

The Square Root of Two Is an ℓ^2 -Betti Number Arising from the Lamplighter Group

Alper Ferudun*

October 14, 2026

Abstract

Grabowski asked whether $\sqrt{2}$ is an ℓ^2 -Betti number arising from the lamplighter group $\mathbb{Z}/2 \wr \mathbb{Z}$, that is, the von Neumann dimension of the kernel of a matrix over its integral group ring (Groups Geom. Dyn. 2016, Problem 4). We show that it is. For an explicit matrix $T' \in M_2(\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}])$ with $8T'$ integral we prove $\dim_{\mathbb{N}} \ker T' = \frac{1}{4} \sum_{N \geq 0} \binom{2N}{N} 8^{-N} = \frac{\sqrt{2}}{4}$, by Pontryagin duality and again by Lück's approximation theorem with an exact formula on the finite quotients $\mathbb{Z}/2 \wr \mathbb{Z}/3P$; four copies of $8T'$ form an integral 8×8 matrix whose kernel has dimension $\sqrt{2}$. Also $k\sqrt{2}/4$ ($k \geq 1$) arises from $\mathbb{Z}/2 \wr \mathbb{Z}$, and $\sqrt{2}$ arises from $\mathbb{Z}/p \wr \mathbb{Z}$ for every even p . The question and the framework are Grabowski's. The mechanism, balanced words counted by central binomial coefficients, is the one by which Ara and Claramunt obtained $\frac{1}{4}\sqrt{2/7}$ and $\frac{1}{8}\sqrt{2^t/(2^t-1)}$, $t \geq 3$, as *virtual* ℓ^2 -Betti numbers of $\mathbb{Z}/2 \wr \mathbb{Z}$, that is, as differences of two ℓ^2 -Betti numbers (J. Algebraic Combin. 2021, Theorem 4.13); in their Remark 4.14 they ask for a genuine witness, a concrete matrix whose kernel has dimension $\frac{1}{4}\sqrt{2/7}$, which we do not provide. It is also the mechanism of our companion note, which shows that $\sqrt{5}/20$ arises from $\mathbb{Z}/2 \wr \mathbb{Z}$ and whose lemmas we use. Lück's theorem and Pontryagin duality are standard. Apart from that companion note, the sources we searched contain no earlier non-virtual irrational algebraic ℓ^2 -Betti number of a group $\mathbb{Z}/p \wr \mathbb{Z}$; a search that finds nothing is not a proof of novelty. A description of the set of ℓ^2 -Betti numbers arising from $\mathbb{Z}/2 \wr \mathbb{Z}$ remains open. This is an unrefereed note.

1 Introduction

1.1 The question

Let G be a countable discrete group. Following Grabowski [Gra10, p. 2366], a non-negative real number r is an ℓ^2 -Betti number arising from G if there is a square matrix $\theta \in M_m(\mathbb{Q}G)$ such that the von Neumann dimension of the kernel of θ , acting on $\ell^2(G)^m$, equals r ; the set of these numbers is denoted by $\mathcal{C}(G)$.

The dimension is not normalized: for $\theta \in M_m(\mathbb{Q}G)$ it is the sum $\sum_i \tau(P_{ii})$ of the traces of the diagonal entries of the kernel projection P , a number in $[0, m]$ (Section 2.1). Both [Gra10, p. 2366] and [Gra16, p. 795] state the Atiyah conjecture for torsion-free groups G as $\mathcal{C}(G) = \mathbb{N}$, which is possible only with this normalization, and [Gra14, §1] uses the trace $\text{tr} \otimes \tau_G$ on $M_k(\mathbb{Q}G)$. With the normalized trace every element of $\mathcal{C}(G)$ would lie in $[0, 1]$, and the question below would have a trivial negative answer. In [Gra16, p. 795] the definition is stated with “a matrix” over the integral group ring, whose shape is not fixed; a square matrix over $\mathbb{Q}G$ has the same kernel as the integral matrix obtained

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

by clearing denominators, so the two definitions give the same answers here, where all matrices are square and $8T'$ is integral.

Grabowski's lamplighter paper asks for a description of $\mathcal{C}(\mathbb{Z}_2 \wr \mathbb{Z})$ substantially different from the definition and, as a more concrete problem, “is $\sqrt{2} \in \mathcal{C}(\mathbb{Z}_2 \wr \mathbb{Z})$?” [Gra16, Problem 4]. Here $\mathbb{Z}_2 \wr \mathbb{Z} = \mathbb{Z}/2 \wr \mathbb{Z} = (\bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/2) \rtimes \mathbb{Z}$ is the lamplighter group. The problem has the same number and the same wording in the arXiv version v4 of [Gra16] (p. 3) and in the journal version (p. 796).

In the UnsolvedMath collection [ula26] (version 1.6.0) the record **OWR-4428-012** is Question 5 of Grabowski's Oberwolfach abstract, “What is $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$?” [Gra10, p. 2368], listed as partially solved; the same question is Question 3 in [Gra14, §1c] (in the numbering of the arXiv version v4, which we use; [GS21, AC22] cite it as Question 1.7). This note answers the concrete question of Problem 4 and gives new elements of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, but it gives no description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$: **the record remains partially solved.**

1.2 Results

Let $G = \mathbb{Z}/2 \wr \mathbb{Z}$, with the lamp generators e_j ($j \in \mathbb{Z}$, $e_j^2 = 1$) and the generator t of $\mathbb{Z}$, so that $te_jt^{-1} = e_{j+1}$. In $\mathbb{Q}G$ put $\pi_j = \frac{1}{2}(1 - e_j)$ and $\pi'_j = \frac{1}{2}(1 + e_j)$, and

$$\begin{aligned} \chi' &= \pi_0, & \alpha' &= 1 + \pi'_0\pi'_1\pi_2, & \beta' &= 1 + \pi'_0\pi_1\pi'_2, & S' &= t^3, \\ T' &= \begin{pmatrix} \chi' - (1 - \chi')\alpha' + (1 - \chi')S' & -\chi' \\ \chi'S' & (1 - \chi')S' - \chi'S' - (1 - \chi')\beta' \end{pmatrix} \in M_2(\mathbb{Q}G). \end{aligned} \quad (1)$$

Theorem A. *The matrix $8T'$ has entries in $\mathbb{Z}G$, and*

$$\dim_{\mathbb{N}} \ker T' = \frac{1}{4} \sum_{N \geq 0} \binom{2N}{N} 8^{-N} = \frac{\sqrt{2}}{4}.$$

Corollary B. *$\sqrt{2} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$: the kernel of the integral 8×8 matrix $\text{diag}(8T', 8T', 8T', 8T')$ has von Neumann dimension $\sqrt{2}$. More generally, $k\sqrt{2}/4 \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ for every integer $k \geq 1$, and $\sqrt{2} \in \mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ for every even $p \geq 2$.*

So the concrete question of Problem 4 of [Gra16] has the answer yes. The matrix T' has the form of the matrix T of the companion note [Fer26, (2)], with other lamp patterns and the shift t^3 instead of t^2 . In the Fourier picture every position carries a symbol of three lamp bits: a *marker* (probability $\frac{1}{2}$), a letter U or D ($\frac{1}{8}$ each), or the idle letter I ($\frac{1}{4}$). Between consecutive markers the fibre operator is a *counter matrix*, with a one-dimensional kernel if U and D occur equally often and a zero kernel otherwise; the balanced words of length N contribute $\frac{1}{4} \cdot 8^{-N} \binom{2N}{N}$ (Lemma 3.7), and $\sum_N \binom{2N}{N} y^N = (1 - 4y)^{-1/2}$ at $y = \frac{1}{8}$ gives $\sqrt{2}$. We prove Theorem A by Pontryagin duality (Section 3) and again by Lück's approximation theorem [Lüc94] with an exact formula on the finite quotients $\mathbb{Z}/2 \wr \mathbb{Z}/3P$ (Section 4).

1.3 What is new, what is not, and what stays open

The question, the definition of $\mathcal{C}(G)$ and the framework are Grabowski's [Gra10, Gra14, Gra16]. The mechanism is not new. Ara and Claramunt [AC21, Theorem 4.13] obtained $\frac{1}{4}\sqrt{2/7}$ from the words with equally many occurrences of two letters and the series $\sum_l \binom{2l}{l} x^l = (1 - 4x)^{-1/2}$, the same balanced-word and central-binomial mechanism, and from other pairs of letters the numbers $\frac{1}{8}\sqrt{2^t/(2^t - 1)}$, $t \geq 3$ (the paragraph after [AC21, Theorem 4.13]). These numbers lie in the subgroup of $\mathbb{R}$ generated by $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$: they are differences of two ℓ^2 -Betti numbers, *virtual* ℓ^2 -Betti numbers

in the words of [AC21, Remark 4.14], where the authors ask for a genuine witness, a concrete matrix over $\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}]$ whose kernel has dimension $\frac{1}{4}\sqrt{2/7}$; their companion paper relates these virtual values to Problem 4 of [Gra16] [AC22, Introduction]. Theorem A gives a genuine witness for the different number $\sqrt{2}/4$; whether $\frac{1}{4}\sqrt{2/7} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ is not decided here. The same mechanism, with symbols of two lamp bits, gives $\sqrt{5}/20 \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ in our companion note [Fer26, Theorem 2], which thereby answered, for $p = 2$, Grabowski's question [Gra14, §1c] whether $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers; with three lamp bits the weighted number of balanced words becomes a central binomial coefficient. We state every lemma we use and refer to [Fer26] for the proofs that are literally the same. Lück's approximation theorem [Lüc94] and Pontryagin duality are standard; the latter was used for such computations by Austin [Aus13] (see [Gra10]) and by Grabowski [Gra16]. For other groups irrational algebraic ℓ^2 -Betti numbers were known: every non-negative number with computable binary expansion, in particular $\sqrt{2}$, arises from some finitely presented group [Gra14, Theorem 1.4], and $\mathcal{C}(\mathbb{Z}/p \wr F_d)$ contains irrational algebraic numbers for free groups F_d with $d \geq 2$ and suitable p [LW13].

To our knowledge Theorem A and Corollary B are new; no priority is claimed, and the literature search is described in the paragraph "Scope and priority" in Section 6. By Lemma 2.2(a), the numbers $a\sqrt{2}/4 + b\sqrt{5}/20$ with integers $a, b \geq 0$ lie in $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ (Theorem A and [Fer26, Theorem 2]); for $a, b \geq 1$ they have degree 4 over $\mathbb{Q}$, with the conjugates $\pm a\sqrt{2}/4 \pm b\sqrt{5}/20$. Open remain: a description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, which is the first part of Problem 4 and the question of the record (we do not know, for instance, whether $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ contains an algebraic number of odd degree greater than 1); whether the virtual values of Ara and Claramunt lie in $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$; the question of Grigorchuk and Simanek for single self-adjoint elements [GS21, §7, Question 6(a)]; and, for odd p , whether $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers (Section 5).

2 Conventions and the operator

2.1 Conventions

We use the conventions of [Fer26, Section 5.1]. Let $A = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/2$, written additively as the group of finitely supported sequences $g = (g_j)_{j \in \mathbb{Z}}$ with $g_j \in \{0, 1\}$; e_j is the sequence with a single 1 at j . Let $G = \mathbb{Z}/2 \wr \mathbb{Z} = A \rtimes \mathbb{Z}$, where the generator t of $\mathbb{Z}$ acts by $te_jt^{-1} = e_{j+1}$. Every element of G is uniquely of the form gt^k with $g \in A$ and $k \in \mathbb{Z}$, and $(gt^k)(ht^l) = (g + \sigma^k h)t^{k+l}$, where $(\sigma h)_j = h_{j-1}$.

The group ring acts on $\ell^2(G)$ by left multiplication, $\lambda(g)\delta_h = \delta_{gh}$, and $M_m(\mathbb{Q}G)$ acts on column vectors in $\ell^2(G)^m$. Let $L(G)$ be the von Neumann algebra generated by $\lambda(G)$, with the trace $\tau(\theta) = \langle \theta\delta_e, \delta_e \rangle$, extended to $M_m(L(G))$ by $\tau_m(\theta) = \sum_i \tau(\theta_{ii})$. For $a \in M_m(\mathbb{Q}G)$ we put

$$\dim_{\text{vN}} \ker a = \tau_m(P_{\ker \lambda(a)}) \in [0, m],$$

where $P_{\ker \lambda(a)} \in M_m(L(G))$ is the orthogonal projection onto the kernel. In [Gra14, §1] the kernel dimension of a matrix that is not self-adjoint is defined through a^*a ; this gives the same number, since $\ker a^*a = \ker a$.

Lemma 2.1 (left and right; [Fer26, Lemma 5.1]). *For a square matrix $a \in M_m(\mathbb{Q}G)$ the kernel of right multiplication $\eta \mapsto \eta a$ on row vectors in $\ell^2(G)^m$ has the same von Neumann dimension as $\ker \lambda(a)$. The same holds for a finite group in place of G , with $\dim_{\mathbb{C}}$ of kernels.*

Proof. See [Fer26, Lemma 5.1]: the unitary $J\delta_g = \delta_{g^{-1}}$ fixes δ_e and conjugates right multiplication by a to $\lambda(a)^*$, and by the polar decomposition $\ker \lambda(a)$ and $\ker \lambda(a)^*$ have the same dimension; for a finite group the ranks of a matrix and of its conjugate transpose agree. $\square$

Lemma 2.2 (sums and subgroups). (a) For $a \in M_m(\mathbb{Q}G)$ and $b \in M_n(\mathbb{Q}G)$,

$$\dim_{\mathbb{N}} \ker \text{diag}(a, b) = \dim_{\mathbb{N}} \ker a + \dim_{\mathbb{N}} \ker b.$$

In particular $\mathcal{C}(G)$ is closed under addition.

(b) Let H be a subgroup of a countable group Γ and let $a \in M_m(\mathbb{Q}H)$. The von Neumann dimension of the kernel of a is the same over H and over Γ ; hence $\mathcal{C}(H) \subseteq \mathcal{C}(\Gamma)$ [Gra10, pp. 2366–2367].

Proof. (a) The operator $\lambda(\text{diag}(a, b))$ is $\lambda(a) \oplus \lambda(b)$ on $\ell^2(G)^m \oplus \ell^2(G)^n$. Its kernel projection is the block-diagonal matrix with the blocks $P_{\ker \lambda(a)}$ and $P_{\ker \lambda(b)}$, and τ_{m+n} adds the traces of the diagonal entries.

(b) As in [Fer26, Remark 7.1]: $\ell^2(\Gamma)^m$ is the orthogonal sum of the subspaces $\ell^2(Hg)^m$ over the right cosets Hg ; each of them is invariant under $\lambda(a)$, and right translation $\delta_y \mapsto \delta_{yg^{-1}}$ maps it unitarily onto $\ell^2(H)^m$ and intertwines the two actions of a . So the kernel projection of a over Γ is the direct sum of copies of the kernel projection over H , and δ_e lies in the summand $\ell^2(H)^m$; the traces agree. $\square$

2.2 The operator

The elements π_j and π'_j are complementary idempotents, $\pi_j \pi'_j = 0$, and all of them commute. Hence $1 - \chi' = \pi'_0$, $\chi' \alpha' = \chi' \beta' = \chi'$, $(1 - \chi') \alpha' = \pi'_0 + \pi'_0 \pi'_1 \pi'_2$, $(1 - \chi') \beta' = \pi'_0 + \pi'_0 \pi_1 \pi'_2$ and $\pi_0 - \pi'_0 = -e_0$, so that the entries of (1) are

$$T'_{11} = -e_0 - \pi'_0 \pi'_1 \pi'_2 + \pi'_0 t^3, \quad T'_{12} = -\pi_0, \quad T'_{21} = \pi_0 t^3, \quad T'_{22} = e_0 t^3 - \pi'_0 - \pi'_0 \pi_1 \pi'_2.$$

Expanding $\pi'_0 \pi'_1 \pi'_2 = \frac{1}{8}(1 + e_0)(1 + e_1)(1 - e_2)$ and $\pi'_0 \pi_1 \pi'_2 = \frac{1}{8}(1 + e_0)(1 - e_1)(1 + e_2)$ gives

$$\begin{aligned} T'_{11} &= -\frac{1}{8}(1 + 9e_0 + e_1 + e_0 e_1 - e_2 - e_0 e_2 - e_1 e_2 - e_0 e_1 e_2) + \frac{1}{2}(1 + e_0)t^3, \\ T'_{12} &= -\frac{1}{2}(1 - e_0), \quad T'_{21} = \frac{1}{2}(1 - e_0)t^3, \\ T'_{22} &= -\frac{1}{8}(5 + 5e_0 - e_1 - e_0 e_1 + e_2 + e_0 e_2 - e_1 e_2 - e_0 e_1 e_2) + e_0 t^3. \end{aligned} \tag{2}$$

In particular $8T' \in M_2(\mathbb{Z}G)$, which is the first statement of Theorem A, and $\ker(8T') = \ker T'$. Only the powers t^0 and t^3 of t occur in T' .

3 First proof of Theorem A: Pontryagin duality

3.1 Fibre operators

Let $X = \{0, 1\}^{\mathbb{Z}}$ with the product μ of the uniform measures on $\{0, 1\}$ (the Haar measure of the compact dual group of $\mathbb{A}$), and $(\tau x)_j = x_{j+1}$.

Lemma 3.1 (fibre operators; [Fer26, Lemma 5.2]). *There is a unitary $W: \ell^2(G) \rightarrow L^2(X \times \mathbb{Z}, \mu \otimes \text{counting})$ with $W\delta_e = 1_X \otimes \delta_0$ such that, for $g \in \mathbb{A}$ and $l \in \mathbb{Z}$,*

$$(W\lambda(gt^l)W^{-1}\Phi)(x, k) = (-1)^{\sum_j g_j x_{j-k}} \Phi(x, k - l). \tag{3}$$

Consequently, for $a \in M_m(\mathbb{Q}G)$ the operator $W\lambda(a)W^{-1}$ (applied componentwise) acts fibrewise: $(W\lambda(a)W^{-1}\Phi)(x, \cdot) = \theta_x(a)\Phi(x, \cdot)$, where $\theta_x(a)$ is the bounded operator on $\ell^2(\mathbb{Z})^m$ obtained by replacing every group element gt^l in the entries of a by the operator $f \mapsto (k \mapsto (-1)^{\sum_j g_j x_{j-k}} f(k - l))$. For every x the map $a \mapsto \theta_x(a)$ is multiplicative.

Proof. The first two statements are [Fer26, Lemma 5.2]; there $W = VW_1$, where $W_1\delta_{gt^k} = \chi_g \otimes \delta_k$ with the characters $\chi_g(x) = (-1)^{\sum_j g_j x_j}$, and $(V\Psi)(x, k) = \Psi(\tau^{-k}x, k)$. For the last statement: $\theta_x(e_j)$ is multiplication by $k \mapsto (-1)^{x_j - k}$, $\theta_x(t)$ is the shift $f \mapsto f(\cdot - 1)$, they satisfy the relations of G (for instance $(\theta_x(t)\theta_x(e_j)\theta_x(t)^{-1}f)(k) = (-1)^{x_{j+1-k}}f(k)$), and the operator that replaces gt^l is $\theta_x(\prod_{j:g_j=1} e_j)\theta_x(t)^l$. $\square$

Lemma 3.2 (trace formula; [Fer26, Lemma 5.3]). *Let $a \in M_m(\mathbb{Q}G)$. Suppose that for μ -almost every x an orthogonal projection P_x on $\ell^2(\mathbb{Z})^m$ with range $\ker \theta_x(a)$ is given, such that $x \mapsto \langle P_x \xi, \eta \rangle$ is measurable for all standard basis vectors ξ, η . Then*

$$\dim_{\text{vN}} \ker a = \sum_{i=1}^m \int_X \langle P_x(\delta_0 \otimes \varepsilon_i), \delta_0 \otimes \varepsilon_i \rangle d\mu(x), \quad (4)$$

where $\varepsilon_1, \dots, \varepsilon_m$ is the standard basis of $\mathbb{C}^m$.

Proof. See [Fer26, Lemma 5.3]: the operator $(P\Phi)(x) = P_x\Phi(x)$ is the orthogonal projection onto $\ker W\lambda(a)W^{-1}$, so $W^{-1}PW = P_{\ker \lambda(a)}$, and $W\delta_e = 1_X \otimes \delta_0$. $\square$

3.2 Classes, symbols and rows

Fix $x \in X$. Since only t^0 and t^3 occur in T' , the operator $\theta_x(T')$ maps the functions supported on a residue class of $\mathbb{Z}$ modulo 3 to functions supported on the same class. For $r \in \{0, 1, 2\}$ we call the index $k = -3m - r$ the *position* $m \in \mathbb{Z}$ of the class r , and, the class being fixed, we write $f_u(m) = f_u(-3m - r)$ and $f_v(m) = f_v(-3m - r)$ for $f = (f_u, f_v) \in \ell^2(\mathbb{Z})^2$. By Lemma 3.1, at the position m of the class r the lamps e_0, e_1, e_2 act by the signs $(-1)^{x_{3m+r}}, (-1)^{x_{3m+r+1}}, (-1)^{x_{3m+r+2}}$, and $(S'f)(-3m - r) = f(-3(m+1) - r)$ is the value at the position $m+1$. The *symbol* at the position m of the class r is

$$\sigma_m^r(x) = (x_{3m+r}, x_{3m+r+1}, x_{3m+r+2}) \in \{0, 1\}^3.$$

It is a *marker* if $x_{3m+r} = 1$ (four symbols); otherwise it is the *letter* U if $\sigma_m^r(x) = (0, 0, 1)$, the letter D if $\sigma_m^r(x) = (0, 1, 0)$, and the letter I if $\sigma_m^r(x) \in \{(0, 0, 0), (0, 1, 1)\}$. Since π_j and π'_j act at the index k by multiplication with $[x_{j-k} = 1]$ and $[x_{j-k} = 0]$, at the position m the element χ' acts by 1 at a marker and by 0 at a letter, α' by 2 at U and by 1 otherwise, and β' by 2 at D and by 1 otherwise.

Lemma 3.3 (rows). *In the class r , the two coordinates of $\theta_x(T')f$ at the position m are*

$$\begin{aligned} f_u(m) - f_v(m) \quad \text{and} \quad f_u(m+1) - f_v(m+1) & \quad \text{if } \sigma_m^r(x) \text{ is a marker,} \\ f_u(m+1) - a f_u(m) \quad \text{and} \quad f_v(m+1) - b f_v(m) & \quad \text{if } \sigma_m^r(x) \text{ is a letter,} \end{aligned}$$

with $(a, b) = (2, 1), (1, 2), (1, 1)$ for U, D, I .

Proof. As in [Fer26, Lemma 5.4]. By (1) and the multiplicativity of θ_x , the first coordinate is $(\chi' - (1 - \chi')\alpha')f_u + (1 - \chi')S'f_u - \chi'f_v$ and the second is $\chi'S'f_u + (1 - \chi')S'f_v - \chi'S'f_v - (1 - \chi')\beta'f_v$, where χ', α' and β' act by multiplication at the position m and S' brings the value from the position $m+1$. Insert $\chi' = 1$ at a marker, and $\chi' = 0, \alpha' = a, \beta' = b$ at a letter. $\square$

3.3 Counter matrices and segments

Definition 3.4 (counter matrices; [Fer26, Definition 3.2]). Let $N \geq 0$ and let $a_0, b_0, \dots, a_{N-1}, b_{N-1}$ be non-zero integers. The *counter matrix* $M((a_i, b_i)_{i < N})$ is the $(2N+2) \times (2N+2)$ matrix which, applied to the column vector $z = (u_0, v_0, u_1, v_1, \dots, u_N, v_N)^\top$, has the rows

$$u_0 - v_0, \quad u_{i+1} - a_i u_i, \quad v_{i+1} - b_i v_i \quad (0 \leq i \leq N-1), \quad u_N - v_N,$$

in this order. For a word $s = s_0 \cdots s_{N-1}$ over the alphabet $\{U, D, I\}$ we write M_s for the counter matrix with $(a_i, b_i) = (2, 1), (1, 2), (1, 1)$ for $s_i = U, D, I$; for example $M_\emptyset = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

Lemma 3.5 (kernel; [Fer26, Lemma 3.5]). *For $s \in \{U, D, I\}^N$ one has $\dim \ker M_s = 1$ if $\#U(s) = \#D(s)$, and $\dim \ker M_s = 0$ otherwise; here $\#U(s)$ and $\#D(s)$ are the numbers of letters U and D in s .*

Proof. See [Fer26, Lemma 3.5]: the first $2N + 1$ equations of $M_s z = 0$ hold if and only if $u_i = A_i u_0$ and $v_i = B_i u_0$ for all i , where $A_i = a_0 \cdots a_{i-1}$ and $B_i = b_0 \cdots b_{i-1}$; the last equation then reads $(A_N - B_N)u_0 = 0$, and $A_N = 2^{\#U(s)}$, $B_N = 2^{\#D(s)}$. $\square$

Lemma 3.6 (segments). *Let $r \in \{0, 1, 2\}$, and let $m < m'$ be positions of markers of the class r with only letters in between, forming the word $s \in \{U, D, I\}^N$, $N = m' - m - 1$. The rows “second coordinate at m ”, “both coordinates at $m + 1, \dots, m' - 1$ ” and “first coordinate at m' ” of the class r involve only the columns $(j, u), (j, v)$ with $m + 1 \leq j \leq m'$, and no other row involves these columns. In the order of the rows just listed and the columns $(m + 1, u), (m + 1, v), \dots, (m', u), (m', v)$ this block is the counter matrix M_s . If x has markers of the class r at infinitely many negative and infinitely many positive positions, the part of $\theta_x(T')$ on the class r is the direct sum of these finite blocks, its kernel is the ℓ^2 -direct sum of their kernels, and the orthogonal projection onto it is the direct sum of the projections $P_{\ker M_s}$ onto the kernels of the blocks.*

Proof. The proof of [Fer26, Lemma 5.5] applies, with Lemma 3.3 in place of [Fer26, Lemma 5.4]. By Lemma 3.3, the second coordinate at the marker m involves only the position $m + 1$, the coordinates at a letter j involve j and $j + 1$, and the first coordinate at the marker m' involves only m' . A position j with $m + 1 \leq j \leq m'$ is therefore involved only in rows at j and at $j - 1$, and those of them that involve j belong to the list. With $u_i = f_u(m + 1 + i)$ and $v_i = f_v(m + 1 + i)$ ($0 \leq i \leq N$) the listed rows are $u_0 - v_0$, then $u_{i+1} - a_i u_i$ and $v_{i+1} - b_i v_i$ for the letters $s_0, \dots, s_{N-1}$, then $u_N - v_N$: this is M_s . Under the hypothesis on x , every row of the class r belongs to exactly one segment (the first coordinate at a marker to the segment ending there, the second to the segment starting there, the coordinates at a letter to the segment containing it), and so does every position. So the part of $\theta_x(T')$ on the class r maps the ℓ^2 -direct sum of the column spaces of the segments into the direct sum of their row spaces, block by block, and a vector lies in its kernel if and only if each of its components lies in the kernel of its block. $\square$

3.4 The weighted count of balanced words

Lemma 3.7. *For every $N \geq 0$,*

$$B_N := \sum_{\substack{s \in \{U, D, I\}^N \\ \#U(s) = \#D(s)}} 2^{\#I(s)} = \sum_{k \geq 0} \binom{N}{2k} \binom{2k}{k} 2^{N-2k} = \binom{2N}{N}.$$

Proof. For the first equality, choose the $2k$ places of the letters U and D , then the k places of U among them; the remaining $N - 2k$ places carry the letter I , each with weight 2. For the second, replace every letter I by one of two new letters I_1, I_2 : then B_N is the number of words $w = w_0 \cdots w_{N-1}$ over $\{U, D, I_1, I_2\}$ with $\#U(w) = \#D(w)$. Put $\mathcal{A}(w) = \{i : w_i \in \{U, I_2\}\}$ and $\mathcal{B}(w) = \{i : w_i \in \{D, I_2\}\}$. The letter w_i is determined by the pair $([i \in \mathcal{A}(w)], [i \in \mathcal{B}(w)])$, namely $U = (1, 0)$, $D = (0, 1)$, $I_1 = (0, 0)$ and $I_2 = (1, 1)$; so $w \mapsto (\mathcal{A}(w), \mathcal{B}(w))$ is a bijection onto the set of pairs of subsets of $\{0, \dots, N - 1\}$, and $|\mathcal{A}(w)| - |\mathcal{B}(w)| = \#U(w) - \#D(w)$. Hence B_N is the number of pairs of subsets of equal size, which is $\sum_j \binom{N}{j}^2 = \binom{2N}{N}$ by Vandermonde’s identity. $\square$

Remark 3.8. $\binom{2N}{N}$ also counts the words of length $2N$ over $\{U, D\}$ with as many letters U as D . The matrices M_s , s a word over $\{U, D\}$, are exactly those recognized by the automaton A_2 of [Fer26, Theorem 1 and Lemma 3.4], so B_N is the coefficient of x^{4N+2} in its series $F_{A_2}(x) = x^2(1 - 4x^4)^{-1/2}$. We do not use this.

3.5 Proof of Theorem A

Under μ the symbols $\sigma_m^r(x)$, $m \in \mathbb{Z}$, of a fixed class r are independent and uniformly distributed on $\{0, 1\}^3$, since they involve the disjoint triples $\{3m + r, 3m + r + 1, 3m + r + 2\}$ of coordinates. So a marker has probability $\frac{1}{2}$, each of the letters U and D has probability $\frac{1}{8}$, and I has probability $\frac{1}{4}$. In particular the set X_0 of the points x that have, in each of the three classes, markers at infinitely many negative and at infinitely many positive positions has full measure.

For $x \in X_0$ let $P_x = P_x^0 \oplus P_x^1 \oplus P_x^2$, where P_x^r is the projection of Lemma 3.6 on the functions supported on the class r . Since $\theta_x(T')$ maps each class into itself and $\ell^2(\mathbb{Z})^2$ is the orthogonal sum of the three corresponding subspaces, P_x is the orthogonal projection onto $\ker \theta_x(T')$; it is a direct sum of finite-rank projections. For $x \notin X_0$ put $P_x = 0$. For $x \in X_0$ a matrix coefficient $\langle P_x(\delta_k \otimes \varepsilon_i), \delta_{k'} \otimes \varepsilon_{i'} \rangle$ vanishes unless k and k' lie in the same segment of the same class, and then it is an entry of $P_{\ker M_s}$ that depends only on the word s of this segment and on the places of k and k' in it. So it is constant on the sets of a countable partition of X_0 into intersections of X_0 with cylinder sets, hence measurable, and Lemma 3.2 applies to $a = T'$.

The index 0 is the position 0 of the class 0, so only the class 0 enters (4). For $N \geq 0$, $s \in \{U, D, I\}^N$ and $1 \leq j \leq N + 1$ let $E_{N,s,j}$ be the set of the $x \in X_0$ that have, in the class 0, a marker at the position $-j$, the letters $s_0, \dots, s_{N-1}$ at the positions $-j + 1, \dots, -j + N$, and a marker at the position $-j + N + 1$; that is, the position 0 is the j -th column position of a segment with word s . These sets are disjoint, since $-j$ is the last marker before the position 0 and $-j + N + 1$ the first marker at or after it, and their union is X_0 . Since $N + 2$ independent symbols are prescribed,

$$\mu(E_{N,s,j}) = \frac{1}{2} \cdot \prod_{i=0}^{N-1} p(s_i) \cdot \frac{1}{2} = \frac{1}{4} \cdot 8^{-N} \cdot 2^{\#I(s)}, \quad p(U) = p(D) = \frac{1}{8}, \quad p(I) = \frac{1}{4}.$$

On $E_{N,s,j}$ the position 0 carries the variables u_{j-1}, v_{j-1} of the block M_s , so the integrand of (4) equals $(P_{\ker M_s})_{2j-1, 2j-1} + (P_{\ker M_s})_{2j, 2j}$ (coordinates numbered $1, \dots, 2N + 2$), and the sum over j is $\text{Tr } P_{\ker M_s} = \dim \ker M_s$. Since the integrand is non-negative and constant on each set $E_{N,s,j}$, Lemmas 3.2, 3.5 and 3.7 give

$$\dim_{\text{vN}} \ker T' = \sum_{N \geq 0} \sum_{s \in \{U, D, I\}^N} \frac{1}{4} 8^{-N} 2^{\#I(s)} \dim \ker M_s = \frac{1}{4} \sum_{N \geq 0} 8^{-N} B_N = \frac{1}{4} \sum_{N \geq 0} \binom{2N}{N} 8^{-N}.$$

Since $\binom{2N}{N} = (-4)^N \binom{-1/2}{N}$, the binomial series gives $\sum_{N \geq 0} \binom{2N}{N} y^N = (1 - 4y)^{-1/2}$ for $|y| < \frac{1}{4}$, with non-negative terms for $y \geq 0$; at $y = \frac{1}{8}$ the sum is $(\frac{1}{2})^{-1/2} = \sqrt{2}$. Hence $\dim_{\text{vN}} \ker T' = \sqrt{2}/4$. Finally $8(\sqrt{2}/4)^2 = 1$, and $\sqrt{2}/4$ is irrational because $\sqrt{2}$ is. Together with Section 2.2 this proves Theorem A. $\square$

3.6 Proof of Corollary B

By Lemma 2.2(a), applied three times, the kernel of $\text{diag}(T', T', T', T')$ has von Neumann dimension $4 \cdot \sqrt{2}/4 = \sqrt{2}$, and $\text{diag}(8T', 8T', 8T', 8T') \in M_8(\mathbb{Z}G)$ has the same kernel. So $\sqrt{2} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, under the definition of [Gra10] (square matrices over $\mathbb{Q}G$) and under that of [Gra16] (matrices over $\mathbb{Z}G$). The block sum of k copies of $8T'$ gives $k\sqrt{2}/4$. For even p , sending the lamp generator e_j of $\mathbb{Z}/2 \wr \mathbb{Z}$

to the element of order 2 of the j -th copy of $\mathbb{Z}/p$, and t to t , identifies $\mathbb{Z}/2 \wr \mathbb{Z}$ with a subgroup of $\mathbb{Z}/p \wr \mathbb{Z}$, since the subgroup of $\bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/p$ formed by the elements of order at most 2 is invariant under the shift. By Lemma 2.2(b), $\sqrt{2}$ and the numbers $k\sqrt{2}/4$ lie in $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$. $\square$

4 Second proof of Theorem A: finite quotients

For $K \geq 1$ let $G_K = \mathbb{Z}/2 \wr \mathbb{Z}/K = (\bigoplus_{r \in \mathbb{Z}/K} \mathbb{Z}/2) \rtimes \mathbb{Z}/K$, of order $K2^K$, and let $\varphi_K: G \rightarrow G_K$ reduce the lamp positions and the exponent of t modulo K : $\varphi_K(gt^k) = \bar{g}t^{k \bmod K}$ with $\bar{g}_r = \sum_{j \equiv r} g_j \bmod 2$. This is a surjective homomorphism, since reduction modulo K commutes with the shift σ . For $a \in M_m(\mathbb{Q}G)$ let a_K be its image in $M_m(\mathbb{Q}G_K)$, acting by left multiplication on $\mathbb{C}[G_K]^m$, and put

$$d_K(a) = \frac{\dim_{\mathbb{C}} \ker a_K}{|G_K|}.$$

Lemma 4.1 (finite Fourier transform; [Fer26, Lemma 6.1]). *For $x \in X_K = \{0, 1\}^{\mathbb{Z}/K}$ let $\theta_x^{(K)}(a)$ be the operator on $\mathbb{C}^{\mathbb{Z}/K} \otimes \mathbb{C}^m$ defined as in Lemma 3.1, with all indices taken modulo K . Then $\dim_{\mathbb{C}} \ker a_K = \sum_{x \in X_K} \dim \ker \theta_x^{(K)}(a)$.*

Proof. See [Fer26, Lemma 6.1]: the proof of Lemma 3.1 applies to the finite group $(\mathbb{Z}/2)^{\mathbb{Z}/K}$, whose characters form an orthogonal basis of $\mathbb{C}^{X_K}$, and $(V\Psi)(x, k) = \Psi(\tau^{-k}x, k)$ is well defined since τ^K is the identity; so left multiplication by a_K is conjugate to the direct sum of the operators $\theta_x^{(K)}(a)$. $\square$

Proposition 4.2 (exact formula). *For every $P \geq 1$, with $B_N = \binom{2N}{N}$,*

$$d_{3P}(T') = \frac{1}{4} \sum_{N=0}^{P-2} \frac{B_N}{8^N} + \frac{4B_{P-1}}{8^P} + \frac{2 \cdot 3^P}{P \cdot 8^P}. \quad (5)$$

Proof. Let $K = 3P$. Since 3 divides K , the three classes of indices modulo 3 are defined on $\mathbb{Z}/K$, and $\theta_x^{(K)}(T')$ preserves each of them. On the class r we use the positions $m \in \mathbb{Z}/P$ (index $-3m - r$) with the symbols $\sigma_m = (x_{3m+r}, x_{3m+r+1}, x_{3m+r+2})$, indices modulo K . These P triples of coordinates partition $\mathbb{Z}/K$, so $x \mapsto (\sigma_m)_{m \in \mathbb{Z}/P}$ is a bijection from X_K onto $(\{0, 1\}^3)^{\mathbb{Z}/P}$, for each r . Lemma 3.3 holds verbatim with the positions taken modulo P . So by Lemma 4.1, $\dim_{\mathbb{C}} \ker T'_K = 3 \sum_{\sigma} \kappa(\sigma)$, where T'_K is the image of T' in $M_2(\mathbb{Q}G_K)$, the sum runs over the 8^P cyclic symbol sequences σ , and $\kappa(\sigma)$ is the kernel dimension of the cyclic operator given by the rows of Lemma 3.3.

If σ contains a marker, the argument of Lemma 3.6 applies on the cycle: for cyclically consecutive markers m, m' the rows “second coordinate at m ”, “both coordinates at the letters in between” and “first coordinate at m' ” and the columns of the positions $m+1, \dots, m'$ form the block M_s of the word s of these letters; when there is only one marker, its segment consists of the other $P-1$ positions and starts and ends at this marker (for $P=1$ the block is $M_{\emptyset}$). Rows and columns are partitioned by the segments. So $\kappa(\sigma)$ is the number of segments whose word s has $\#U(s) = \#D(s)$. We count the pairs $(\sigma, \text{marker } m \text{ of } \sigma)$ for which the segment starting at m is balanced. Fix m . For a segment of $N \leq P-2$ letters with word s , the symbol at m is a marker (4 choices), the symbols at $m+1, \dots, m+N$ are letters forming s ($2^{\#I(s)}$ choices, since I has two symbols and U, D one each), the symbol at $m+N+1$ is a marker (4 choices), and the remaining $P-N-2$ symbols are free (8 choices each); summing $16 \cdot 2^{\#I(s)} \cdot 8^{P-N-2}$ over the balanced words s of length N gives $16B_N 8^{P-N-2}$ by Lemma 3.7. For $N = P-1$ the marker at m is the only one, and σ is determined by s and the two free bits of the marker: $4B_{P-1}$ pairs. So the pairs number $P(\sum_{N=0}^{P-2} 16B_N 8^{P-N-2} + 4B_{P-1})$.

If σ contains no marker, every row is of the second kind, and the two tracks decouple into $f_u(m+1) = a_m f_u(m)$ and $f_v(m+1) = b_m f_v(m)$ around the cycle; they have a non-zero solution if and only

if $\prod_m a_m = 1$, resp. $\prod_m b_m = 1$, that is, if no U , resp. no D , occurs. So $\kappa(\sigma) = [\text{no } U] + [\text{no } D]$. There are 3^P marker-free sequences without U (symbols in $\{000, 010, 011\}$) and 3^P without D , which contribute $2 \cdot 3^P$.

Dividing $3(P \sum_{N=0}^{P-2} 16B_N 8^{P-N-2} + 4PB_{P-1} + 2 \cdot 3^P)$ by $|G_{3P}| = 3P \cdot 8^P$ gives (5). $\square$

Second proof of Theorem A. The kernels $N_i = \ker \varphi_{3 \cdot 2^i}$, $i \geq 0$, are normal subgroups of finite index. They are nested, since $\varphi_{3 \cdot 2^i}$ factors through $\varphi_{3 \cdot 2^{i+1}}$, and their intersection is trivial: if gt^k lies in all of them, then $3 \cdot 2^i$ divides k for every i , so $k = 0$, and every residue class modulo $3 \cdot 2^i$ contains an even number of the lamps of g ; once $3 \cdot 2^i$ exceeds the diameter of the support of g , each class contains at most one of them, so $g = 0$. We use Lück's approximation theorem [Lüc94] (see [KKL25, Theorem 6.2]) in its form for matrices over $\mathbb{Q}G$ and a nested chain of normal subgroups of finite index with trivial intersection. This form is [KKL25, Conjecture 7.1(1)], and by [KKL25, Remark 7.2] it holds for our chain, since G is amenable and the quotients are finite, so that all these groups are sofic and satisfy the Determinant Conjecture. Applied to the square matrix $8T'$ and this chain, it gives $\dim_{\mathbb{N}} \ker T' = \lim_{i \rightarrow \infty} d_{3 \cdot 2^i}(T')$; by Lemma 2.1 it does not matter whether the theorem is formulated with left or with right multiplication, on G and on the finite quotients. In (5) with $P = 2^i$ the middle term is at most $4 \cdot 4^{P-1} 8^{-P} = 2^{-P}$, since $B_N \leq 4^N$, the last term tends to 0, and the sum tends to $\frac{1}{4} \sum_{N \geq 0} B_N 8^{-N}$, which equals $\sqrt{2}/4$ by the binomial series, as in Section 3.5. $\square$

This proof uses the finite Fourier transform and elementary counting instead of the direct integral of Lemma 3.2, and Lück's theorem as a citation.

Remark 4.3 (the error on the finite quotients). Subtracting $\sqrt{2}/4 = \frac{1}{4} \sum_{N \geq 0} B_N 8^{-N}$ from (5) gives

$$d_{3P}(T') - \frac{\sqrt{2}}{4} = \frac{2B_{P-1}}{8^P} - \frac{1}{4} \sum_{N \geq P} \frac{B_N}{8^N} + \frac{2 \cdot 3^P}{P \cdot 8^P}.$$

The first term comes from the configurations with a single marker, whose segment of $P - 1$ letters has the weight $4 \cdot 8^{-P}$ instead of $\frac{1}{4} \cdot 8^{-P+1}$, and the second from the segments that are too long for the cycle. Both lie in $[0, 2^{-P-1}]$, since $B_N \leq 4^N$, and they have opposite signs, so $|d_{3P}(T') - \sqrt{2}/4| \leq 2^{-P-1} + 2(3/8)^P/P$; the last term comes from the marker-free configurations. Table 1 lists some values.

$K = 3P$	$d_K(T')$	decimal	$d_K(T') - \sqrt{2}/4$
3	5/4	1.2500000	$9.0 \cdot 10^{-1}$
6	33/64	0.5156250	$1.6 \cdot 10^{-1}$
9	101/256	0.3945313	$4.1 \cdot 10^{-2}$
12	2993/8192	0.3653564	$1.2 \cdot 10^{-2}$
15	29263/81920	0.3572144	$3.7 \cdot 10^{-3}$
18	92995/262144	0.3547478	$1.2 \cdot 10^{-3}$
21	2598067/7340032	0.3539585	$4.1 \cdot 10^{-4}$
30		0.3535724	$1.9 \cdot 10^{-5}$
60		0.3535534	$3.2 \cdot 10^{-9}$

Table 1: The exact values (5); $\sqrt{2}/4 = 0.3535534\dots$. The values for $K \leq 18$ were also obtained from Lemma 4.1 by computing the fibre operators, those for $K \leq 9$ directly in the group ring, and the value for $K = 21$ by two of the verification runs (Section 6).

5 Remarks

Virtual values. The number $\sqrt{2}/4$ is none of the virtual values of Ara and Claramunt: $(\sqrt{2}/4)^2 = \frac{1}{8}$, while $(\frac{1}{8}\sqrt{2^t/(2^t-1)})^2 = 2^t/(64(2^t-1))$ lies in $(\frac{1}{64}, \frac{1}{56}]$, and $\frac{1}{4}\sqrt{2/7}$ is the case $t = 3$. So Theorem A does not decide the question of [AC21, Remark 4.14], where it is also said that no group G is known to the authors for which $\mathcal{C}(G)$ differs from the set of non-negative elements of the subgroup of $\mathbb{R}$ generated by $\mathcal{C}(G)$.

Single self-adjoint elements. Grigorchuk and Simanek ask for the values at 0 of the spectral measures of the self-adjoint elements of $\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}]$ [GS21, §7, Question 6(a)], that is, of *single* elements. One has $\ker T' = \ker T'^*T'$ with $T'^*T' \in M_2(\mathbb{Q}G)$ self-adjoint, but we do not reduce the matrix to a single element of $\mathbb{Q}G$; that question is not answered here.

Odd p . For odd p the group $\mathbb{Z}/p \wr \mathbb{Z}$ has no element of order 2, so it does not contain $\mathbb{Z}/2 \wr \mathbb{Z}$; we claim nothing for odd p .

6 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation. The computations test the definitions and the statements on finite instances; they do not test, and do not use, the statements about the infinite group in Section 3 or Lück's theorem. The reproducibility package contains the exact program of this note, the programs with which the result was first obtained (`original/`), and those of the three independent verification runs (`verification_run_A/`, `verification_run_B/`, `independent_run_2/`). They use exact arithmetic, or ranks modulo two large primes, which are compared with exact ranks where both are computed.

The exact program of this note (`check_note.py`; standard library, exact arithmetic, about two minutes) checks: the kernels of all 29,524 counter matrices M_s with $s \in \{U, D, I\}^N$, $N \leq 9$ (Lemma 3.5); the automaton A_2 of [Fer26], run on all 625 letters by a depth-first search, whose recognized matrices of size at most 22 are the M_s with $s \in \{U, D\}^N$ (Remark 3.8); Lemma 3.7 by enumeration ($N \leq 10$), through the bijection ($N \leq 6$) and by Vandermonde's identity ($N \leq 300$); rational enclosures of $\sqrt{2}/4$ and $\sqrt{2}$; the expansion (2) of T' , built from its definition in $\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}]$; the kernels of left and of right multiplication by $8T'$ on $\mathbb{Q}[G_K]^2$, computed directly in the group ring for $K \leq 7$, and of the block sum of four copies for $K \leq 4$; the fibre operators of Lemma 4.1 for all x and $K \leq 14$, whose kernels agree with the group-ring values and with Proposition 4.2; for all 299,592 configurations with at most 6 positions and each of the three classes, the rows of Lemma 3.3, the segment blocks (2,632,140 of them, each equal to $8M_s$) and the kernel counts in the proof of Proposition 4.2; the same rows and blocks in 1,000 random windows of the line; and Table 1, the identity of Remark 4.3 ($P \leq 40$) and its bound ($P \leq 60$). The programs in `original/` treat the group ring for $K \leq 9$, the fibre operators for $K \leq 18$ and the blocks for $P \leq 6$. When the note was written, every program of the package with a recorded output, except `check_note.py`, whose output was recorded then, was run again from an extracted copy of the package (`RERUN_LOG.txt`); the third verification run ran all of them again, `check_note.py` included, from an extracted copy of the archive (`independent_run_2/rerun_package_v1.txt`). In both re-runs the outputs agree with the recorded ones up to running times.

Verification. The result was first obtained, with both proofs, together with the programs in `original/`. Two rounds of independent verification, all AI-assisted, followed; each run wrote its own programs from the definitions. In the first round two runs examined the first written version, and neither saw the other's report. The first run (A) compared Problem 4 and the definitions with

both versions of [Gra16] and with [Gra10, Gra14], and re-derived both proofs. Its programs compute the kernels in the group ring for $K \leq 9$, a Fourier model of their own for $K \leq 16$, equal to the group-ring values for $K \leq 9$, and the parts of the fibre operators on one class up to $K = 21$, which confirm (5) there; they check 98,382 segment blocks in random windows of the line, and, as a test on a known value, they give finite-quotient values that approach the number $1/3$ of Grigorchuk and Żuk [GŻ01]. The second run (B) wrote its programs before reading the original ones: the kernels in the group ring for $K \leq 9$, over the eigenspaces of right multiplication by t (a decomposition the note does not use) for $K \leq 12$, and through the fibre operators for $K \leq 20$, all in agreement and with (5); the block structure on all configurations with $P \leq 6$ (one class for $P = 6$); three deliberately altered operators, all detected; and a Monte-Carlo estimate of the integrand of (4) on the line, without markers or segments, 0.35354 ± 0.00130 from 100,000 samples. It also searched the literature and ran the original programs again, with identical outputs up to running times. Neither run of the first round found an error in a theorem, proposition or lemma. The corrections they required are made here: the page and wording of Problem 4 in the journal version, the evidence for the unnormalized trace, and the shape of the matrices in [Gra16] (Section 1.1); the fact that [AC21, Remark 4.14] does not cite Problem 4, which is cited in [AC22]; the presentation of the balanced-word mechanism and of the virtual values of Ara and Claramunt as the closest prior work (abstract, Section 1.3, “Scope and priority”); the definition of P_x on all three classes (Section 3.5); the description of a database search; and the bibliographic data. When the note was written, the common lemmas were cited from [Fer26], the multiplicativity of θ_x was added to Lemma 3.1, the bound of Remark 4.3 was sharpened, and `check_note.py` was written.

In the second round a third run examined the final text line by line, the parts changed at the writing first. It read the sources again: [Gra10], both versions of [Gra16], with the page of Problem 4 rendered, [Gra14], [AC21] also in its journal version, [AC22, GS21], [KKL25] for the form of Lück’s theorem, and [Fer26] for every statement cited from it; it re-derived all proofs and read `check_note.py` completely. It found no error in a theorem, proposition or lemma. The corrections it required are made here: a sentence in Section 1.3 which said that it was not known whether $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ contains algebraic numbers of degree greater than 2, although $\sqrt{2}/4 + \sqrt{5}/20$ is such a number of degree 4; the numbering of the question of [Gra14] (Section 1.1); the form of Lück’s theorem in Section 4; the description of the re-run at the writing; and the record of this run. Its three programs (`independent_run_2/`) compute the kernels of left multiplication by $8T'$ on $\mathbb{Q}[G_K]^2$ in the group ring for $K \leq 9$ (exactly over $\mathbb{Q}$ for $K \leq 5$, modulo two primes otherwise), of right multiplication for $K \leq 8$, and of the image of $8T'$ in $\mathbb{Q}[\mathbb{Z}/4 \wr \mathbb{Z}/K]^2$ for $K \leq 4$; build the change of basis of Lemma 4.1 explicitly for $K \leq 5$ and find the matrix of $8T'$ block diagonal, with the blocks given by (3); compute all fibre operators for $K \leq 15$ (exactly for $K \leq 10$), which agree with the group-ring values for $K \leq 9$, and their parts on the three classes for all x up to $K = 21$, which agree with (5); check Lemmas 3.3 and 3.6 on all configurations with $P \leq 5$, on the class 0 for $P = 6$, and on 1,500 random windows of the line in all three classes; check the bookkeeping of Section 3.5 with the exact kernel projections of the M_s for $N \leq 7$, and the trace formula (4) on the finite quotients for $K \leq 9$; and check Lemmas 3.5 and 3.7, Remarks 3.8 and 4.3, Table 1 and Section 5. Operators with other letter probabilities or with the shift t^2 , and a wrong sign rule in (3), are detected. It also ran every program of the package again from an extracted copy of its archive; all recorded outputs were reproduced up to running times.

Scope and priority. The note answers the concrete question of Problem 4 of [Gra16]: $\sqrt{2} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ (Corollary B). It does not describe $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, as asked by the first part of Problem 4, by Question 5 of [Gra10] and by Question 3 of [Gra14], so the corpus record OWR-4428-012 remains partially solved; nor does it decide whether the virtual values of Ara and Claramunt are ℓ^2 -Betti numbers, answer Question 6(a) of [GS21], or treat odd p . The question, the definitions and the framework

are Grabowski’s [Gra10, Gra14, Gra16]. The mechanism, balanced words counted by central binomial coefficients, is the one by which Ara and Claramunt obtained the irrational algebraic *virtual* ℓ^2 -Betti numbers $\frac{1}{4}\sqrt{2/7}$ and $\frac{1}{8}\sqrt{2^t/(2^t-1)}$, $t \geq 3$ [AC21, Theorem 4.13]; in [AC21, Remark 4.14] they ask for a genuine witness for $\frac{1}{4}\sqrt{2/7}$. The form of the operator, the Fourier picture and the decomposition into segments are those of our companion note [Fer26]. Lück’s approximation theorem [Lüc94] and Pontryagin duality are standard. Only Theorem A and Corollary B are, to our knowledge, new; no priority is claimed, and the proofs are short once the counter of [Fer26] is known.

What was read: [Gra10] in the publisher’s open PDF (pp. 2366–2369); [Gra16] in its arXiv version v4 (pp. 1–5) and in the journal version (pp. 795–798); in their arXiv versions, whose numbering we use, [Gra14] (v4, Section 1), [AC21] (v2, the introduction and Section 4.3; the journal version, read by the third run, has the same numbers there), [AC22] (v2, introduction), [GS21] (v2, Section 7) and [KKL25] (v1, Sections 6 and 7.1); and [Fer26]. The journal version of [Gra14] is not openly accessible and was not read. The theorem of [Lüc94] is cited as a standard result; what we say about [Aus13, GŻ01, LW13] is taken from [Gra10, Gra14, AC22]. The searches were made on 14 October 2026, when the result was first obtained and by the two runs of the first round: the arXiv API (keyword searches combining “lamplighter” with Betti numbers, the Atiyah problem, von Neumann dimension, kernels or spectral measures; Betti numbers of wreath products with “algebraic” or “irrational”; “Atiyah” with “square root” or “irrational algebraic”; author searches; and the 100 most recent titles containing “lamplighter”), OpenAlex (the works citing [Gra10, Gra14, Gra16, AC21, AC22, LW13, GS21], by title and abstract), Semantic Scholar (the 20 works citing the arXiv version of [Gra16], the most recent of them from 2026, and those citing the arXiv version of [AC21]), Crossref, zbMATH Open, and four web searches; the texts of [AC21, AC22, GS21, KKL25] were searched for the lamplighter group and for $\sqrt{2}$. The third run repeated the searches in part on the same day: the arXiv API (abstracts with “lamplighter” and Betti numbers, the Atiyah problem, von Neumann algebras or spectral measures; “Betti” with “irrational algebraic”; author searches; the 40 most recent titles containing “lamplighter”, the latest of September 2026), Crossref, OpenAlex (the works citing [Gra16, AC21, AC22], and works on lamplighter groups and Betti numbers since 2024), zbMATH Open and one web search; it found nothing new. In these sources we found no statement that $\sqrt{2} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, and no earlier non-virtual irrational algebraic ℓ^2 -Betti number of a lamplighter group $\mathbb{Z}/p \wr \mathbb{Z}$ other than $\sqrt{5}/20$ of [Fer26]: the values known there are transcendental and other irrational numbers [Gra16, AC21], irrational algebraic numbers for the free lamplighters $\mathbb{Z}/p \wr F_d$ [LW13], and irrational algebraic virtual ℓ^2 -Betti numbers of $\mathbb{Z}/2 \wr \mathbb{Z}$ [AC21, AC22]. A search that finds nothing is not a proof of novelty; in particular theses, talks and unindexed preprints are not covered.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AC21] Pere Ara and Joan Claramunt. L^2 -Betti numbers arising from the lamplighter group. *Journal of Algebraic Combinatorics*, 54(4):1201–1245, 2021. [arXiv:2005.12814](#), [doi:10.1007/s10801-021-01044-8](#).
- [AC22] Pere Ara and Joan Claramunt. Approximating the group algebra of the lamplighter by infinite matrix products. *Forum Mathematicum*, 34(4):851–891, 2022. [arXiv:2005.12374](#), [doi:10.1515/forum-2020-0141](#).

- [Aus13] Tim Austin. Rational group ring elements with kernels having irrational dimension. *Proceedings of the London Mathematical Society*, 107(6):1424–1448, 2013. [arXiv:0909.2360](#), [doi:10.1112/plms/pdt029](#).
- [Fer26] Alper Ferudun. Irrational algebraic values of Grabowski’s automaton series and an irrational algebraic ℓ^2 -Betti number of the lamplighter group. Zenodo, unrefereed preprint, version 1.0, 2026. [doi:10.5281/zenodo.23377729](#).
- [Gra10] Łukasz Grabowski. On the Atiyah problem for the lamplighter groups. *Oberwolfach Reports*, 7(3):2366–2369, 2010. Abstract in Report 40/2010, *Actions and Invariants of Residually Finite Groups: Asymptotic Methods*, workshop of 5–11 September 2010, pp. 2335–2391. [doi:10.4171/OWR/2010/40](#).
- [Gra14] Łukasz Grabowski. On Turing dynamical systems and the Atiyah problem. *Inventiones Mathematicae*, 198(1):27–69, 2014. [arXiv:1004.2030](#), [doi:10.1007/s00222-013-0497-5](#).
- [Gra16] Łukasz Grabowski. Irrational l^2 invariants arising from the lamplighter group. *Groups, Geometry, and Dynamics*, 10(2):795–817, 2016. [arXiv:1009.0229](#), [doi:10.4171/GGD/366](#).
- [GS21] Rostislav Grigorchuk and Brian Simanek. Spectra of Cayley graphs of the lamplighter group and random Schrödinger operators. *Transactions of the American Mathematical Society*, 374(4):2421–2445, 2021. [arXiv:1902.10129](#), [doi:10.1090/tran/8156](#).
- [GŻ01] Rostislav I. Grigorchuk and Andrzej Żuk. The lamplighter group as a group generated by a 2-state automaton, and its spectrum. *Geometriae Dedicata*, 87(1–3):209–244, 2001. [doi:10.1023/A:1012061801279](#).
- [KKL25] Dominik Kirstein, Christian Kremer, and Wolfgang Lück. Some problems and conjectures about L^2 -invariants. In *Geometry and Topology of Aspherical Manifolds*, volume 816 of *Contemporary Mathematics*, pages 3–43. American Mathematical Society, Providence, RI, 2025. [arXiv:2311.17830](#), [doi:10.1090/conm/816/16349](#).
- [Lüc94] Wolfgang Lück. Approximating L^2 -invariants by their finite-dimensional analogues. *Geometric and Functional Analysis*, 4(4):455–481, 1994. [doi:10.1007/BF01896404](#).
- [LW13] Franz Lehner and Stephan Wagner. Free lamplighter groups and a question of Atiyah. *American Journal of Mathematics*, 135(3):835–849, 2013. [arXiv:1005.2347](#), [doi:10.1353/ajm.2013.0029](#).
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record OWR-4428-012.