L}V$.

(b) $V \geq \delta N - 2\kappa M$; in particular V is bounded below.

(c) $|V(s') - V(s)| \leq J := 1 + \delta + \eta + \kappa(1 + 2M/L)$ for every transition $s \rightarrow s'$.

(d) $\gamma(Y) = M^2(M-1-Y)/((M+Y-1)(M+Y)(M+Y+1))$.

Proof. (a) h is even, N and $A+B$ are invariant under σ , and σ exchanges $g(Y)w(Q)$ and $g(X)\bar{w}(Q)$. (b) $h \geq 0$ and the bracket in (5) is less than $2M$. (c) In one transition Q , N and $A+B$ change by at most one, and at most one of X, Y changes, by one. Writing $g(Y')w(Q') - g(Y)w(Q) = (g(Y') - g(Y))w(Q') + g(Y)(w(Q') - w(Q))$ and likewise for X , the bracket changes by less than $1 + 2M/L$. (d) $(Y+1)(M+Y-1) - Y(M+Y+1) = M-1-Y$. $\square$

4.2 The master inequality

Lemma 4.2. At every state,

$$\begin{aligned} \mathcal{L}V \leq (r_5 - r_6)\varphi_+(Q) + (r_6 + \frac{\lambda}{2})(\varphi_+(Q) + \varphi_-(Q)) + \delta(\lambda - T) + \eta(\lambda - r_3 - r_4) \\ - \kappa w(Q)G_Y - \kappa \bar{w}(Q)G_X + \frac{\kappa M(\lambda + T)}{L}. \end{aligned} \quad (6)$$

Proof. We compute the generator of each term of (5). The value of Q moves by -1 under r_1 and r_6 , by $+1$ under r_2 and r_5 , and not under r_3, r_4 . So $\mathcal{L}h(Q) = \frac{\lambda}{2}(\varphi_+ + \varphi_-) + r_5\varphi_+ + r_6\varphi_-$, which equals the first two terms of (6). Next $\mathcal{L}(\delta N) = \delta(\lambda - T)$ by Lemma 2.1, and $\mathcal{L}(\eta(A+B)) = \eta(\lambda - r_3 - r_4)$.

Let $U = g(Y)w(Q)$. Under r_1 , U changes by $g(Y)(w(Q-1) - w(Q)) \geq -M/L$; under r_2 and r_5 by $g(Y)(w(Q+1) - w(Q)) \geq 0$; under r_3 not at all; under r_4 by $\Delta^+g(Y)w(Q)$; under r_6 (where Y and Q both decrease) by $-\Delta^-g(Y)w(Q-1) + g(Y)(w(Q-1) - w(Q))$. Since $w(Q-1) \leq w(Q)$,

$$\mathcal{L}U \geq w(Q)(r_4 \Delta^+g(Y) - r_6 \Delta^-g(Y)) - \frac{M}{L} \left(\frac{\lambda}{2} + r_6 \right) = w(Q)G_Y - \frac{M}{L} \left(\frac{\lambda}{2} + r_6 \right).$$

By the symmetry σ the same computation gives $\mathcal{L}(g(X)\bar{w}(Q)) \geq \bar{w}(Q)G_X - \frac{M}{L}(\frac{\lambda}{2} + r_5)$. Adding up, with $r_5 + r_6 = T$, gives (6). $\square$

Lemma 4.3. For every real $M \geq 32$:

(g1) $\gamma(Y) \geq 0$ for $0 \leq Y \leq M-1$;

(g2) $\gamma(Y) \geq 0.13$ for $0 \leq Y \leq M/2$;

(g3) $-\gamma(Y) \leq M^2/(27(M-1)^2) < 0.0395$ for all $Y \geq 0$; hence $\kappa\gamma_-(Y) \leq 0.64$, where $\gamma_- = \max(-\gamma, 0)$ and $\gamma_+ = \max(\gamma, 0)$.

Proof. (g1) is the sign of $M-1-Y$ in Lemma 4.1(d). (g2) On $[0, M-1]$ the numerator of γ decreases and the denominator increases, so $\gamma(Y) \geq \gamma(M/2) = \frac{4M^2-8M}{27M^2-12}$. The derivative of this function of M has the sign of $216M^2 - 96M + 96 > 0$, so it is at least its value $3840/27636 > 0.1389$ at $M = 32$. (g3) Only $Y > M-1$ matters. With $s = Y+1-M > 0$ the three factors of the denominator are $2M-2+s$, $2M-1+s$ and $2M+s$, so $-\gamma(Y) \leq M^2s/(2M-2+s)^3$. The maximum of $s/(c+s)^3$ over $s > 0$ is $4/(27c^2)$, attained at $s = c/2$; with $c = 2M-2$ this gives $M^2/(27(M-1)^2)$, which decreases in M and equals $1024/25947 < 0.0395$ at $M = 32$. $\square$

5 Drift estimates and the proof of the theorem

By Lemma 4.1(a) it suffices to treat states with $Y \leq X$ or, in the corner, with $Y < M_0$; the mirror images follow by the symmetry σ .

Proposition 5.1 (interior). *If $\min(X, Y) \geq M_0$, then $\mathcal{L}V \leq 1.8 - \frac{\delta}{2} \min(X, Y) \leq -4094$.*

Proof. Let $X \geq Y \geq M_0$. In (6): $|r_5 - r_6| = (X - Y)/D < 1$ and $r_6 = (X + 1)Y/D \leq Y$, so the h -terms are at most $1 + (Y + \lambda/2)/L'$; by Lemma 3.1(c), $Y \leq T \leq 2Y + 1$; and, as $r_6 \leq Y$ and $r_5 \leq X$,

$$-G_Y \leq r_6 \Delta^- g(Y) \leq \frac{YM^2}{(M + Y - 1)(M + Y)} \leq \frac{M^2}{M_0}, \quad -G_X \leq r_5 \Delta^- g(X) \leq \frac{M^2}{M_0},$$

which we use with $w, \bar{w} \leq 1$. Hence $\mathcal{L}V \leq 1 + \frac{\lambda}{2L'} + \frac{Y}{L'} + (\delta + \eta)\lambda - \delta Y + \frac{2\kappa M^2}{M_0} + \frac{\kappa M(\lambda + 1)}{L} + \frac{2\kappa MY}{L}$, and (4) gives $\mathcal{L}V \leq 1 + \frac{1}{64} + \frac{1}{4} + \frac{1}{2} + \frac{1}{32} - \frac{\delta}{2}Y \leq 1.8 - \frac{\delta}{2}Y$. Finally $\frac{\delta}{2}M_0 \geq \frac{65536\Lambda^2}{16\Lambda} = 4096\Lambda$. $\square$

Proposition 5.2 (face). *If $X \geq X_1$ and $Y \leq M_0 - 1$, or $Y \geq X_1$ and $X \leq M_0 - 1$, then $\mathcal{L}V \leq -0.7$, whatever A and B are.*

Proof. Let $X \geq X_1$ and $Y \leq M_0 - 1$. Then $X > Y$, $D \geq X_1$ and $Y + 1 \leq M_0$. Put $e_1 = (2M_0^2 + 2M_0 + 1)/X_1$, $e_2 = M_0 M^2/X_1^2$, $e_3 = (M_0 + L')M_0/X_1$ and $e_4 = (M_0 + L' + L)M_0/X_1$. From (4) and $X_1 \geq 2^{50}\Lambda^4$, with $\Lambda \geq 1$, one gets $e_1 \leq 7.7 \cdot 10^{-6}$, $e_2 \leq 10^{-20}$, $e_3 \leq 3.9 \cdot 10^{-6}$, $e_4 \leq 5.8 \cdot 10^{-6}$, $(2M_0 + 1)/X_1 \leq 1.2 \cdot 10^{-10}$ and $M_0/X_1 \leq 6 \cdot 10^{-11}$, so that

$$\delta e_1 + \kappa(e_2 + e_3 + e_4) + (2M_0 + 1)/X_1 \leq 0.001. \quad (7)$$

We use four estimates. (i) By Lemma 3.1(c), $2Y + 1 - e_1 \leq T \leq 2Y + 1$. (ii) $-\kappa \bar{w}(Q)G_X \leq \kappa r_5 \Delta^- g(X) \leq \kappa(Y + 1)M^2/(M + X - 1)^2 \leq \kappa e_2$, since $X/D \leq 1$. (iii) By (4), $(Y + \lambda/2)/L' \leq \frac{\delta}{4}Y + \frac{1}{64}$ and $\kappa M(\lambda + T)/L \leq \frac{\kappa M(\lambda + 1)}{L} + \frac{2\kappa MY}{L} \leq \frac{1}{32} + \frac{\delta}{4}Y$. (iv) Let $c \in \{L', L' + L\}$ and suppose $B \geq X - Y - c$ (note $X - Y - c > 0$). Then $r_4 \geq (X - Y - c)(Y + 1)/D$, and writing $(X - Y - c)(Y + 1) = (X + 1)(Y + 1) - (Y + 1 + c)(Y + 1)$ and using the definition of γ ,

$$G_Y \geq \frac{(X + 1)\gamma(Y) - (Y + 1 + c)(Y + 1)\Delta^+ g(Y)}{D} \geq \frac{X + 1}{D} \gamma(Y) - \frac{(M_0 + c)M_0}{X_1},$$

where $\frac{X+1}{D} = 1 - \frac{Y}{D} \in [1 - 6 \cdot 10^{-11}, 1]$. The last term is e_3 for $c = L'$ and e_4 for $c = L' + L$.

Case F1: $Q \geq -L'$. Then $w(Q) = 1$, and $B = Q + A + X - Y \geq X - Y - L'$ because $A \geq 0$; so (iv) applies with $c = L'$ and gives $-\kappa G_Y \leq -\kappa \gamma_+(Y)(1 - 6 \cdot 10^{-11}) + \kappa \gamma_-(Y) + \kappa e_3$. The h -terms of (6) are at most $1 + (Y + \lambda/2)/L'$, as in the interior. With (i)–(iii), (4) and (7),

$$\mathcal{L}V \leq 1 + \frac{1}{64} + \frac{1}{4} + \frac{1}{32} + 0.001 - \delta(2Y + 1) + \frac{\delta}{2}Y - \kappa \gamma_+(Y)(1 - 6 \cdot 10^{-11}) + \kappa \gamma_-(Y),$$

that is, $\mathcal{L}V \leq 1.298 - \delta(1.5Y + 1) - \kappa \gamma_+(Y)(1 - 6 \cdot 10^{-11}) + \kappa \gamma_-(Y)$. (a) If $Y \leq M/2$, then $\gamma(Y) \geq 0.13$ by (g2) and $\mathcal{L}V \leq 1.298 - 16 \cdot 0.13(1 - 6 \cdot 10^{-11}) < -0.78$. (b) If $Y > M/2$ and $\gamma(Y) \geq 0$, then $\mathcal{L}V \leq 1.298 - \frac{3}{4}\delta M = 1.298 - 3 < -1.7$. (c) If $\gamma(Y) < 0$, then $Y > M - 1$ by (g1) and $\kappa \gamma_-(Y) \leq 0.64$ by (g3), so $\mathcal{L}V \leq 1.298 + 0.64 - \delta(1.5(M - 1) + 1) \leq 1.938 - (6 - \frac{1}{16}) < -3.9$.

Case F2: $Q \leq -L' - 1$. Then $\varphi_+(Q) = -1$ and $\varphi_+(Q) + \varphi_-(Q) = 0$, so the h -terms equal $-(X - Y)/D = -1 + (2Y + 1)/D \leq -1 + (2M_0 + 1)/X_1$. If $w(Q) > 0$, then $Q > -L' - L$, so $B > X - Y - L' - L$, and (iv) with $c = L' + L$ gives $-\kappa w(Q)G_Y \leq \kappa \gamma_-(Y) + \kappa e_4$, because $w(Q) \leq 1$ and $-\frac{X+1}{D}\gamma(Y) \leq \gamma_-(Y)$; this bound also holds when $w(Q) = 0$. With (i)–(iii) and (7),

$$\mathcal{L}V \leq -1 + \frac{1}{4} + \frac{1}{32} + 0.001 - \delta(2Y + 1) + \frac{\delta}{4}Y + \kappa \gamma_-(Y) \leq -0.7177 - \delta(1.75Y + 1) + \kappa \gamma_-(Y).$$

If $\gamma(Y) \geq 0$, this is at most -0.7177 . Otherwise $Y > M - 1$, so $1.75\delta Y \geq 1.75(4 - \delta) \geq 6.78$ and $\mathcal{LV} \leq -0.7177 + 0.64 - 6.78 < -6$.

Since L' is an integer, the two cases cover all integers Q . Only $A \geq 0$ and $B \geq 0$ were used, so the bound holds for all A, B . The mirror face follows by symmetry. $\square$

Proposition 5.3 (corner). *If $X < X_1$ and $Y < M_0$, or $Y < X_1$ and $X < M_0$, then*

$$\mathcal{LV} \leq (\lambda + 2M_0)J - \frac{\eta(A + B)}{X_1 + M_0}.$$

In particular $\mathcal{LV} \leq -\frac{1}{2}$ if $A + B \geq K := (X_1 + M_0)((\lambda + 2M_0)J + \frac{1}{2})/\eta$.

Proof. Let $X < X_1$ and $Y < M_0$. By Lemma 3.1(c), $T < 2\min(X, Y) + 1 \leq 2M_0$. Under r_3 only A and X change, and V changes by $-\eta - \kappa\Delta^+g(X)\bar{w}(Q) \leq -\eta$; likewise under r_4 by $-\eta - \kappa\Delta^+g(Y)w(Q) \leq -\eta$. The other four transitions have total rate $\lambda + T \leq \lambda + 2M_0$, and each changes V by at most J (Lemma 4.1(c)). Finally $r_3 + r_4 = (A(X + 1) + B(Y + 1))/D \geq (A + B)/D$ and $D < X_1 + M_0$. $\square$

Proof of Theorem 1.1. Irreducibility and non-explosion are Lemma 3.1(a), (b). Let $s \in \mathbb{N}^4$. If $\min(X, Y) \geq M_0$, then $\mathcal{LV}(s) \leq -4094$ (Proposition 5.1). Otherwise, say $Y < M_0$ (the case $X < M_0$ is the mirror image): if $X \geq X_1$, then $\mathcal{LV}(s) \leq -0.7$ (Proposition 5.2); if $X < X_1$, then $\mathcal{LV}(s) \leq -\frac{1}{2}$ unless $A + B < K$ (Proposition 5.3). Hence $\mathcal{LV} \leq -\frac{1}{2}$ outside the finite set

$$F = \{X < X_1, Y < M_0, A + B < K\} \cup \{Y < X_1, X < M_0, A + B < K\}.$$

Every state has at most six possible jumps and total rate at least $\lambda > 0$, and V is bounded below (Lemma 4.1(b)). By Lemma 3.2 with $\varepsilon = \frac{1}{2}$, $\mathbb{E}_s\tau_F \leq 2(V(s) + 2\kappa M)$ for $s \notin F$, and every state is positive recurrent. $\square$

6 Remarks and open questions

The constants and the simulations. The constants in (3) are crude; they were chosen so that every estimate closes with room to spare. The threshold X_1 and the bound K are astronomically large, and the finite set F is not meant to be sharp. Gillespie simulations, which only illustrate the theorem, give time averages of N over $[0, 2 \cdot 10^5]$ of about 4.5, 34, 156, $1.2 \cdot 10^3$ and $1.2 \cdot 10^4$ for $\lambda = 1, 4, 10, 30, 100$ (seeds 1, 2 and 7). A run with $\lambda = 10$ from $(5, 20000, 40000, 0)$ shows the mechanism; in the scaling of (2) this point ($y = 0, a = \lambda/2$) lies on the boundary of the open set of escaping trajectories in the proof of [NRE09, Prop. 3.3]. Q rises from -20005 at rate about 0.99, N first grows to about 10^5 , and the system empties once Q is near 0, between the times $2 \cdot 10^4$ and $3 \cdot 10^4$ (seeds 3 and 7).

Related models. We cite these only to place the result. In the model of Hajek and Zhu [HZ11], as summarised in [Haj10, pp. 2762–2763], the seed uploads at a fixed rate U_s , and with immediate departures and empty arrivals the system is positive recurrent if $\lambda < U_s$ and transient if $\lambda > U_s$ (the missing piece syndrome). In the DFC system the seed is pulled at the total rate $(A + B + X + Y)/(X + Y + 1)$, which is not fixed, and half of the arrivals are committed to each chunk, so that theorem does not apply. The two-chunk stability proof of Oguz, Anantharam and Norros [OAN15] concerns, by the passages of their preprints that we read, their own rare-chunk rule, not DFC. Of the paper of Fogelklou, Konstantopoulos and Tucker [FKT12] on the global stability of a peer-to-peer network model only the title was read (no abstract was available to us), so we cannot say whether it bears on the DFC chain.

The variant with contact population $A+B+X+Y+1$. If empty peers are also contacted, which is a possible reading of the wording of [Nor10], the denominators D in Table 1 become $A+B+X+Y+1$. The large system limit of that chain is also not stable in the sense of [NRE09]: it has the same first integral $(b+y) - (a+x)$, and its equilibria form the unbounded curve $a = y$, $b = x$, $xy = \lambda(x+y)$, $y > \lambda$. So the restriction to the chain of [NRE09, Fig. 2] matters. Our proof does not cover the variant: there $\mathcal{LQ} = (X-Y)/(A+B+X+Y+1)$, which is no longer of order one when $A+B$ is large. Whether the variant is positive recurrent is open.

7 Computations, verification and scope

No step of the proof depends on a computation. The reproducibility package contains *the exact program of this note*, `check_note.py` (standard library only, exact rational arithmetic, about 20 seconds), which checks the identities of Lemmas 2.1, 2.2, 3.1 and 4.1 and Remark 2.3 (the polynomial ones on grids large enough to prove them), all constants of Sections 4 and 5 and, as a guard, $\mathcal{LV}$ exactly on 10,000 random states and a face sweep for each of $\lambda = \frac{1}{10}, 1, \frac{37}{10}, 10, 100$, with no failure (worst face values $-1.1, -0.875, -0.784, -0.763, -0.751$); the programs of the first stage (`original/`); the simulations with recorded seeds (`run_sims.sh`); and the programs and outputs of the three verification runs described next (`verification_run_A/`, `verification_run_B/`, `independent_run_2/`).

Verification. The result was first obtained with a written proof and the programs in `original/`. Two independent verification runs, both AI-assisted, examined that first written version, not the text of this note; each re-derived the whole proof line by line, compared the statement with [Nor10, NRE09] and wrote its own exact programs, and neither read the other’s report. Neither found a mathematical error or a gap. Their programs tested (6), the jump bound and the regional bounds for eight or nine values of λ in $[0.01, 1000]$ (up to 3.5 million face states for one λ), searched adversarially outside F (no state with $\mathcal{LV} > -\frac{1}{2}$), and detected eight of nine deliberately wrong inputs (such as $\kappa = 1$). The runs required corrections, none mathematical, and all were made: a page range, the attribution of the refuted conjecture to arXiv:0910.5577v1, the rate bound in Lemma 3.1(b), wording in Lemma 3.1(c), the remarks on the variant and on the constants, and recorded seeds for the simulations (those of the first stage were not recorded; seeds 1, 2, 3 reproduce them line by line). When this note was written, one constant was corrected (case F2: -0.71775 , not -0.718 ; the bound $\mathcal{LV} \leq -0.7$ is unchanged).

A further independent verification run, also AI-assisted, then examined the text of this note. It read [Nor10, pp. 2782–2783] and Sections 2 and 3.1–3.3 of [NRE09] again, re-derived every lemma and proposition, and wrote its own exact programs (`independent_run_2/`): the generator identities; (6) transition by transition against the exact generator; Lemma 4.3 symbolically; (4) and (7) uniformly in $\Lambda \geq 1$; and $\mathcal{LV}$ on interior, face and corner states for $\lambda \in \{\frac{1}{100}, \frac{1}{10}, 1, \frac{37}{10}, 10, 100, 1000\}$ (up to 4.7 million face states for one λ), with the face sweep at $\min(A, B) = 0$ justified by the exact identity $\mathcal{LV}(A+t, B+t, X, Y) - \mathcal{LV}(A, B, X, Y) = t \left[\frac{X+1}{D} (-\eta - \kappa \Delta^+ g(X) \bar{w}(Q)) + \frac{Y+1}{D} (-\eta - \kappa \Delta^+ g(Y) w(Q)) \right] < 0$. It found no failure and the same worst face values; its mutation tests ($\kappa = 1$, $\kappa = 0$, no seed in the departure rates) produced states outside F with $\mathcal{LV} > -\frac{1}{2}$, so the tests are not vacuous. It re-ran the package from the source archive with identical outputs, found no mathematical error, and required corrections of presentation only: Hajek’s page range, the description of [FKT12], two sentences on the large system limit in Section 1, and the description of the starting point of the simulation in Section 6.

Scope and priority. The note answers the question of the corpus record for the chain of [NRE09, Fig. 2], for every $\lambda > 0$. Not covered are the variant with contact population $A+B+X+Y+1$, the other systems of [NRE09], more chunks, the example in the opposite direction asked for in [Nor10,

p. 2783], and the stationary law. The question and the stability conjecture are due to Norros and Reittu [Nor10, pp. 2782–2783]. The DFC model, the limit (2) and its instability (Proposition 3.3), the definition of stability and the finite-set criterion (Section 2), and the remark that λ scales the limit but not the chain are due to Norros, Reittu and Eirola [NRE09]; Theorem 1.1 refutes their Conjecture 3.4 of arXiv:0910.5577v1. The Foster–Lyapunov method is standard [MT93].

What was read: [NRE09] in full; the abstracts of Norros and of Hajek in the open file of the report [Nor10, Haj10]; of [NRE11] only the abstract, on the publisher’s page as served to one anonymous request on 19 October 2026, 22:26 UTC (the other anonymous requests, on 19 and 20 October, returned a short automated-access page), and the zbMATH review [Zak11], which calls DFC “very unsatisfactory” where arXiv v1 says “found unstable”. The full text of [NRE11] was not read, so whether it keeps Conjecture 3.4 is unknown; a copy listed in HAL (hal-00781341) could not be retrieved, as the server refused automated access. Of [MT93] only the abstract was read (Lemma 3.2 is proved here); [HZ11] was not read (we rely on [Haj10]); of [OAN15] and its preprints we read the title and the passages citing [NRE11] shown by Semantic Scholar, of [FKT12] the title (Crossref, OpenAlex, Semantic Scholar and zbMATH show no abstract); the theses of Reittu, Oguz (2012) and Fogelklou (2012) were not read. The searches (19–20 October 2026) used the arXiv API (keyword queries on first-chunk and two-chunk systems, positive recurrence, missing pieces, random peer sampling and large system limits, and the later papers of Norros and of Reittu), zbMATH, Crossref, OpenAlex, Semantic Scholar and five web searches. OpenAlex lists 31 works citing [NRE11], whose titles were read (on 19 and again on 20 October; other repetitions of this request were refused with HTTP status 429); Semantic Scholar lists 29 of them, with citation contexts for 13, and none of these mentions a proof or a disproof for the stochastic DFC chain. No printed answer to the question was found in the sources searched (those named above); a search that finds nothing is not a proof of novelty, and no priority is claimed beyond the statement that no earlier proof was found there. The likeliest places for an overlooked proof are the full text of [NRE11] and the three theses.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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