

Genuine Witnesses for Virtual ℓ^2 -Betti Numbers of Ara and Claramunt and Irrational Algebraic ℓ^2 -Betti Numbers of $\mathbb{Z}/p \wr \mathbb{Z}$

Alper Ferudun*

October 15, 2026

Abstract

Grabowski asked what the set $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ of ℓ^2 -Betti numbers arising from the lamplighter group is, that is, of von Neumann dimensions of kernels of square matrices over its rational group ring, and whether $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers (Invent. Math. 2014). Ara and Claramunt obtained $\frac{1}{4}\sqrt{2/7}$ and $\frac{1}{8}\sqrt{2^t/(2^t-1)}$, $t \geq 3$, as *virtual* ℓ^2 -Betti numbers of $\mathbb{Z}/2 \wr \mathbb{Z}$, that is, as differences of two ℓ^2 -Betti numbers, and asked for a concrete matrix whose kernel has dimension $\frac{1}{4}\sqrt{2/7}$ (J. Algebraic Combin. 2021, Remark 4.14). We give genuine witnesses for these numbers. An explicit $T \in M_2(\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}])$ built from five lamp bits and the shift t^5 , with $32T$ integral, has $\dim_{\text{vN}} \ker T = \frac{1}{16} \sum_{N \geq 0} W_N 32^{-N} = \frac{1}{4}\sqrt{2/7}$, where $W_N = \sum_k \binom{N}{2k} \binom{2k}{k} 25^k 14^{N-2k}$; a family $T^{(n)}$ with $n+2$ lamp bits gives $\frac{1}{8}\sqrt{2^n/(2^n-1)}$ for every $n \geq 2$. So every number of the family listed by Ara and Claramunt lies in $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$; the other elements of their group of virtual values are not treated. With $\rho_j = \frac{1}{p}(1+e_j+\dots+e_j^{p-1})$ the same template gives, for every integer $p \geq 2$, an explicit $T_p \in M_2(\mathbb{Q}[\mathbb{Z}/p \wr \mathbb{Z}])$ with $p^2 T_p$ integral and $\dim_{\text{vN}} \ker T_p = (\frac{p-1}{p})^2 \sqrt{(p-1)/(p+3)}$, which is irrational of degree 2 because $(p-1)(p+3) = (p+1)^2 - 4$ is never a square; for $p = 3$ it is $4\sqrt{3}/27$. All values are proved twice: by Pontryagin duality with a decomposition into counter matrices, and by Lück's approximation theorem with an exact formula on finite quotients. The questions and the frameworks are Grabowski's and Ara and Claramunt's; the balanced-word counter is the mechanism of Ara and Claramunt and of our two companion notes, which gave $\sqrt{5}/20$ and $\sqrt{2}$ for $p = 2$. Apart from these notes, the sources we searched contain no earlier non-virtual irrational algebraic ℓ^2 -Betti number of a group $\mathbb{Z}/p \wr \mathbb{Z}$; a search that finds nothing is not a proof of novelty. A description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ remains open. This is an unrefereed note.

1 Introduction

1.1 The questions

Let G be a countable discrete group. Following Grabowski [Gra10, p. 2366], [Gra14, §1], a non-negative real number r is an *ℓ^2 -Betti number arising from G* if there is a square matrix $\theta \in M_m(\mathbb{Q}G)$ whose kernel, acting on $\ell^2(G)^m$, has von Neumann dimension r ; the set of these numbers is $\mathcal{C}(G)$. The dimension is the unnormalized one, a number in $[0, m]$ (Section 2.1); this is the trace $\text{tr} \otimes \tau_G$ of [Gra14, §1], and only with it can the Atiyah conjecture for torsion-free groups be stated as $\mathcal{C}(G) = \mathbb{N}$, as in [Gra10, p. 2366] and [Gra16, p. 795]; in [Gra16] the matrices have entries in $\mathbb{Z}G$, and our witnesses are square and become integral after scaling. Ara and Claramunt write $\mathcal{C}(G, \mathbb{Q})$ for the same set [AC21, Definition 2.2] (matrices over $\mathbb{Q}G$ acting on the left, unnormalized trace) and $\mathcal{G}(G, \mathbb{Q})$ for

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

the subgroup of $(\mathbb{R}, +)$ that it generates; they call the elements of $\mathcal{G}(G, \mathbb{Q})$ *virtual ℓ^2 -Betti numbers* [AC21, Remark 4.14].

In the UnsolvedMath collection [ula26] (version 1.6.0) the record **OWR-4428-012** is Question 5 of Grabowski’s Oberwolfach abstract, “What is $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$?” [Gra10, p. 2368], listed as partially solved; the same question is Question 3 in [Gra14, §1c] (in the numbering of the arXiv version v4, which we use; [GS21, AC22] cite it as Question 1.7). This note gives new elements of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ but no description of the set, so **the record remains partially solved**. It answers two published questions about such sets.

Grabowski’s question. In [Gra14, §1c], after recalling that Lehner and Wagner found irrational algebraic numbers in $\mathcal{C}(\mathbb{Z}/p \wr F_d)$ for free groups F_d [LW13] and that $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains transcendental numbers for every $p > 1$ (see [Gra16, Theorem 2]), Grabowski writes: “This raises the question whether $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers.” The passage is on p. 5 of the arXiv version v4; Ara and Claramunt cite it as [Gra14, p. 32], a page of the journal version, which we could not compare. The next sentence announces algebraic numbers of degree k in $\mathcal{C}((\mathbb{Z}/p \wr \mathbb{Z})^k)$, to be shown in a future version of [Gra16]; this concerns the direct powers, and neither the arXiv version v4 nor the journal version of [Gra16] contains such a result. For $p = 2$, and hence for every even p , the question was answered by our companion notes [Fer26a, Fer26b]; in the literature we found it open for every p . For odd p the group $\mathbb{Z}/p \wr \mathbb{Z}$ has no element of order 2, so it contains no copy of $\mathbb{Z}/2 \wr \mathbb{Z}$ and the notes do not help.

The question of Ara and Claramunt. Ara and Claramunt obtained $\frac{1}{4}\sqrt{2/7} \in \mathcal{G}(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Q})$ from the language of the words in two letters x_1, x_2 with $|w|_{x_1} = |w|_{x_2}$ [AC21, Theorem 4.13], and, from other pairs of letters, the numbers $\frac{1}{8}\sqrt{2^t/(2^t - 1)}$, $t \geq 3$, the case $t = 3$ being $\frac{1}{4}\sqrt{2/7}$ (the paragraph after [AC21, Theorem 4.13]; we call these numbers *the listed family*). In [AC21, Remark 4.14] they explain that $\frac{1}{4}\sqrt{2/7}$ is a difference $\alpha_1 - \alpha_2$ of two ℓ^2 -Betti numbers, ask whether it is itself one, and, in that case, for “a concrete matrix over $\mathbb{Q}\Gamma$ witnessing this fact”; this would answer the question of Grabowski quoted above, and they know no group G with $\mathcal{C}(G, \mathbb{Q}) \neq \mathcal{G}(G, \mathbb{Q}) \cap \mathbb{R}_+$. Their Proposition 4.11 and Theorem 1.1 give many further elements of $\mathcal{G}(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Q})$; we do not treat them. The journal version has the same numbers (pp. 1238–1239).

1.2 Results

Let $p \geq 2$ be an integer and $G_p = \mathbb{Z}/p \wr \mathbb{Z} = (\bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/p) \rtimes \mathbb{Z}$, with the lamp generators e_j ($e_j^p = 1$) and the generator t of $\mathbb{Z}$, $te_jt^{-1} = e_{j+1}$. In $\mathbb{Q}G_p$ put

$$\rho_j = \frac{1}{p}(1 + e_j + e_j^2 + \cdots + e_j^{p-1}), \quad \pi'_j = \rho_j, \quad \pi_j = 1 - \rho_j;$$

for $p = 2$ these are $\pi'_j = \frac{1}{2}(1 + e_j)$ and $\pi_j = \frac{1}{2}(1 - e_j)$, as in [Fer26a, Fer26b]. For $\chi, \alpha, \beta \in \mathbb{Q}G_p$ and an integer $r \geq 1$ let

$$T[\chi, \alpha, \beta; r] = \begin{pmatrix} \chi - (1 - \chi)\alpha + (1 - \chi)t^r & -\chi \\ \chi t^r & (1 - \chi)t^r - \chi t^r - (1 - \chi)\beta \end{pmatrix} \in M_2(\mathbb{Q}G_p), \quad (1)$$

the form of the matrices of [Fer26a, (2)] ($r = 2$) and [Fer26b, (1)] ($r = 3$). All dimensions are unnormalized (Section 2.1); with the normalized trace on M_2 every value below would be halved.

Theorem A. *Let $p = 2$ and*

$$\chi = \pi_0\pi_1, \quad \alpha = 1 + (1 - \pi_0\pi_1)\pi'_2\pi'_3 - \pi'_0\pi'_1\pi'_2\pi'_3\pi'_4, \quad \beta = 1 + (1 - \pi_0\pi_1)\pi_2\pi_3 - \pi'_0\pi'_1\pi_2\pi_3\pi'_4,$$

and $T = T[\chi, \alpha, \beta; 5]$. Then $32T \in M_2(\mathbb{Z}/2 \wr \mathbb{Z})$, and

$$\dim_{\mathbb{V}\mathbb{N}} \ker T = \frac{1}{16} \sum_{N \geq 0} \frac{W_N}{32^N} = \frac{1}{4} \sqrt{\frac{2}{7}} = \frac{\sqrt{14}}{28}, \quad W_N = \sum_{k \geq 0} \binom{N}{2k} \binom{2k}{k} 25^k 14^{N-2k}.$$

This number is a root of $56X^2 - 1$, which is irreducible over $\mathbb{Q}$.

Corollary B. The numbers $\frac{1}{4}\sqrt{2/7}$ and $\frac{1}{8}\sqrt{2^n/(2^n-1)}$, $n \geq 2$, of Theorems A and C lie in $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, with the square integral witnesses $32T$ and $2^{n+2}T^{(n)}$; so do their sums with non-negative integer coefficients, and they lie in $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ for every even p . In particular every number $\frac{1}{8}\sqrt{2^t/(2^t-1)}$, $t \geq 3$, of the family listed by Ara and Claramunt is a genuine ℓ^2 -Betti number arising from $\mathbb{Z}/2 \wr \mathbb{Z}$; for $\frac{1}{4}\sqrt{2/7}$ ($t = 3$) this answers the question of [AC21, Remark 4.14], with a concrete matrix over $\mathbb{Q}\Gamma$, indeed over $\mathbb{Z}\Gamma$, where $\Gamma = \mathbb{Z}/2 \wr \mathbb{Z}$.

Theorem C. Let $p = 2$, $n \geq 2$ and $r = n + 2$, let

$$\alpha_n = 1 + (1 - \pi_0 \pi_1) \pi'_2 \pi'_3 - \prod_{j=0}^{r-1} \pi'_j, \quad \beta_n = 1 + (1 - \pi_0 \pi_1) \pi_2 \pi_3 - \pi'_0 \pi'_1 \pi_2 \pi_3 \prod_{j=4}^{r-1} \pi'_j,$$

and $T^{(n)} = T[\pi_0 \pi_1, \alpha_n, \beta_n; r]$, so that $T^{(3)} = T$. Then $2^r T^{(n)} \in M_2(\mathbb{Z}/2 \wr \mathbb{Z})$, and

$$\dim_{\mathbb{V}\mathbb{N}} \ker T^{(n)} = \frac{1}{16} \sum_{N \geq 0} \frac{W_N^{(n)}}{2^{rN}} = \frac{1}{8} \sqrt{\frac{2^n}{2^n-1}}, \quad W_N^{(n)} = \sum_{k \geq 0} \binom{N}{2k} \binom{2k}{k} u_n^{2k} i_n^{N-2k},$$

where $u_n = 3 \cdot 2^{n-2} - 1$ and $i_n = 3 \cdot 2^{n-1} + 2$. This number is a root of $64(2^n - 1)X^2 - 2^n$, which is irreducible over $\mathbb{Q}$; for $n = 2$ it is $\sqrt{3}/12$, for $n = 3$ it is $\frac{1}{4}\sqrt{2/7}$.

Theorem D. Let $p \geq 2$ be an integer, $a = \rho_0$, $b = \rho_1$, and $T_p = T[\pi_0 \pi_1, 1 + \pi'_0 \pi_1, 1 + \pi_0 \pi'_1; 2]$, that is,

$$\begin{aligned} (T_p)_{11} &= 1 - 3a - 2b + 3ab + (a + b - ab)t^2, & (T_p)_{12} &= -(1 - a)(1 - b), \\ (T_p)_{21} &= (1 - a)(1 - b)t^2, & (T_p)_{22} &= -a - 2b + 2ab - (1 - 2a - 2b + 2ab)t^2. \end{aligned}$$

Then $p^2 T_p \in M_2(\mathbb{Z}[\mathbb{Z}/p \wr \mathbb{Z}])$, and

$$\dim_{\mathbb{V}\mathbb{N}} \ker T_p = \left(\frac{p-1}{p}\right)^4 \sum_{N \geq 0} \frac{B_N}{p^{2N}} = \left(\frac{p-1}{p}\right)^2 \sqrt{\frac{p-1}{p+3}}, \quad B_N = \sum_{k \geq 0} \binom{N}{2k} \binom{2k}{k} (p-1)^{2k}.$$

This number is irrational and algebraic of degree 2, a root of $p^4(p+3)X^2 - (p-1)^5$, because $(p-1)(p+3) = (p+1)^2 - 4$ lies strictly between p^2 and $(p+1)^2$.

Corollary E. For every integer $p \geq 2$ the set $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers of degree 2, together with all their positive integer multiples; for instance $4\sqrt{3}/27$ for $p = 3$, $8\sqrt{2}/25$ for $p = 5$ and $36\sqrt{15}/245$ for $p = 7$. So Grabowski's question of [Gra14, §1c] has a positive answer for every $p \geq 2$.

The primality of p is not used. For $p = 2$, T_2 is the matrix T of [Fer26a, Theorem 2] and Theorem D gives $\sqrt{5}/20$ again; so Theorem D adds no new element of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, while Theorems A and C do; and the question of [Gra14, §1c] is not itself a record of [ula26].

The three theorems are instances of one statement, Theorem 3.7. For a partition of the bit patterns $b \in \{0, 1\}^r$ into *markers*, a non-empty set M , and the *letters* U, D, I , let $\chi, \alpha - 1$ and $\beta - 1$ be the

sums of the idempotents $\varepsilon_b = \prod_i (\pi_i \text{ or } \pi'_i)$ over M , U and D . In the Fourier picture every position carries a random pattern b , with probability $q(b) = (p-1)^{|b|}/p^r$; between two consecutive markers the fibre operator is a *counter matrix*, which has a one-dimensional kernel if U and D occur equally often in between and a zero kernel otherwise; and

$$\dim_{\text{vN}} \ker T[\chi, \alpha, \beta; r] = q_M^2 ((1 - q_I)^2 - 4q_U q_D)^{-1/2} \leq q_M, \quad (2)$$

where q_X is the probability of the class X . We prove this by Pontryagin duality (Section 3.4) and again by Lück's approximation theorem with an exact formula on the finite quotients $\mathbb{Z}/p \wr \mathbb{Z}/rP$ (Section 3.5).

1.3 What is new, what is not, and what stays open

The questions, the definition of $\mathcal{C}(G)$ and the framework are Grabowski's [Gra10, Gra14, Gra16]; the virtual values, the listed family and the question answered in Corollary B are Ara and Claramunt's [AC21]. The mechanism is not new: counting balanced words through $\sum_k \binom{2k}{k} z^k = (1 - 4z)^{-1/2}$ is how Ara and Claramunt obtained the listed family as virtual values [AC21, Theorem 4.13], and the matrix form (1), the Fourier picture and the decomposition into counter matrices are those of our companion notes [Fer26a, Fer26b], which treat $p = 2$ with two and three lamp bits; Theorem 3.7 is their common proof, written once for all p , all numbers of lamp bits and all partitions. For general p the bit $[x_j \neq 0]$ corresponds to the splitting of $\mathbb{Z}/p$ into $\{0\}$ and $\{1, \dots, p-1\}$ used by Grabowski [Gra16, §4]. Pontryagin duality, used for such computations by Austin [Aus13] (see [Gra10, p. 2366]) and by Grabowski [Gra14, Gra16], and Lück's approximation theorem [Lüc94] are standard. To our knowledge Theorems A and C, which give genuine witnesses for the listed family, and Theorem D for odd p are new; no priority is claimed, and the proofs are short once the counter of [Fer26a] is known. The literature search is described in the paragraph "Scope and priority" in Section 6.

Open remain: a description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, which is the question of the record and the first part of [Gra16, Problem 4, p. 796] (its concrete part, $\sqrt{2} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, was answered in [Fer26b]); whether the other elements of $\mathcal{G}(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Q})$ of [AC21, Proposition 4.11 and Theorem 1.1] lie in $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$; a description of $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ for odd p ; whether some $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains an algebraic number of odd degree greater than 1; and the question of Grigorchuk and Simanek for single self-adjoint elements of $\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}]$ [GS21, §7, Question 6(a)], since our witnesses are 2×2 matrices.

2 Conventions and common lemmas

2.1 Groups, traces, kernels

Let $p \geq 2$ and $A_p = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/p$, the group of finitely supported sequences $g = (g_j)_{j \in \mathbb{Z}}$, written additively; e_j has a single 1 at j , and $g = \prod_j e_j^{g_j}$ multiplicatively. Let $G_p = A_p \rtimes \mathbb{Z}$ with $te_j t^{-1} = e_{j+1}$; every element is uniquely gt^k , and $(gt^k)(ht^l) = (g + \sigma^k h)t^{k+l}$, $(\sigma h)_j = h_{j-1}$.

For a countable group G , the group ring acts on $\ell^2(G)$ by left multiplication, $\lambda(x)\delta_y = \delta_{xy}$, and $M_m(\mathbb{Q}G)$ acts on column vectors in $\ell^2(G)^m$. Let $L(G)$ be the von Neumann algebra generated by $\lambda(G)$, with the trace $\tau(\theta) = \langle \theta \delta_e, \delta_e \rangle$ and $\tau_m(\theta) = \sum_i \tau(\theta_{ii})$ on $M_m(L(G))$. For $a \in M_m(\mathbb{Q}G)$ we put $\dim_{\text{vN}} \ker a = \tau_m(P_{\ker \lambda(a)}) \in [0, m]$, where $P_{\ker \lambda(a)} \in M_m(L(G))$ is the orthogonal projection onto the kernel. This is the dimension of [Gra10, p. 2366], [Gra14, §1] (pp. 2–3 of the arXiv version v4, before §1a) and [AC21, Definition 2.2]; in [Gra14, §1] the kernel dimension of a matrix that is not self-adjoint is defined through a^*a , which gives the same number, since $\ker a^*a = \ker a$.

Lemma 2.1 (left and right; [Fer26a, Lemma 5.1]). *For a square matrix $a \in M_m(\mathbb{Q}G)$ the kernel of right multiplication $\eta \mapsto \eta a$ on row vectors in $\ell^2(G)^m$ has the same von Neumann dimension as $\ker \lambda(a)$. The same holds for a finite group in place of G , with $\dim_{\mathbb{C}}$ of kernels.*

Proof. The unitary $J\delta_x = \delta_{x^{-1}}$ fixes δ_e and satisfies $J(\xi b) = \lambda(b^*)J\xi$ for $b = \sum_x b_x x \in \mathbb{Q}G$, $b^* = \sum_x b_x x^{-1}$; so it conjugates right multiplication by a to $\lambda(a^*)$, $(a^*)_{ji} = (a_{ij})^*$. By the polar decomposition in $M_m(L(G))$ the projections onto $(\ker \lambda(a))^{\perp}$ and onto $\overline{\text{ran}} \lambda(a) = (\ker \lambda(a)^*)^{\perp}$ have the same trace, so $\tau_m(P_{\ker \lambda(a^*)}) = \tau_m(P_{\ker \lambda(a)})$. For a finite group, a matrix and its conjugate transpose have the same rank. $\square$

Remark 2.2 (a third reading). If each entry acts on column vectors by convolution from the right, $\xi \mapsto (\sum_j \xi_j a_{ij})_i$, then J conjugates this operator to $\lambda(\bar{a})$, $\bar{a}_{ij} = (a_{ij})^*$. As $a \mapsto \bar{a}$ is a bijection of $M_m(\mathbb{Q}G)$, this reading gives the same set $\mathcal{C}(G)$, but the witness for a value is then the entrywise adjoint $\bar{T}$ of our T . For T_3 itself this reading gives other values on the finite quotients, for instance $\frac{5}{9}$ instead of $d_4(T_3) = \frac{29}{81}$ (Sections 2.2 and 5.1), as a verification run found and the second independent verification run confirmed; its value on $\mathbb{Z}/3 \wr \mathbb{Z}$ was not computed.

Lemma 2.3 (sums and subgroups). (a) *For $a \in M_m(\mathbb{Q}G)$ and $b \in M_n(\mathbb{Q}G)$, $\dim_{\text{vN}} \ker \text{diag}(a, b) = \dim_{\text{vN}} \ker a + \dim_{\text{vN}} \ker b$; so $\mathcal{C}(G)$ is closed under addition.*

(b) *If H is a subgroup of a countable group Γ and $a \in M_m(\mathbb{Q}H)$, then the von Neumann dimension of the kernel of a is the same over H and over Γ ; hence $\mathcal{C}(H) \subseteq \mathcal{C}(\Gamma)$ [Gra10, pp. 2366–2367]. For even p , the map $e_j \mapsto e_j^{p/2}$, $t \mapsto t$ embeds $\mathbb{Z}/2 \wr \mathbb{Z}$ into $\mathbb{Z}/p \wr \mathbb{Z}$, so $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z}) \subseteq \mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$.*

Proof. As in [Fer26b, Lemma 2.2]. (a) $\lambda(\text{diag}(a, b)) = \lambda(a) \oplus \lambda(b)$, and τ_{m+n} adds the traces of the diagonal entries. (b) $\ell^2(\Gamma)^m$ is the orthogonal sum of the $\lambda(a)$ -invariant spaces $\ell^2(Hy)^m$, Hy a right coset, each mapped onto $\ell^2(H)^m$ by the right translation $\delta_z \mapsto \delta_{zy^{-1}}$, which intertwines the actions of a ; and $\delta_e \in \ell^2(H)^m$. The elements of order at most 2 of A_p form a shift-invariant subgroup isomorphic to A_2 . $\square$

2.2 Pontryagin duality

Let $X = (\mathbb{Z}/p)^{\mathbb{Z}}$ with the product μ of the uniform probability measures on $\mathbb{Z}/p$ (the Haar measure of the compact dual group of A_p), let $(\tau x)_j = x_{j+1}$, and let $\zeta = e^{2\pi i/p}$.

Lemma 2.4 (fibre operators). *There is a unitary $W: \ell^2(G_p) \rightarrow L^2(X \times \mathbb{Z}, \mu \otimes \text{counting})$ with $W\delta_e = 1_X \otimes \delta_0$ such that, for $g \in A_p$ and $l \in \mathbb{Z}$,*

$$(W\lambda(gt^l)W^{-1}\Phi)(x, k) = \zeta^{\sum_j g_j x_{j-k}} \Phi(x, k - l). \quad (3)$$

Consequently, for $a \in M_m(\mathbb{Q}G_p)$ the operator $W\lambda(a)W^{-1}$, applied componentwise, acts fibrewise: $(W\lambda(a)W^{-1}\Phi)(x, \cdot) = \theta_x(a)\Phi(x, \cdot)$, where $\theta_x(a)$ is the operator on $\ell^2(\mathbb{Z})^m$ obtained by replacing every group element gt^l in the entries of a by $f \mapsto (k \mapsto \zeta^{\sum_j g_j x_{j-k}} f(k - l))$. For every x the map $a \mapsto \theta_x(a)$ is multiplicative, $\theta_x(\rho_j)$ is the multiplication by $k \mapsto [x_{j-k} = 0]$, and $\theta_x(\pi_j)$ is the multiplication by $k \mapsto [x_{j-k} \neq 0]$.

Proof. As in [Fer26a, Lemma 5.2] for $p = 2$. The characters $\chi_g(x) = \zeta^{\sum_j g_j x_j}$, $g \in A_p$, are orthonormal in $L^2(X, \mu)$, since $\int \chi_g \bar{\chi}_h d\mu = \prod_j \frac{1}{p} \sum_{y \in \mathbb{Z}/p} \zeta^{(g_j - h_j)y} = [g = h]$, and their span is a unital $*$ -subalgebra of $C(X)$ that separates points, hence dense by the Stone–Weierstrass theorem; so they form an orthonormal basis, and $W_1: \delta_{gt^k} \mapsto \chi_g \otimes \delta_k$ is unitary. From $e_j \cdot gt^k = (g + e_j)t^k$ and $\chi_{g+e_j} = \zeta^{x_j} \chi_g$ we get $(W_1\lambda(e_j)W_1^{-1}\Psi)(x, k) = \zeta^{x_j} \Psi(x, k)$; from $t \cdot gt^k = (\sigma g)t^{k+1}$ and $\chi_{\sigma g}(x) = \chi_g(\tau x)$ we get

$(W_1\lambda(t)W_1^{-1}\Psi)(x, k) = \Psi(\tau x, k-1)$. Let $(V\Psi)(x, k) = \Psi(\tau^{-k}x, k)$, a unitary since τ preserves μ , and $W = VW_1$. Since $(\tau^{-k}x)_j = x_{j-k}$,

$$(W\lambda(t)W^{-1}\Phi)(x, k) = (V^{-1}\Phi)(\tau^{1-k}x, k-1) = \Phi(x, k-1), \quad (W\lambda(e_j)W^{-1}\Phi)(x, k) = \zeta^{x_{j-k}}\Phi(x, k),$$

and $W\delta_e = V(1_X \otimes \delta_0) = 1_X \otimes \delta_0$. The operators $\theta_x(e_j)$ (multiplication by $k \mapsto \zeta^{x_{j-k}}$) and $\theta_x(t)$ (the shift $f \mapsto f(\cdot - 1)$) satisfy the relations of G_p , for instance $(\theta_x(t)\theta_x(e_j)\theta_x(t)^{-1}f)(k) = \zeta^{x_{j+1-k}}f(k)$; so θ_x is a representation of G_p , extended linearly to $\mathbb{Q}G_p$ and to matrices, and the operator that replaces gt^l is $\theta_x(\prod_j e_j^{g_j})\theta_x(t)^l$. This gives (3) and the fibrewise action. Finally $\theta_x(\rho_j)$ multiplies by $\frac{1}{p} \sum_{c=0}^{p-1} \zeta^{cx_{j-k}} = [x_{j-k} = 0]$, which holds for every integer $p \geq 2$. $\square$

Lemma 2.5 (trace formula; [Fer26a, Lemma 5.3]). *Let $a \in M_m(\mathbb{Q}G_p)$. Suppose that for μ -almost every x an orthogonal projection P_x on $\ell^2(\mathbb{Z})^m$ with range $\ker \theta_x(a)$ is given, such that $x \mapsto \langle P_x \xi, \eta \rangle$ is measurable for all standard basis vectors ξ, η . Then*

$$\dim_{\text{vN}} \ker a = \sum_{i=1}^m \int_X \langle P_x(\delta_0 \otimes \varepsilon_i), \delta_0 \otimes \varepsilon_i \rangle d\mu(x), \quad (4)$$

where $\varepsilon_1, \dots, \varepsilon_m$ is the standard basis of $\mathbb{C}^m$.

Proof. The operator $(P\Phi)(x) = P_x\Phi(x)$ on $L^2(X; \ell^2(\mathbb{Z})^m)$ is an orthogonal projection, whose range consists of the Φ with $\Phi(x) \in \ker \theta_x(a)$ for almost every x ; by Lemma 2.4 this is $\ker W\lambda(a)W^{-1}$. So $W^{-1}PW = P_{\ker \lambda(a)}$, and $W\delta_e = 1_X \otimes \delta_0$ gives (4). $\square$

For $K \geq 1$ let $G_{p,K} = \mathbb{Z}/p \wr \mathbb{Z}/K = (\bigoplus_{\mathbb{Z}/K} \mathbb{Z}/p) \rtimes \mathbb{Z}/K$, of order Kp^K , and let $\varphi_K: G_p \rightarrow G_{p,K}$, $\varphi_K(gt^k) = \bar{g}t^{k \bmod K}$ with $\bar{g}_s = \sum_{j \equiv s \pmod{K}} g_j$; this is a surjective homomorphism, since reduction modulo K commutes with the shift. For $a \in M_m(\mathbb{Q}G_p)$ let a_K be its image, acting by left multiplication on $\mathbb{C}[G_{p,K}]^m$, and

$$d_K(a) = \frac{\dim_{\mathbb{C}} \ker a_K}{|G_{p,K}|}.$$

Lemma 2.6 (finite Fourier transform; [Fer26a, Lemma 6.1]). *For $x \in X_K = (\mathbb{Z}/p)^{\mathbb{Z}/K}$ let $\theta_x^{(K)}(a)$ be defined as in Lemma 2.4, with all indices taken modulo K . Then $\dim_{\mathbb{C}} \ker a_K = \sum_{x \in X_K} \dim \ker \theta_x^{(K)}(a)$.*

Proof. The proof of Lemma 2.4 applies to the finite group $(\mathbb{Z}/p)^{\mathbb{Z}/K}$: the characters χ_g , $g \in (\mathbb{Z}/p)^{\mathbb{Z}/K}$, as functions of $x \in X_K$, form an orthogonal basis of $\mathbb{C}^{X_K}$, and $(V\Psi)(x, k) = \Psi(\tau^{-k}x, k)$ is well defined since τ^K is the identity. So left multiplication by a_K is conjugate to the direct sum of the operators $\theta_x^{(K)}(a)$. $\square$

3 The counter template

3.1 The operator of a partition

Fix $p \geq 2$ and $r \geq 1$. The elements π_i and $\pi'_i = \rho_i$ are complementary idempotents ($e_i \rho_i = \rho_i$, hence $\rho_i^2 = \rho_i$), and all of them commute. For $b \in \{0, 1\}^r$ let

$$\varepsilon_b = \prod_{i=0}^{r-1} \begin{cases} \pi_i & \text{if } b_i = 1, \\ \pi'_i & \text{if } b_i = 0; \end{cases}$$

the ε_b are pairwise orthogonal idempotents with $\sum_b \varepsilon_b = 1$. Let $\{0, 1\}^r = M \sqcup U \sqcup D \sqcup I$ be a partition with $M \neq \emptyset$; the patterns in M are *markers*, those in U, D, I the *letters* U, D, I . Put

$$\chi = \sum_{b \in M} \varepsilon_b, \quad \alpha = 1 + \sum_{b \in U} \varepsilon_b, \quad \beta = 1 + \sum_{b \in D} \varepsilon_b, \quad T = T[\chi, \alpha, \beta; r], \quad (5)$$

with $T[\cdot]$ as in (1). For $b \in \{0, 1\}^r$ let $|b| = \sum_i b_i$ and $q(b) = (p-1)^{|b|}/p^r$, so that $\sum_b q(b) = 1$, and for $X \in \{M, U, D, I\}$ let $q_X = \sum_{b \in X} q(b)$.

Lemma 3.1 (the entries). *The elements χ , $\alpha - 1$ and $\beta - 1$ are pairwise orthogonal idempotents, $\chi\alpha = \chi\beta = \chi$, and*

$$T_{11} = (2\chi - 1) - (\alpha - 1) + (1 - \chi)t^r, \quad T_{12} = -\chi, \quad T_{21} = \chi t^r, \quad T_{22} = (1 - 2\chi)t^r - (1 - \chi) - (\beta - 1).$$

Only t^0 and t^r occur, and $p^r T \in M_2(\mathbb{Z}G_p)$.

Proof. The three elements are sums of the ε_b over disjoint sets. So $\chi\alpha = \chi + \chi(\alpha - 1) = \chi$, and $(1 - \chi)\alpha = (1 - \chi) + (\alpha - 1)$ since $\chi(\alpha - 1) = 0$; likewise for β . Inserting this into (1) gives the four entries. Every ρ_i lies in $\frac{1}{p}\mathbb{Z}G_p$, so every ε_b lies in $p^{-r}\mathbb{Z}G_p$. $\square$

3.2 Classes, symbols and rows

Fix $x \in X$. Since only t^0 and t^r occur in T , the operator $\theta_x(T)$ of Lemma 2.4 maps the functions supported on a residue class of $\mathbb{Z}$ modulo r to functions supported on the same class. For $c \in \{0, \dots, r-1\}$ we call the index $k = -rm - c$ the *position* $m \in \mathbb{Z}$ of the class c , and, the class being fixed, we write $f_u(m) = f_u(-rm - c)$ and $f_v(m) = f_v(-rm - c)$ for $f = (f_u, f_v) \in \ell^2(\mathbb{Z})^2$. Let $\nu(y) = [y \neq 0]$ for $y \in \mathbb{Z}/p$. The *symbol* at the position m of the class c is

$$\sigma_m^c(x) = (\nu(x_{rm+c}), \nu(x_{rm+c+1}), \dots, \nu(x_{rm+c+r-1})) \in \{0, 1\}^r.$$

By Lemma 2.4, at the index $k = -rm - c$ the element π_i acts by $\nu(x_{i-k}) = \nu(x_{rm+c+i})$ and π'_i by $1 - \nu(x_{rm+c+i})$, so ε_b acts at the position m by $[\sigma_m^c(x) = b]$; and $(\theta_x(t^r)f)(-rm - c) = f(-r(m+1) - c)$ is the value at the position $m+1$. Hence at the position m the element χ acts by 1 if $\sigma_m^c(x)$ is a marker and by 0 otherwise, α by 2 if $\sigma_m^c(x) \in U$ and by 1 otherwise, and β by 2 if $\sigma_m^c(x) \in D$ and by 1 otherwise.

Lemma 3.2 (rows). *In the class c , the two coordinates of $\theta_x(T)f$ at the position m are*

$$\begin{aligned} f_u(m) - f_v(m) \quad \text{and} \quad f_u(m+1) - f_v(m+1) & \quad \text{if } \sigma_m^c(x) \text{ is a marker,} \\ f_u(m+1) - a f_u(m) \quad \text{and} \quad f_v(m+1) - b f_v(m) & \quad \text{if } \sigma_m^c(x) \text{ is a letter,} \end{aligned}$$

with $(a, b) = (2, 1), (1, 2), (1, 1)$ for the letters U, D, I .

Proof. As in [Fer26a, Lemma 5.4]. By (1) and the multiplicativity of θ_x , the first coordinate is $(\chi - (1 - \chi)\alpha)f_u + (1 - \chi)t^r f_u - \chi f_v$ and the second is $\chi t^r f_u + (1 - \chi)t^r f_v - \chi t^r f_v - (1 - \chi)\beta f_v$, where χ, α, β act by multiplication at the position m and t^r brings the value from the position $m+1$. Insert $\chi = 1$ at a marker, and $\chi = 0, \alpha = a, \beta = b$ at a letter. $\square$

3.3 Counter matrices and segments

Definition 3.3 (counter matrices; [Fer26a, Definition 3.2]). For $N \geq 0$ and a word $s = s_0 \cdots s_{N-1}$ over $\{U, D, I\}$, the *counter matrix* M_s is the $(2N+2) \times (2N+2)$ matrix which, applied to $z = (u_0, v_0, u_1, v_1, \dots, u_N, v_N)^T$, has the rows

$$u_0 - v_0, \quad u_{i+1} - a_i u_i, \quad v_{i+1} - b_i v_i \quad (0 \leq i \leq N-1), \quad u_N - v_N,$$

in this order, where $(a_i, b_i) = (2, 1), (1, 2), (1, 1)$ for $s_i = U, D, I$; for example $M_\emptyset = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

Lemma 3.4 (kernel; [Fer26a, Lemma 3.5]). $\dim \ker M_s = 1$ if $\#U(s) = \#D(s)$, and $\dim \ker M_s = 0$ otherwise, where $\#U(s)$ and $\#D(s)$ are the numbers of letters U and D in s .

Proof. The first $2N + 1$ equations of $M_s z = 0$ hold if and only if $u_i = A_i u_0$ and $v_i = B_i u_0$ for all i , where $A_i = a_0 \cdots a_{i-1}$ and $B_i = b_0 \cdots b_{i-1}$; the last one then reads $(A_N - B_N)u_0 = 0$, and $A_N = 2^{\#U(s)}$, $B_N = 2^{\#D(s)}$. $\square$

Lemma 3.5 (segments). *Let $m < m'$ be positions of markers of the class c with only letters in between, forming the word $s \in \{U, D, I\}^N$, $N = m' - m - 1$. The rows “second coordinate at m ”, “both coordinates at $m + 1, \dots, m' - 1$ ” and “first coordinate at m' ” of the class c involve only the columns (j, u) , (j, v) with $m + 1 \leq j \leq m'$, and no other row involves these columns. In the order of the rows just listed and of the columns $(m + 1, u)$, $(m + 1, v), \dots, (m', u)$, (m', v) this block is M_s . If x has markers of the class c at infinitely many negative and at infinitely many positive positions, then the part of $\theta_x(T)$ on the class c is the direct sum of these finite blocks, and the orthogonal projection onto its kernel is the direct sum of the projections $P_{\ker M_s}$.*

Proof. As in [Fer26a, Lemma 5.5], with Lemma 3.2: the second coordinate at the marker m involves only the position $m + 1$, the coordinates at a letter j involve j and $j + 1$, and the first coordinate at m' involves only m' ; so a position $j \in [m + 1, m']$ occurs only in rows at j and $j - 1$, all of them listed. With $u_i = f_u(m + 1 + i)$, $v_i = f_v(m + 1 + i)$ the listed rows are those of M_s . Under the hypothesis on x every row and every position of the class belong to exactly one segment (the first coordinate at a marker to the segment ending there, the second to the one starting there), so the class part is block diagonal with these finite blocks, and its kernel is the ℓ^2 -direct sum of the block kernels. $\square$

3.4 The value

Lemma 3.6 (weighted count). *For $N \geq 0$ let*

$$\omega_N = \sum_{\substack{s \in \{U, D, I\}^N \\ \#U(s) = \#D(s)}} q_U^{\#U(s)} q_D^{\#D(s)} q_I^{\#I(s)} = \sum_{k \geq 0} \binom{N}{2k} \binom{2k}{k} (q_U q_D)^k q_I^{N-2k}.$$

Then $0 \leq \omega_N \leq (1 - q_M)^N$, and

$$\sum_{N \geq 0} \omega_N = ((1 - q_I)^2 - 4q_U q_D)^{-1/2}, \quad (1 - q_I)^2 - 4q_U q_D = q_M^2 + 2q_M(q_U + q_D) + (q_U - q_D)^2 \geq q_M^2.$$

Proof. For the second expression of ω_N choose the $2k$ places of the letters U and D , and then the k places of U among them. Summing over all words instead of the balanced ones gives $\omega_N \leq (q_U + q_D + q_I)^N = (1 - q_M)^N$. The identity for $(1 - q_I)^2 - 4q_U q_D$ follows from $1 - q_I = q_M + q_U + q_D$; since $q_M > 0$, it shows that $4q_U q_D < (1 - q_I)^2$ and $q_I < 1$. All terms being non-negative, we may sum in any order: $\sum_{N \geq 2k} \binom{N}{2k} q_I^{N-2k} = (1 - q_I)^{-2k-1}$, and with the binomial series $\sum_k \binom{2k}{k} z^k = (1 - 4z)^{-1/2}$ ($0 \leq z < \frac{1}{4}$) at $z = q_U q_D / (1 - q_I)^2$,

$$\sum_{N \geq 0} \omega_N = \frac{1}{1 - q_I} \sum_{k \geq 0} \binom{2k}{k} \frac{(q_U q_D)^k}{(1 - q_I)^{2k}} = \frac{1}{1 - q_I} \left(1 - \frac{4q_U q_D}{(1 - q_I)^2}\right)^{-1/2} = ((1 - q_I)^2 - 4q_U q_D)^{-1/2}. \quad \square$$

Theorem 3.7 (the counter template). *Let $p \geq 2$, $r \geq 1$, and let T be the matrix (5) of a partition with $M \neq \emptyset$. Then $p^r T \in M_2(\mathbb{Z}G_p)$ and*

$$\dim_{\mathbb{V}N} \ker T = q_M^2 \sum_{N \geq 0} \omega_N = q_M^2 ((1 - q_I)^2 - 4q_U q_D)^{-1/2} \leq q_M.$$

First proof. Integrality is Lemma 3.1. Under μ the coordinates x_j are independent and uniform on $\mathbb{Z}/p$, so the bits $\nu(x_j)$ are independent and equal to 1 with probability $\frac{p-1}{p}$. The symbols $\sigma_m^c(x)$, $m \in \mathbb{Z}$, of a fixed class c involve the disjoint blocks $\{rm + c, \dots, rm + c + r - 1\}$ of coordinates, which partition $\mathbb{Z}$; so they are independent, and $\sigma_m^c(x) = b$ with probability $(\frac{p-1}{p})^{|b|}(\frac{1}{p})^{r-|b|} = q(b)$. Since $q_M > 0$, the set X_0 of the x that have, in each class, markers at infinitely many negative and at infinitely many positive positions has full measure.

For $x \in X_0$ let P_x be the direct sum over the r classes of the projections of Lemma 3.5; since $\theta_x(T)$ maps each class into itself and $\ell^2(\mathbb{Z})^2$ is the orthogonal sum of the class subspaces, P_x is the orthogonal projection onto $\ker \theta_x(T)$. Put $P_x = 0$ for $x \notin X_0$. A matrix coefficient $\langle P_x(\delta_k \otimes \varepsilon_i), \delta_{k'} \otimes \varepsilon_{i'} \rangle$ vanishes unless k and k' lie in the same segment of the same class, and then it is an entry of $P_{\ker M_s}$ that depends only on the word s of this segment and on the places of k and k' in it. So it is constant on the sets of a countable partition of X_0 into intersections of X_0 with cylinder sets, hence measurable, and Lemma 2.5 applies.

The index 0 is the position 0 of the class 0, so only this class enters (4). For $N \geq 0$, $s \in \{U, D, I\}^N$ and $1 \leq j \leq N + 1$ let $E_{N,s,j}$ be the set of the $x \in X_0$ that have, in the class 0, a marker at the position $-j$, the letters $s_0, \dots, s_{N-1}$ at the positions $-j + 1, \dots, -j + N$, and a marker at the position $-j + N + 1$: the position 0 is the j -th column position of a segment with word s . These sets are disjoint, since $-j$ is the last marker before the position 0 and $-j + N + 1$ the first marker at or after it, and their union is X_0 . Since $N + 2$ independent symbols are prescribed,

$$\mu(E_{N,s,j}) = q_M^2 q_U^{\#U(s)} q_D^{\#D(s)} q_I^{\#I(s)},$$

independently of j . On $E_{N,s,j}$ the position 0 carries the variables u_{j-1}, v_{j-1} of the block M_s , which are its coordinates number $2j - 1$ and $2j$ in the order $u_0, v_0, \dots, u_N, v_N$ of Definition 3.3; so the integrand of (4) equals $(P_{\ker M_s})_{2j-1, 2j-1} + (P_{\ker M_s})_{2j, 2j}$, and the sum over $j = 1, \dots, N + 1$ is $\text{Tr } P_{\ker M_s} = \dim \ker M_s$. The integrand is non-negative and constant on each $E_{N,s,j}$, so by Lemmas 2.5, 3.4 and 3.6

$$\dim_{\nu N} \ker T = \sum_{N \geq 0} \sum_{s \in \{U, D, I\}^N} q_M^2 q_U^{\#U(s)} q_D^{\#D(s)} q_I^{\#I(s)} \dim \ker M_s = q_M^2 \sum_{N \geq 0} \omega_N.$$

The closed form and the bound $\leq q_M^2 \cdot q_M^{-1} = q_M$ follow from Lemma 3.6. $\square$

3.5 Finite quotients and the second proof

Proposition 3.8 (exact formula). *In the situation of Theorem 3.7, for every $P \geq 1$,*

$$d_{rP}(T) = q_M^2 \sum_{N=0}^{P-2} \omega_N + q_M \omega_{P-1} + \frac{(q_D + q_I)^P + (q_U + q_I)^P}{P}. \quad (6)$$

Proof. Let $K = rP$. The classes of indices modulo r are defined on $\mathbb{Z}/K$, and $\theta_x^{(K)}(T)$ preserves each of them. On the class c we use the positions $m \in \mathbb{Z}/P$ (index $-rm - c$) with the symbols $\sigma_m = (\nu(x_{rm+c+i}))_{i < r}$, indices modulo K ; Lemma 3.2 holds verbatim with positions modulo P . The P blocks $\{rm + c, \dots, rm + c + r - 1\}$ partition $\mathbb{Z}/K$, and a bit 0 comes from one value of a coordinate, a bit 1 from $p - 1$ values; so exactly $\prod_m w(\sigma_m)$ points $x \in X_K$ have a given cyclic symbol sequence $\sigma \in (\{0, 1\}^r)^{\mathbb{Z}/P}$ in the class c , where $w(b) = (p - 1)^{|b|}$. Let $\kappa(\sigma)$ be the kernel dimension of the cyclic operator with the rows of Lemma 3.2. By Lemma 2.6,

$$\dim_{\mathbb{C}} \ker T_K = \sum_{x \in X_K} \sum_{c=0}^{r-1} \kappa(\sigma^c(x)) = r \sum_{\sigma} \left(\prod_m w(\sigma_m) \right) \kappa(\sigma).$$

Write $w_X = \sum_{b \in X} w(b) = p^r q_X$ and $\Omega_N = \sum_s \prod_i w(s_i) = p^{rN} \omega_N$, the sum over the balanced words s of length N .

If σ contains a marker, the argument of Lemma 3.5 applies on the cycle: for cyclically consecutive markers m, m' the rows and columns described there form the block M_s of the word s of the letters in between; when there is only one marker, its segment consists of the other $P - 1$ positions and starts and ends at this marker (for $P = 1$ the block is $M_\emptyset$). Rows and columns are partitioned by the segments, so $\kappa(\sigma)$ is the number of balanced segments, that is, of markers m whose segment, starting at m , is balanced. Fix m . A segment of $N \leq P - 2$ letters with word s prescribes the symbols at $m, \dots, m + N + 1$, with weight $w_M^2 \prod_i w(s_i)$, and leaves $P - N - 2$ symbols free, of total weight $(p^r)^{P-N-2}$; for $N = P - 1$ the marker at m is the only one, and the weight is $w_M \prod_i w(s_i)$. Summing over the balanced words and over the P choices of m gives $P(\sum_{N=0}^{P-2} w_M^2 \Omega_N p^{r(P-N-2)} + w_M \Omega_{P-1})$.

If σ contains no marker, the two tracks decouple into $f_u(m+1) = a_m f_u(m)$ and $f_v(m+1) = b_m f_v(m)$ around the cycle, which have a non-zero solution if and only if $\prod_m a_m = 1$, resp. $\prod_m b_m = 1$, that is, if no U , resp. no D , occurs. So $\kappa(\sigma) = [\text{no } U] + [\text{no } D]$, and these σ contribute $(w_D + w_I)^P + (w_U + w_I)^P$.

Dividing r times the sum of the two contributions by $|G_{p,K}| = rP p^{rP}$ gives (6). $\square$

We use Lück's approximation theorem in the following form. Let Γ be a countable group and $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \dots$ normal subgroups of finite index with $\bigcap_i \Gamma_i = \{1\}$. For $a \in M_m(\mathbb{Q}\Gamma)$ let $a[i]$ be its image in $M_m(\mathbb{Q}[\Gamma/\Gamma_i])$, and let r_a and $r_{a[i]}$ denote right multiplication on row vectors in $\ell^2(\Gamma)^m$ and $\mathbb{C}[\Gamma/\Gamma_i]^m$. If Γ is amenable, then

$$\dim_{\mathbb{N}} \ker r_a = \lim_{i \rightarrow \infty} \frac{\dim_{\mathbb{C}} \ker r_{a[i]}}{[\Gamma : \Gamma_i]}. \quad (7)$$

This is Lück's approximation theorem [Lüc94] (see [KKL25, Theorem 6.2] for finite CW-complexes); in this matrix form it is [KKL25, Conjecture 7.1(1)], which holds for such a chain whenever Γ and all Γ/Γ_i satisfy the Determinant Conjecture [KKL25, Remark 7.2], for instance when they are sofic, as amenable and finite groups are [KKL25, Section 5].

Second proof of Theorem 3.7. The kernels $N_i = \ker \varphi_{r2^i}$, $i \geq 0$, are normal subgroups of finite index in G_p . They are nested, since φ_{r2^i} factors through $\varphi_{r2^{i+1}}$, and their intersection is trivial: if gt^k lies in all of them, then $r2^i$ divides k for every i , so $k = 0$, and the lamps of g in every residue class modulo $r2^i$ add up to 0 in $\mathbb{Z}/p$; once $r2^i$ exceeds the diameter of the support of g , each class contains at most one of them, so $g = 0$. The group G_p is amenable (abelian-by-cyclic). By (7), applied to the square matrix $p^r T$, and by Lemma 2.1 on G_p and on the finite quotients, which lets us pass from right to left multiplication, $\dim_{\mathbb{N}} \ker T = \lim_i d_{r2^i}(T)$. In (6) with $P = 2^i$ the second term is at most $q_M(1 - q_M)^{P-1}$ and the third at most $2(1 - q_M)^P/P$, by Lemma 3.6; both tend to 0, and the first tends to $q_M^2 \sum_N \omega_N$. $\square$

Remark 3.9 (the error on the finite quotients). Let $v = q_M^2 \sum_N \omega_N$. Subtracting it from (6) gives

$$d_{rP}(T) - v = q_M(1 - q_M) \omega_{P-1} - q_M^2 \sum_{N \geq P} \omega_N + \frac{(q_D + q_I)^P + (q_U + q_I)^P}{P}.$$

The first term comes from the single-marker configurations (weight q_M instead of q_M^2), the second from the segments too long for the cycle; by Lemma 3.6 both lie in $[0, q_M(1 - q_M)^P]$, with opposite signs, so $|d_{rP}(T) - v| \leq q_M(1 - q_M)^P + ((q_D + q_I)^P + (q_U + q_I)^P)/P$.

4 Genuine witnesses for the values of Ara and Claramunt

Throughout this section $p = 2$, so that $q(b) = 2^{-r}$ for every pattern, and patterns $b = (b_0, \dots, b_{r-1})$ are written as words $b_0 b_1 \dots b_{r-1}$.

4.1 Five lamp bits: proof of Theorem A

Lemma 4.1 (the patterns of Theorem A). *The elements χ , α , β of Theorem A are those of (5) with $r = 5$ and*

$$\begin{aligned} M &= \{b : b_0 = b_1 = 1\}, \\ U &= \{00001, 01000, 01001, 10000, 10001\}, \\ D &= \{00111, 01110, 01111, 10110, 10111\}; \end{aligned}$$

the other 14 patterns form I . Hence $q_M = \frac{1}{4}$, $q_U = q_D = \frac{5}{32}$ and $q_I = \frac{7}{16}$.

Proof. Since $\pi_i + \pi'_i = 1$, a product of some of the factors π_i, π'_i is the sum of the ε_b over the patterns that agree with it in the bits it prescribes. So $\pi_0 \pi_1 = \sum_{b_0=b_1=1} \varepsilon_b$, and $1 - \pi_0 \pi_1 = \pi'_0 \pi'_1 + \pi'_0 \pi_1 + \pi_0 \pi'_1$ is the sum over the patterns with $(b_0, b_1) \neq (1, 1)$. Hence $(1 - \pi_0 \pi_1) \pi'_2 \pi'_3$ is the sum of ε_b over the six patterns with $(b_0, b_1) \neq (1, 1)$ and $b_2 = b_3 = 0$, among them 00000, while $\pi'_0 \pi'_1 \pi'_2 \pi'_3 \pi'_4 = \varepsilon_{00000}$; so $\alpha - 1 = \sum_{b \in U} \varepsilon_b$. Likewise $(1 - \pi_0 \pi_1) \pi_2 \pi_3$ is the sum over the six patterns with $(b_0, b_1) \neq (1, 1)$ and $b_2 = b_3 = 1$, among them 00110, and $\pi'_0 \pi'_1 \pi_2 \pi_3 \pi'_4 = \varepsilon_{00110}$; so $\beta - 1 = \sum_{b \in D} \varepsilon_b$. $\square$

Proof of Theorem A. By Lemma 4.1 and Theorem 3.7, $32T \in M_2(\mathbb{Z}[\mathbb{Z}/2 \wr \mathbb{Z}])$ and $\dim_{\mathbb{N}} \ker T = \frac{1}{16} \sum_N \omega_N$, where $\omega_N = \sum_k \binom{N}{2k} \binom{2k}{k} \left(\frac{25}{1024}\right)^k \left(\frac{14}{32}\right)^{N-2k} = W_N/32^N$. Moreover $(1 - q_I)^2 - 4q_U q_D = \frac{81}{256} - \frac{100}{1024} = \frac{7}{32}$, so

$$\dim_{\mathbb{N}} \ker T = \frac{1}{16} \left(\frac{32}{7}\right)^{1/2} = \frac{\sqrt{2}}{4\sqrt{7}} = \frac{1}{4} \sqrt{\frac{2}{7}} = \frac{\sqrt{14}}{28}.$$

Finally $56(\sqrt{14}/28)^2 = 1$, and $56X^2 - 1$ has no rational root, so it is irreducible over $\mathbb{Q}$ and the value is irrational. $\square$

Equivalently, $\sum_N W_N y^N = ((1 - 4y)(1 - 24y))^{-1/2}$ for $0 \leq y < \frac{1}{24}$, and $W_0, W_1, W_2, \dots = 1, 14, 246, 4844, 100966, \dots$

For $p = 2$, $r = 5$ Proposition 3.8 reads

$$d_{5P}(T) = \frac{1}{16} \sum_{N=0}^{P-2} \frac{W_N}{32^N} + \frac{8W_{P-1}}{32^P} + \frac{2 \cdot 19^P}{P \cdot 32^P}. \quad (8)$$

For $P = 1$ the sum is empty, and $d_5(T) = \frac{8}{32} + \frac{2 \cdot 19}{32} = \frac{1}{4} + \frac{19}{16} = \frac{23}{16}$. Table 1 lists some values.

4.2 The family: proof of Theorem C

Lemma 4.2 (the patterns of Theorem C). *Let $n \geq 2$ and $r = n + 2$. The elements $\pi_0 \pi_1$, α_n , β_n of Theorem C are those of (5) with*

$$\begin{aligned} M &= \{b : b_0 = b_1 = 1\}, \\ U &= \{b : (b_0, b_1) \neq (1, 1), b_2 = b_3 = 0\} \setminus \{0^r\}, \\ D &= \{b : (b_0, b_1) \neq (1, 1), b_2 = b_3 = 1\} \setminus \{0011 0^{r-4}\}, \end{aligned}$$

and $|M| = 2^n$, $|U| = |D| = u_n = 3 \cdot 2^{n-2} - 1$, $|I| = i_n = 3 \cdot 2^{n-1} + 2$. For $n = 3$ these are the sets of Lemma 4.1, and $T^{(3)} = T$.

$K = 5P$	$d_K(T)$	decimal	$d_K(T) - \frac{1}{4}\sqrt{2/7}$
5	23/16	1.4375000	$1.3 \cdot 10^0$
10	537/1024	0.5244141	$3.9 \cdot 10^{-1}$
15	14227/49152	0.2894491	$1.6 \cdot 10^{-1}$
20	427729/2097152	0.2039571	$7.0 \cdot 10^{-2}$
60		0.1341394	$5.1 \cdot 10^{-4}$
100		0.1336430	$1.2 \cdot 10^{-5}$
200		0.1336306	$1.1 \cdot 10^{-8}$

Table 1: The exact values (8); $\frac{1}{4}\sqrt{2/7} = 0.1336306\dots$. The values for $K = 5, 10, 15$ were also obtained from the fibre operators of Lemma 2.6, the value for $K = 5$ also directly in the group ring, and the value for $K = 20$ by programs of the original computation and of both verification runs of Theorems A and C; the second independent verification run obtained the values for $K \leq 15$ again (Section 6).

Proof. As for Lemma 4.1: $(1 - \pi_0\pi_1)\pi'_2\pi'_3$ is the sum of the ε_b over the $3 \cdot 2^{r-4} = 3 \cdot 2^{n-2}$ patterns with $(b_0, b_1) \neq (1, 1)$ and $b_2 = b_3 = 0$, and $\prod_{j < r} \pi'_j = \varepsilon_{0^r}$ is one of them; likewise for β_n with the pattern 00110^r . Then $|I| = 2^r - 2^n - 2u_n = 3 \cdot 2^{n-1} + 2$. $\square$

Proof of Theorem C. By Lemma 4.2 and Theorem 3.7, $2^r T^{(n)}$ is integral, $q_M = \frac{1}{4}$, $q_U = q_D = u_n/2^r$, $q_I = i_n/2^r$, and $\dim_{\mathbb{N}} \ker T^{(n)} = \frac{1}{16} \sum_N \omega_N$ with $\omega_N = W_N^{(n)}/2^{rN}$. Since $i_n + 2u_n = 3 \cdot 2^n$ and $i_n - 2u_n = 4$,

$$(1 - q_I)^2 - 4q_U^2 = (1 - q_I - 2q_U)(1 - q_I + 2q_U) = \left(1 - \frac{3}{4}\right)(1 - 2^{-n}),$$

so $\dim_{\mathbb{N}} \ker T^{(n)} = \frac{1}{16} \cdot 2(1 - 2^{-n})^{-1/2} = \frac{1}{8} \sqrt{2^n/(2^n - 1)}$. The polynomial $64(2^n - 1)X^2 - 2^n$ has a rational root only if $2^n/(2^n - 1)$ is the square of a rational number; as 2^n and $2^n - 1$ are coprime, both would be squares, but $2^n - 1 \equiv 3 \pmod{4}$ for $n \geq 2$. So the polynomial is irreducible over $\mathbb{Q}$ and the value is irrational. $\square$

By Proposition 3.8, for instance $d_4, d_8, d_{12}, d_{16} = \frac{3}{2}, \frac{37}{64}, \frac{251}{768}, \frac{1889}{8192}$ for $n = 2$, and $d_6, d_{12} = \frac{45}{32}, \frac{2041}{4096}$ for $n = 4$.

Proof of Corollary B. The square integral matrices $32T$ and $2^r T^{(n)}$ have the kernels of T and $T^{(n)}$, so the numbers lie in $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ in the sense of [Gra10, Gra14, Gra16] and of [AC21, Definition 2.2]. Sums are block sums (Lemma 2.3(a)), even p is Lemma 2.3(b), and for $t \geq 3$ the number $\frac{1}{8}\sqrt{2^t/(2^t - 1)}$ is the value of Theorem C with $n = t$. $\square$

4.3 Remarks

Remark 4.3 (why five bits). For $p = 2$ the value in Theorem 3.7 depends only on the numbers $m = |M|$, $n_U = |U|$ and $n_D = |D|$, and it equals $\frac{1}{4}\sqrt{2/7}$ if and only if $4^r((m + n_U + n_D)^2 - 4n_U n_D) = 56m^4$. For $r = 5$ and $m = 8$ ($q_M = \frac{1}{4}$) this is $16s + d^2 = 160$ with $s = n_U + n_D$ and $d = n_U - n_D$; then d is divisible by 4, has the parity of s and satisfies $|d| \leq s$, which leaves only $d = 0$, $s = 10$, that is, $n_U = n_D = 5$. The program of Section 6 finds no solution with $r \leq 4$ (for any number of markers), only $(m, n_U, n_D) = (8, 5, 5)$ for $r = 5$, and $(16, 10, 10)$, $(16, 5, 13)$, $(16, 13, 5)$ for $r = 6$; there are three for $r = 7$. So five is the least number of lamp bits for the value $\frac{1}{4}\sqrt{2/7}$; with five bits the counts 8, 5, 5, 14 are forced, with more bits they are not. The patterns themselves can be chosen freely.

Remark 4.4 (virtual and genuine values). Ara and Claramunt know no group G with $\mathcal{C}(G, \mathbb{Q}) \neq \mathcal{G}(G, \mathbb{Q}) \cap \mathbb{R}_+$ [AC21, Remark 4.14]. Corollary B removes the listed family as candidates for such a difference for $G = \mathbb{Z}/2 \wr \mathbb{Z}$; it says nothing about the other elements of $\mathcal{G}(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Q})$ obtained in [AC21] (the numbers α_L of [AC21, Proposition 4.11] and those of [AC21, Theorem 1.1]), nor about the equality in general. Since the value in Theorem 3.7 is at most $q_M \leq 1$, a single matrix of the template has kernel dimension at most 1; the witness for $\sqrt{2}$ in [Fer26b] is a block sum of four copies.

5 Quadratic irrationals for every $\mathbb{Z}/p \wr \mathbb{Z}$

Let $p \geq 2$. With $r = 2$ the four patterns are 11, 01, 10, 00, and $\varepsilon_{11} = \pi_0\pi_1$, $\varepsilon_{01} = \pi'_0\pi_1$, $\varepsilon_{10} = \pi_0\pi'_1$. So T_p is the matrix (5) of the partition

$$M = \{11\}, \quad U = \{01\}, \quad D = \{10\}, \quad I = \{00\},$$

with $q_M = (\frac{p-1}{p})^2$, $q_U = q_D = \frac{p-1}{p^2}$ and $q_I = \frac{1}{p^2}$. The bit b_i of a symbol is $\nu(x_j) = [x_j \neq 0]$, the splitting of $\mathbb{Z}/p$ into $\{0\}$ and $\{1, \dots, p-1\}$ of [Gra16, §4].

Proof of Theorem D. With $a = \rho_0$ and $b = \rho_1$ we have $\chi = (1-a)(1-b)$, $\alpha - 1 = a(1-b)$ and $\beta - 1 = (1-a)b$, and Lemma 3.1 gives the entries displayed in Theorem D: for instance $T_{11} = 2(1-a)(1-b) - 1 - a(1-b) + (a+b-ab)t^2$ and $T_{22} = (1-2(1-a)(1-b))t^2 - (a+b-ab) - (b-ab)$. As $a, b \in \frac{1}{p}\mathbb{Z}G_p$, the matrix p^2T_p is integral. By Theorem 3.7, $\dim_{\mathbb{V}N} \ker T_p = (\frac{p-1}{p})^4 \sum_N \omega_N$, where $\omega_N = \sum_k \binom{N}{2k} \binom{2k}{k} \frac{(p-1)^{2k}}{p^{4k}} \frac{1}{p^{2(N-2k)}} = B_N/p^{2N}$, and

$$(1 - q_I)^2 - 4q_Uq_D = \left(1 - \frac{1}{p^2}\right)^2 - \frac{4(p-1)^2}{p^4} = \frac{(p-1)^2((p+1)^2 - 4)}{p^4} = \frac{(p-1)^3(p+3)}{p^4}.$$

Hence $\dim_{\mathbb{V}N} \ker T_p = \frac{(p-1)^4}{p^4} \cdot \frac{p^2}{(p-1)^{3/2}(p+3)^{1/2}} = (\frac{p-1}{p})^2 \sqrt{\frac{p-1}{p+3}} =: \delta_p$.

If δ_p were rational, then $\frac{p-1}{p+3} = (p^2\delta_p/(p-1)^2)^2$ would be the square of a rational number, hence so would be the integer $(p-1)(p+3) = (p+3)^2 \cdot \frac{p-1}{p+3}$, which would then be a perfect square. But $(p-1)(p+3) = (p+1)^2 - 4$ and $p^2 < (p+1)^2 - 4 < (p+1)^2$ for $p \geq 2$. So δ_p is irrational; since $\delta_p^2 = (p-1)^5/(p^4(p+3))$ is rational, δ_p is algebraic of degree 2, a root of $p^4(p+3)X^2 - (p-1)^5$. $\square$

Proof of Corollary E. Theorem D and Lemma 2.3(a); the values are $\delta_3 = \frac{4}{9} \cdot \frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{27}$, $\delta_5 = \frac{16}{25} \cdot \frac{1}{\sqrt{2}} = \frac{8\sqrt{2}}{25}$ and $\delta_7 = \frac{36}{49} \sqrt{\frac{3}{5}} = \frac{36\sqrt{15}}{245}$. $\square$

5.1 Finite quotients

For $r = 2$ Proposition 3.8 reads

$$d_{2P}(T_p) = \sum_{N=0}^{P-2} \frac{(p-1)^4 B_N}{p^{2N+4}} + \frac{(p-1)^2 B_{P-1}}{p^{2P}} + \frac{2}{P p^P}, \quad (9)$$

since $q_D + q_I = q_U + q_I = \frac{1}{p}$. For $p = 2$ this is the formula of [Fer26a, Proposition 6.2]. For $p = 3$ it gives $d_2, d_4, d_6, d_8, d_{10}, d_{12} = \frac{10}{9}, \frac{29}{81}, \frac{214}{729}, \frac{3449}{13122}, \frac{76666}{295245}, \frac{136711}{531441}$, values that were also obtained from the fibre operators and, for $K \leq 4$, directly in the group ring (Section 6); $d_{12} - \delta_3 = 6.5 \cdot 10^{-4}$, $d_{40} - \delta_3 = 1.8 \cdot 10^{-8}$ and $d_{120} - \delta_3 = 2.3 \cdot 10^{-19}$. The program of Section 6 determines the sign of the error $d_{2P}(T_p) - \delta_p$ (Remark 3.9) for $P \leq 40$ and $p \in \{2, 3, 5, 7, 11\}$: it is positive for all these P when $p = 2$ or $p = 3$; for $p = 5$ it alternates for $4 \leq P \leq 12$ and is positive afterwards, for $p = 7$ it alternates for $4 \leq P \leq 24$, and for $p = 11$ it alternates for all $P \leq 40$. Other p were not examined.

5.2 Remarks

Remark 5.1 (composite p , subgroups). Nothing uses that p is prime: Lemma 2.4 needs only $\frac{1}{p} \sum_c \zeta^{cy} = [y = 0]$. For $q \mid p$, $e_j \mapsto e_j^{p/q}$, $t \mapsto t$ embeds $\mathbb{Z}/q \wr \mathbb{Z}$ into $\mathbb{Z}/p \wr \mathbb{Z}$, so also $\delta_q \in \mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ (Lemma 2.3(b)).

Remark 5.2 (the encoding matters). For $p = 2$ all four patterns have the probability $\frac{1}{4}$, and it does not matter which pattern is the marker. For odd p it does. With the marker 00, the letters $U = 01$ and $D = 10$ and the idle letter $I = 11$, Theorem 3.7 gives $q_M = \frac{1}{p^2}$, $q_I = (\frac{p-1}{p})^2$ and the value $1/(p^2 \sqrt{4p-3})$, which is rational when $4p-3$ is a square, for instance $\frac{1}{27}$ for $p = 3$ and $\frac{1}{245}$ for $p = 7$. This formula belongs to this choice of I : with the marker 00, $U = 11$, $D = 01$ and $I = 10$ the value for $p = 3$ is $1/(9\sqrt{17})$. The encoding of Theorem D gives $(p-1)(p+3) = (p+1)^2 - 4$ under the root, which is never a square.

6 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation. The computations test the definitions and the statements on finite instances; they do not test, and do not use, the statements about the infinite group in Section 3.4 or Lück's theorem. The reproducibility package contains the exact program of this note, the programs with which the results were first obtained (`part1_original/`, `part2_original/`), those of the four independent verification runs of the first versions and those of the second independent verification run of the final text (`independent_run_2/`); its `README.md` gives the command lines, the running times and what each program checks. The programs use exact arithmetic, or ranks modulo two large primes compared with exact ranks where both are computed. *The exact program of this note* (`check_note.py`; standard library, exact arithmetic over $\mathbb{Q}$ and $\mathbb{Q}(\zeta_p)$, about five minutes) checks, for T , $T^{(2)}$, $T^{(4)}$, T_3 and T_5 , the matrices built from their definitions, the pattern sets, Lemma 3.1 and the expansion in Theorem D; the kernels on $\mathbb{Q}[G_{p,K}]^2$ directly in the group ring for $K \leq 7, 7, 6, 4, 3$ and through the fibre operators of Lemma 2.6 for all x and $K \leq 15, 16, 12, 10, 6$, in agreement with each other and, for $K = rP$, with Proposition 3.8; Lemmas 3.2 and 3.5 on all configurations with $K = rP \leq 15, 16, 12, 10, 6$ and on random windows of the line for T and T_3 ; Proposition 3.8 for 24 random partitions; and Lemmas 3.4 and 3.6, the values, Remark 3.9, the signs in Section 5.1 and the facts used in Remarks 4.3 and 5.2.

Verification. The results were first obtained, with both proofs, together with the programs in `part1_original/` and `part2_original/`. Each part was then examined by two independent verification runs, all AI-assisted; each wrote its own programs from the definitions, and the two runs of a part did not see each other's reports. All four compared the definitions and the questions with the sources and re-derived both proofs. For Theorems A and C, run A computed the kernels on $\mathbb{Z}/2 \wr \mathbb{Z}/K$ without Fourier transform for $K \leq 8$ and $K = 10$, a fibre model for $K \leq 15$, (6) for $K \leq 40$ by enumerating class sequences, and Monte-Carlo estimates of $d_{200}(T)$ without the segments (0.13381 ± 0.00029 , against 0.133631); run B computed the kernels in the group ring for $K \leq 8$ and through an eigenspace decomposition of its own for $K \leq 16$ and $K = 20$, and the family for $n = 2, 4, 5$. For Theorem D, run A computed the kernels in the group ring for $p = 2, 3, 5, 7$, a Fourier model up to $K = 16$, and Monte-Carlo estimates of the integrand of (4) on the line (0.25657 ± 0.00140 for $p = 3$, against 0.256600), and it found the reading of Remark 2.2; run B computed the kernels for $p = 2, \dots, 7$ in the group ring and in $\mathbb{Q}(\zeta_p)$ through the fibre operators, for $p = 3$ up to $K = 12$, and tested five altered operators. Both runs B ran the original programs again (identical outputs up to running times), searched the literature and checked the bibliographic data. No run found an error in a theorem, proposition or lemma. The corrections they required are made here: five lamp bits are the least number, not a forced one (Remark 4.3); the statement about Ara and Claramunt is

restricted to the listed family (Section 1.1 and Remark 4.4); the value formula and the bound $\leq q_M$ are proved (Lemma 3.6); the question of [Gra14] is quoted with its location and the following sentence; Remark 4.14, Problem 4 and Lück’s theorem have precise references; Remarks 2.2, 5.1 and 5.2 were added; and the sign change in Section 5.1 is stated for the examined p only. When the note was written, the three constructions were presented as instances of Theorem 3.7, whose proofs are the common proofs of the instances written once for all p , r and partitions (run A of Theorem A had re-derived the value formula for general probabilities, run A of Theorem D that for other encodings), and `check_note.py` was written; the 75 runs of the package with a recorded output were repeated from an extracted copy of the package, and 70 outputs agree with the recorded ones up to running times, the other five in every exact value (`RERUN_LOG.txt`).

The final text was then examined by a second independent verification run, also AI-assisted, which had not seen the note before; it was the first to examine Theorem 3.7. It re-derived both proofs of Theorem 3.7, Proposition 3.8 and Remark 3.9 line by line, then all other proofs; read [Gra10, Gra14, Gra16, AC21, KKL25] again in the places listed below, and [GS21, AC22] for the numbering of the questions; checked the cited numbers of the companion notes and the bibliographic data; and repeated the literature search. Its own programs (`independent_run_2/`) build the operators from their definitions in the group ring and compute the kernels on $\mathbb{Z}/p \wr \mathbb{Z}/K$ without Fourier transform for $K \leq 7$ (T , $T^{(2)}$, $T^{(4)}$), $K \leq 5$ (T_3) and $K \leq 3$ (T_5), and through the fibre operators, after a numerical check of the conjugation in Lemma 2.6, for $K \leq 15, 16, 12$ (T , $T^{(2)}$, $T^{(4)}$), for $K \leq 10$ and $K = 12$ (T_3) and for $K \leq 6$ (T_5); they test the integrand of the first proof on windows of the line for all cells $E_{N,s,j}$ with $N \leq 4$ (T , $T^{(2)}$) or $N \leq 5$ (T_3 , T_5), and Proposition 3.8 on 24 random partitions with $p \in \{2, 3, 4, 5, 6\}$, on T_4 and T_6 , on the encodings of Remark 5.2 and on all class sequences with $P \leq 8$; and they recompute the numbers of Table 1, Sections 4.2 and 5.1 and Remarks 3.9, 4.3 and 5.2. All results agree with the note. The run also repeated all 76 runs of the package from an extracted copy of the archive: 14 outputs are identical byte for byte, 60 up to running times, and two differ only in the printed diagnostics described in `RERUN_LOG.txt`. It found no error in a theorem, proposition or lemma. It required five corrections, which were made: the attribution of Pontryagin duality in Section 1.3, this account, the searches in the next paragraph, the caption of Table 1 and the last sentence of Remark 2.2.

Scope and priority. The note answers the question of [AC21, Remark 4.14] for $\frac{1}{4}\sqrt{2/7}$, shows that every number $\frac{1}{8}\sqrt{2^t/(2^t-1)}$, $t \geq 3$, listed after [AC21, Theorem 4.13] is a genuine ℓ^2 -Betti number arising from $\mathbb{Z}/2 \wr \mathbb{Z}$ (Corollary B), and answers Grabowski’s question of [Gra14, §1c] for every $p \geq 2$ (Corollary E; for $p = 2$ the answer was given in [Fer26a]). It does not describe $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, the question of the record OWR-4428-012, which remains partially solved, and it does not treat the other elements of $\mathcal{G}(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Q})$ found in [AC21], algebraic numbers of odd degree greater than 1, or single self-adjoint elements [GS21]. The questions, the definitions and the frameworks are Grabowski’s [Gra10, Gra14, Gra16] and Ara and Claramunt’s [AC21]; the balanced-word counter is the mechanism by which Ara and Claramunt obtained the listed family as virtual values [AC21, Theorem 4.13]; the matrix form, the Fourier picture and the segments are those of our companion notes [Fer26a, Fer26b]; Lück’s approximation theorem [Lüc94] and Pontryagin duality are standard. Only Theorems A and C, which give genuine witnesses for the listed family, and Theorem D for odd p are, to our knowledge, new; no priority is claimed. What was read: [Gra10] (pp. 2366–2369); [Gra14] in the arXiv version v4 (Section 1, whose numbering we use; the journal version was not compared); [Gra16] in both versions (pp. 1–3 and the beginning of Section 4; journal pp. 795–797); [AC21] in both versions (Sections 1, 2.2, 4.3; journal Section 4.3); [AC22, GS21, KKL25] in part; our companion notes. [Lüc94] was not read; what we say about [LW13, Aus13] is taken from [Gra10, Gra14]. The second independent verification run read these sources again in the same places. The searches were made on 15 October

2026, when the results were first obtained, by the four verification runs, when the note was written and again by the second independent verification run: the arXiv API (keyword and author searches, and the most recent titles containing “lamplighter”, up to 6 October 2026), OpenAlex (the works citing [Gra14, Gra16, AC21, AC22, LW13, GS21]), Semantic Scholar (partly refused for the rate limit), Crossref, zbMATH Open, and thirteen web searches. In these sources we found no answer to [AC21, Remark 4.14], and no earlier non-virtual irrational algebraic ℓ^2 -Betti number of a lamplighter group $\mathbb{Z}/p \wr \mathbb{Z}$ other than those of our companion notes [Fer26a, Fer26b]; known there are transcendental and other irrational numbers [Gra16, AC21], irrational algebraic numbers for the free lamplighters $\mathbb{Z}/p \wr F_d$ [LW13], and irrational algebraic virtual ℓ^2 -Betti numbers of $\mathbb{Z}/2 \wr \mathbb{Z}$ [AC21, AC22]. A search that finds nothing is not a proof of novelty; theses, talks and unindexed preprints are not covered.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AC21] Pere Ara and Joan Claramunt. L^2 -Betti numbers arising from the lamplighter group. *Journal of Algebraic Combinatorics*, 54(4):1201–1245, 2021. [arXiv:2005.12814](#), [doi:10.1007/s10801-021-01044-8](#).
- [AC22] Pere Ara and Joan Claramunt. Approximating the group algebra of the lamplighter by infinite matrix products. *Forum Mathematicum*, 34(4):851–891, 2022. [arXiv:2005.12374](#), [doi:10.1515/forum-2020-0141](#).
- [Aus13] Tim Austin. Rational group ring elements with kernels having irrational dimension. *Proceedings of the London Mathematical Society*, 107(6):1424–1448, 2013. [arXiv:0909.2360](#), [doi:10.1112/plms/pdt029](#).
- [Fer26a] Alper Ferudun. Irrational algebraic values of Grabowski’s automaton series and an irrational algebraic ℓ^2 -Betti number of the lamplighter group. Zenodo, unrefereed preprint, version 1.0, 2026. [doi:10.5281/zenodo.23377729](#).
- [Fer26b] Alper Ferudun. The square root of two is an ℓ^2 -Betti number arising from the lamplighter group. Zenodo, unrefereed preprint, version 1.0, 2026. [doi:10.5281/zenodo.23380620](#).
- [Gra10] Łukasz Grabowski. On the Atiyah problem for the lamplighter groups. *Oberwolfach Reports*, 7(3):2366–2369, 2010. Abstract in Report 40/2010, pp. 2335–2391. [doi:10.4171/OWR/2010/40](#).
- [Gra14] Łukasz Grabowski. On Turing dynamical systems and the Atiyah problem. *Inventiones Mathematicae*, 198(1):27–69, 2014. [arXiv:1004.2030](#), [doi:10.1007/s00222-013-0497-5](#).
- [Gra16] Łukasz Grabowski. Irrational l^2 invariants arising from the lamplighter group. *Groups, Geometry, and Dynamics*, 10(2):795–817, 2016. [arXiv:1009.0229](#), [doi:10.4171/GGD/366](#).
- [GS21] Rostislav Grigorchuk and Brian Simanek. Spectra of Cayley graphs of the lamplighter group and random Schrödinger operators. *Transactions of the American Mathematical Society*, 374(4):2421–2445, 2021. [arXiv:1902.10129](#), [doi:10.1090/tran/8156](#).

- [KKL25] Dominik Kirstein, Christian Kremer, and Wolfgang Lück. Some problems and conjectures about L^2 -invariants. In *Geometry and Topology of Aspherical Manifolds*, volume 816 of *Contemporary Mathematics*, pages 3–43. American Mathematical Society, Providence, RI, 2025. `arXiv:2311.17830`, `doi:10.1090/conm/816/16349`.
- [Lüc94] Wolfgang Lück. Approximating L^2 -invariants by their finite-dimensional analogues. *Geometric and Functional Analysis*, 4(4):455–481, 1994. `doi:10.1007/BF01896404`.
- [LW13] Franz Lehner and Stephan Wagner. Free lamplighter groups and a question of Atiyah. *American Journal of Mathematics*, 135(3):835–849, 2013. `arXiv:1005.2347`, `doi:10.1353/ajm.2013.0029`.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record OWR-4428-012.