

Irrational Algebraic Values of Grabowski's Automaton Series and an Irrational Algebraic ℓ^2 -Betti Number of the Lamplighter Group

Alper Ferudun*

October 14, 2026

Abstract

In an Oberwolfach report of 2010, Ł. Grabowski asked three questions about the set $\mathcal{C}(G)$ of ℓ^2 -Betti numbers arising from a group G , that is, of von Neumann dimensions of kernels of square matrices over $\mathbb{Q}G$. Question 4 asks whether $\mathcal{C}(G) \not\subseteq \mathbb{Q}$ holds exactly when G contains a wreath product $\mathbb{Z}/p \wr K$ with K infinite and finitely generated; Question 5 asks what $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ is; Question 6 asks which values the series $F_A(x) = \sum_i \dim \ker(M_i) x^{d(M_i)}$, summed over the band matrices M_i recognized by a finite automaton A whose letters are integer $n \times n$ matrices, takes at rational points, and in particular whether such a value can be irrational and algebraic. We answer the last part of Question 6: an explicit automaton A_2 with $n = 2$ and entries in $\{-2, \dots, 2\}$ recognizes tridiagonal matrices that encode a two-track multiplicative counter, $F_{A_2}(x) = x^2(1 - 4x^4)^{-1/2}$, and $F_{A_2}(\frac{1}{2}) = \sqrt{3}/6$. Every non-negative rational number is a value, and the values obtained from regular languages with one linear letter-count constraint are algebraic by Furstenberg's theorem on diagonals; the set of values is not determined. Transplanting the counter into the group ring, we construct an explicit $T \in M_2(\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}])$ with $4T$ integral and $\dim_{\mathbb{N}} \ker T = \sqrt{5}/20$, with two proofs: by Pontryagin duality, and by Lück's approximation theorem with an exact formula on the finite quotients $\mathbb{Z}/2 \wr \mathbb{Z}/K$. This answers, for $p = 2$, Grabowski's question (Invent. Math. 2014) whether $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers; it does not decide whether $\sqrt{2} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ (Groups Geom. Dyn. 2016, Problem 4). The questions and the automaton framework are Grabowski's. Ara and Claramunt obtained irrational algebraic *virtual* ℓ^2 -Betti numbers of $\mathbb{Z}/2 \wr \mathbb{Z}$, that is, differences of two ℓ^2 -Betti numbers, from a closely related letter-count language. In the sources we searched we found no earlier non-virtual irrational algebraic ℓ^2 -Betti number of a group $\mathbb{Z}/p \wr \mathbb{Z}$; a search that finds nothing is not a proof of novelty. Question 4, a description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, and the set of values of the series F_A remain open. This is an unrefereed note.

1 Introduction

1.1 The questions, and the status of the two corpus records

Let G be a countable discrete group. Following Grabowski [Gra10, p. 2366], a non-negative real number r is an ℓ^2 -Betti number arising from G if there is a square matrix $\theta \in M_m(\mathbb{Q}G)$ such that the von Neumann dimension of the kernel of θ , acting on $\ell^2(G)^m$, equals r ; the set of these numbers is denoted by $\mathcal{C}(G)$. (The conventions are recalled in Section 5.1.) Grabowski's abstract [Gra10] in the report of the Oberwolfach workshop of September 2010 recalls that every non-negative real number is an ℓ^2 -Betti number arising from some finitely generated group (shown independently in [Gra14] and [PSŻ15]), that $\mathcal{C}((\mathbb{Z}/2 \wr \mathbb{Z})^3) \not\subseteq \mathbb{Q}$ [Gra14], and that Lehner and Wagner found irrational algebraic numbers in $\mathcal{C}(\mathbb{Z}/p \wr F_d)$ for free groups F_d with $d \geq 2$ and $p \geq 2d - 1$ [LW13]; it records that $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

contains transcendental numbers for every $p \geq 2$ (Theorem 2 there; the proof is in [Gra16]), and asks the following questions [Gra10, pp. 2367–2368].

Q4. Is $\mathcal{C}(G) \not\subseteq \mathbb{Q}$ equivalent to $\mathbb{Z}/p \wr K \subseteq G$ for some p and some infinite finitely generated group K ?

Q5. “What is $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$?”

Q6. For a finite automaton A whose letters are $n \times n$ integer matrices with entries in $\{-C, \dots, C\}$, let $M_1, M_2, \dots$ be the matrices recognized by A (Definition 2.1) and $F_A(x) = \sum_{i \geq 1} \dim \ker M_i \cdot x^{d(M_i)}$, where $d(M)$ is the size of M . What are the possible values of F_A at rational numbers? In particular, can $F_A(q)$ be an irrational algebraic number for some automaton A and some rational q ?

Question 5 is motivated in [Gra10, p. 2368] by the observation that the families of finite graphs which occur in Grabowski’s transcendental example “can be encoded by a regular language”; Question 6 is the concrete related problem posed there.

In the UnsolvedMath collection [ula26] (version 1.6.0) these questions form two records: **OWR-4428-011** bundles Questions 4, 5 and 6 (it writes “for some prime p ” in Question 4, which is equivalent, since $\mathbb{Z}/q \wr K \subseteq \mathbb{Z}/p \wr K$ for a prime q dividing p), and **OWR-4428-012** is Question 5 alone. Both are listed there as partially solved, and **both remain partially solved after this note**: we answer the “in particular” part of Question 6 (Theorem 1), give partial information on the values of F_A (Section 4), and exhibit one new element of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ (Theorem 2); we do not describe $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, and we do not address Question 4.

1.2 Results

Theorem 1. *There is a deterministic finite automaton A_2 over the alphabet of 2×2 integer matrices with entries in $\{-2, \dots, 2\}$ (described in Section 3), all of whose recognized matrices are tridiagonal, such that*

$$F_{A_2}(x) = \sum_{k \geq 0} \binom{2k}{k} x^{4k+2} = \frac{x^2}{\sqrt{1-4x^4}} \quad (|x| < 2^{-1/2}).$$

In particular $F_{A_2}(\frac{1}{2}) = \frac{\sqrt{3}}{6}$, a root of the irreducible polynomial $12t^2 - 1$; it is irrational and algebraic.

The recognized matrices encode a *two-track multiplicative counter*: the kernel equations of the matrix M_s attached to a word $s \in \{U, D\}^N$ are $u_0 = v_0$, $u_{i+1} = a_i u_i$, $v_{i+1} = b_i v_i$ and $u_N = v_N$, with $(a_i, b_i) = (2, 1)$ for the letter U and $(1, 2)$ for D , so that M_s has a non-zero kernel exactly when $2^{\#U(s)} = 2^{\#D(s)}$. The balanced words are counted by central binomial coefficients, whose generating function is a square root. Section 4 shows that every non-negative rational number is a value of some F_A and that a whole class of values is algebraic (Propositions 4.1 and 4.2 and Corollary 4.3).

The same counter can be built into the lamplighter group. Let $G = \mathbb{Z}/2 \wr \mathbb{Z} = (\bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/2) \rtimes \mathbb{Z}$, with lamp generators e_j and $te_j t^{-1} = e_{j+1}$.

Theorem 2. *Let $T \in M_2(\mathbb{Q}G)$ be the matrix (2) of Section 5.2. Then $4T \in M_2(\mathbb{Z}G)$, and*

$$\dim_{\mathbb{N}} \ker T = \sum_{N \geq 0} \frac{T_N}{4^{N+2}} = \frac{\sqrt{5}}{20}, \quad T_N = \sum_{k \geq 0} \binom{N}{2k} \binom{2k}{k},$$

where the T_N are the central trinomial coefficients. The number $\sqrt{5}/20$ is a root of the irreducible polynomial $80t^2 - 1$; in particular $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ contains an irrational algebraic number.

We give two proofs. The first (Section 5) computes $\dim_{\text{vN}} \ker T$ by Pontryagin duality: the fibre operators of T split into finite blocks, one for each stretch between two consecutive “markers” of a random configuration, and each block is a matrix M_s of the kind used for Theorem 1, now over the alphabet $\{U, D, I\}$. The second (Section 6) computes the normalized kernel dimensions on the finite quotients $\mathbb{Z}/2 \wr \mathbb{Z}/K$ exactly (Proposition 6.2) and applies Lück’s approximation theorem [Lüc94].

1.3 What is new, what is not, and what stays open

The questions, the definition of F_A and the idea that families of components of such operators are described by regular languages are Grabowski’s [Gra10]; see also [Gra16, Remark 14(ii)]. The method of the first proof of Theorem 2, which turns the group ring into operators over a space of lamp configurations by Pontryagin duality and integrates the kernel dimensions of finite components, is the one used by Dicks–Schick, Austin and Grabowski, among others [DS02, Aus13, Gra14, Gra16] (we follow the description in [Gra10] and [Gra16, §§1, 3]). The counting idea, equal numbers of occurrences of two letters, is the one used by Ara and Claramunt [AC21, Theorem 4.13]: in their framework of rational series the language $\{w : |w|_{x_1} = |w|_{x_2}\}$ yields $\frac{1}{4}\sqrt{2/7}$, and other pairs of letters yield $\frac{1}{8}\sqrt{2^t/(2^t-1)}$ for $t \geq 3$, as elements of the subgroup of $\mathbb{R}$ generated by $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, that is, as differences of two ℓ^2 -Betti numbers. They call such numbers *virtual ℓ^2 -Betti numbers* [AC21, Remark 4.14], and their companion paper [AC22, Introduction] presents them as the first irrational algebraic virtual ℓ^2 -Betti numbers of the lamplighter group. Lück’s approximation theorem [Lüc94], Pontryagin duality and Furstenberg’s theorem on diagonals [Fur67] are standard tools and are used as such. Irrational algebraic ℓ^2 -Betti numbers arising from other groups were known: every non-negative number with computable binary expansion, in particular every non-negative algebraic number, arises from some finitely presented group [Gra14, Theorem 1.4], and every non-negative real number from some finitely generated group [Gra14, Theorem 1.3], [PSŽ15]. The question of [Gra14, §1c] concerns the groups $\mathbb{Z}/p \wr \mathbb{Z}$ themselves.

To our knowledge the following are new: Theorem 1, which answers the “in particular” part of Question 6, and Theorem 2, which shows that $\sqrt{5}/20 \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$. Theorem 2 answers, for $p = 2$, the question of Grabowski [Gra14, §1c] “whether $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers”. It does not answer the more concrete Problem 4 of [Gra16], whether $\sqrt{2} \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, and it is not a description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$. Both arguments are short once the counter is seen. The following remain open: Question 4; a description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$, in particular Problem 4 of [Gra16]; the set of values of the series F_A , in particular whether it contains transcendental numbers (Remark 4.5); and irrational algebraic elements of $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ for odd p (Section 7). The literature search behind the word “new” is described in the paragraph “Scope and priority” at the end of Section 8, which also describes the computations and the independent verification runs.

2 The automaton series

2.1 The definitions of the source

The source [Gra10, p. 2368] fixes integers n and C and an automaton whose letters are $n \times n$ integer matrices with entries in $\{-C, -C+1, \dots, C\}$. A matrix M of dimension at least n is recognized by A if its entries vanish outside “the n -neighbourhood of the diagonal” and A recognizes the word of “subsequent $n \times n$ minors of M along the diagonal”. With $M_1, M_2, \dots$ the sequence of all recognized matrices and $d(M)$ the dimension of M , the series is $F_A(x) = \sum_{i \geq 1} \dim \ker M_i \cdot x^{d(M_i)}$. We use these definitions in the following form; the choices are discussed in Section 2.2.

Definition 2.1 (statement card). Let $n \geq 1$ and $C \geq 0$ be integers, and let $\Sigma_{n,C}$ be the finite set of $n \times n$ integer matrices with entries in $\{-C, \dots, C\}$. An *automaton* is a finite automaton A with input alphabet $\Sigma_{n,C}$; $\mathcal{L}(A) \subseteq \Sigma_{n,C}^*$ is the language it accepts. For a $d \times d$ matrix M with $d \geq n$ and $1 \leq r \leq d - n + 1$, the *window* $W_r(M) = (M_{j,k})_{r \leq j, k \leq r+n-1}$ is the $n \times n$ submatrix with upper left corner (r, r) , and the *window word* of M is $w(M) = W_1(M)W_2(M) \cdots W_{d-n+1}(M)$. The matrix M is *recognized* by A if

- (i) $M_{j,k} = 0$ whenever $|j - k| \geq n$, and
- (ii) every window of M lies in $\Sigma_{n,C}$, and $w(M) \in \mathcal{L}(A)$.

The *series* of A is $F_A(x) = \sum_M \dim \ker M \cdot x^{d(M)}$, the sum over all recognized matrices M , where $d(M)$ is the size of M and $\dim \ker M$ is the dimension of the kernel of M over $\mathbb{Q}$ (equivalently over $\mathbb{R}$ or $\mathbb{C}$: an integer matrix has the same rank over every field of characteristic zero). For $q \in \mathbb{Q}$ the number $F_A(q)$ counts as a *value* of the series if the series converges at q .

The windows of a $d \times d$ matrix cover exactly the positions (j, k) with $|j - k| \leq n - 1$. By (i), a recognized matrix is therefore determined by its window word, and there are at most $|\Sigma_{n,C}|^{d-n+1}$ recognized matrices of size d . Hence F_A is a power series with non-negative integer coefficients $c_d \leq d |\Sigma_{n,C}|^{d-n+1}$, its radius of convergence is at least $1/|\Sigma_{n,C}|$, and the order of summation is irrelevant. Whether A is deterministic does not matter, since deterministic and non-deterministic finite automata accept the same languages. The question is: *which numbers are values $F_A(q)$, and can a value be irrational and algebraic?* Theorem 1 answers the second part; for the first part see Section 4.

2.2 Readings

Sliding windows. Since the letters are matrices, a “minor” is a submatrix. We read “subsequent” minors along the diagonal as the consecutive windows $W_1, W_2, \dots$, each shifted by one step along the diagonal. They exist for every size $d \geq n$, in accordance with the source’s “matrix M of dimension at least n ”.

The block-diagonal reading, and why we reject it. One could also read “subsequent minors” as the disjoint diagonal blocks $W_1, W_{n+1}, W_{2n+1}, \dots$, and the region allowed by (i) as the union of these blocks. Then every recognized matrix would be block diagonal, its size would be a multiple of n , and $\dim \ker M$ would be the sum of the kernel dimensions of the blocks, an additive weight $\kappa(w) = \sum_i \kappa(w_i)$ of the word. For a regular language $\mathcal{L}$ recognized by a deterministic automaton with initial vector e , final vector f and transition matrices E_ℓ , $E = \sum_\ell E_\ell$, one has

$$\sum_{w \in \mathcal{L}} \kappa(w) y^{|w|} = e^\top (I - yE)^{-1} \left(y \sum_\ell \kappa(\ell) E_\ell \right) (I - yE)^{-1} f, \quad (1)$$

a rational function with integer coefficients; $F_A(x)$ would be this function at $y = x^n$, so that under this reading every value $F_A(q)$ would be rational and the answer to the “in particular” part would be negative. We reject this reading: the source admits every dimension at least n , it speaks of a neighbourhood of the diagonal, which is a band, and under this reading the question would be settled by the two-line argument just given. (The variant with disjoint blocks but a band-shaped region (i) leaves the band entries between the blocks unconstrained by A ; then infinitely many matrices of one size can be recognized, and the series is in general not defined.)

The band. With sliding windows, the entries seen by the automaton are those with $|j - k| \leq n - 1$, and we read “the n -neighbourhood of the diagonal” as this set, the positions at distance less than n from the diagonal. If it meant $|j - k| \leq n$, the entries at distance exactly n would lie in no window,

every accepted window word of length at least 2 would belong to infinitely many integer matrices, and the coefficients of the series would in general be infinite. (If one bounded these invisible entries by C , the series would be defined but would not be determined by the window words; we do not treat this reading.) Theorem 1 is robust within the reading we use: its matrices are tridiagonal, and Remark 3.6 gives the same conclusion for every window size $n \geq 2$.

The case $n = 1$. For $n = 1$ the recognized matrices are diagonal, $\dim \ker M$ is the number of zero diagonal entries, again an additive weight, and (1) with $y = x$ shows that F_A is a rational function with integer coefficients. So irrational values require $n \geq 2$.

Exponent, field, values. The exponent of x is the size $d(M)$, not the length $d - n + 1$ of the window word (Remark 3.7 treats the other choice); kernels are taken over $\mathbb{Q}$; all values used below are taken at points inside the disc of convergence, where the series converges absolutely.

3 Proof of Theorem 1

3.1 The automaton A_2

Let $n = 2$ and $C = 2$, so that the alphabet $\Sigma = \Sigma_{2,2}$ consists of the $5^4 = 625$ integer 2×2 matrices with entries in $\{-2, \dots, 2\}$. For an integer k put

$$X_k = \begin{pmatrix} 1 & -1 \\ -k & 0 \end{pmatrix}, \quad P_k = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

The deterministic automaton A_2 has the states q_0 (initial), $q_A, q_B, q_E, q_{\text{acc}}$ (the only accepting state) and q_{dead} , and the transitions of Table 1; every other pair of a state and a letter, in particular every letter other than X_2, X_1, P_2, P_1, E, Z , and every letter read in q_{acc} or in q_{dead} , leads to q_{dead} .

state	X_2	X_1	P_2	P_1	E	Z
q_0	q_A	q_B	—	—	—	q_{acc}
q_A	—	—	—	q_E	—	—
q_B	—	—	q_E	—	—	—
q_E	—	—	q_A	q_B	q_{acc}	—

Table 1: The eight transitions of A_2 that do not lead to q_{dead} (a dash stands for q_{dead}).

Lemma 3.1 (the language). $\mathcal{L}(A_2)$ consists of the one-letter word Z and of the words

$$X_{a_0} P_{b_0} P_{a_1} P_{b_1} \cdots P_{a_{N-1}} P_{b_{N-1}} E \quad (N \geq 1, (a_i, b_i) \in \{(2, 1), (1, 2)\}).$$

Proof. Since q_{dead} is absorbing and not accepting, an accepted word uses only the transitions of Table 1. From q_0 the letter Z leads to q_{acc} , from which no such transition leaves; so Z is accepted, and Z occurs in no other accepted word. Otherwise the first letter is X_2 (to q_A) or X_1 (to q_B), and the next letter is forced: P_1 from q_A , P_2 from q_B . So the first two letters are $X_{a_0} P_{b_0}$ with $(a_0, b_0) \in \{(2, 1), (1, 2)\}$, and the automaton is then in q_E . From q_E either E ends the word in q_{acc} , or P_2 , resp. P_1 , followed by the forced P_1 , resp. P_2 , gives a further pair $P_{a_i} P_{b_i}$ with $(a_i, b_i) \in \{(2, 1), (1, 2)\}$ and returns to q_E . $\square$

3.2 Counter matrices

Definition 3.2. Let $N \geq 0$ and let $a_0, b_0, \dots, a_{N-1}, b_{N-1}$ be non-zero integers. The *counter matrix* $M = M((a_i, b_i)_{i < N})$ is the $(2N + 2) \times (2N + 2)$ matrix which, applied to the column vector $z =$

$(u_0, v_0, u_1, v_1, \dots, u_N, v_N)^\top$, has the rows

$$u_0 - v_0, \quad u_{i+1} - a_i u_i, \quad v_{i+1} - b_i v_i \quad (0 \leq i \leq N-1), \quad u_N - v_N,$$

in this order (row 1, rows $2i+2$ and $2i+3$, row $2N+2$). For a word $s = s_0 \cdots s_{N-1}$ over the alphabet $\{U, D, I\}$ we write M_s for the counter matrix with

$$(a_i, b_i) = (2, 1) \text{ if } s_i = U, \quad (1, 2) \text{ if } s_i = D, \quad (1, 1) \text{ if } s_i = I.$$

For example, $M_\emptyset = Z$ and

$$M_U = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix};$$

the letter I is used from Section 4 on.

Lemma 3.3 (windows). *Every counter matrix $M = M((a_i, b_i)_{i < N})$ is tridiagonal. Its window word is $w(M) = Z$ if $N = 0$, and*

$$w(M) = X_{a_0} P_{b_0} P_{a_1} P_{b_1} \cdots P_{a_{N-1}} P_{b_{N-1}} E \quad \text{if } N \geq 1.$$

Proof. For $N = 0$, $M = Z$. Let $N \geq 1$ and $d = 2N + 2$. By definition the non-zero entries of M are $M_{1,1} = 1$, $M_{1,2} = -1$; $M_{2i+2,2i+1} = -a_i$, $M_{2i+2,2i+3} = 1$, $M_{2i+3,2i+2} = -b_i$, $M_{2i+3,2i+4} = 1$ for $0 \leq i \leq N-1$; and $M_{d,d-1} = 1$, $M_{d,d} = -1$. All of them satisfy $|j-k| \leq 1$. Along the band: $M_{r,r} = 0$ for $2 \leq r \leq d-1$; $M_{r,r+1} = 1$ for $2 \leq r \leq d-1$; $M_{r+1,r}$ equals $-a_i$ for $r = 2i+1$, equals $-b_i$ for $r = 2i+2$ ($0 \leq i \leq N-1$), and equals 1 for $r = d-1$. The window $W_r = \begin{pmatrix} M_{r,r} & M_{r,r+1} \\ M_{r+1,r} & M_{r+1,r+1} \end{pmatrix}$ is therefore X_{a_0} for $r = 1$, P_{a_i} for $r = 2i+1$ with $1 \leq i \leq N-1$, P_{b_i} for $r = 2i+2$ with $0 \leq i \leq N-1$, and E for $r = d-1 = 2N+1$. $\square$

Lemma 3.4 (recognized matrices). *The matrices recognized by A_2 are exactly the matrices M_s with $s \in \{U, D\}^N$ and $N \geq 0$; the map $s \mapsto M_s$ is injective, and $d(M_s) = 2N + 2$.*

Proof. By Lemmas 3.1 and 3.3, each M_s with $s \in \{U, D\}^N$ is tridiagonal, its windows lie in $\Sigma_{2,2}$, and $w(M_s) \in \mathcal{L}(A_2)$; so M_s is recognized. The word $w(M_s)$ determines the pairs (a_i, b_i) , hence s . Conversely, let M be recognized. Then M is tridiagonal by (i), of size $d \geq 2$, and $w(M) \in \mathcal{L}(A_2)$. By Lemma 3.1 and Lemma 3.3, $w(M) = w(M_s)$ for some $s \in \{U, D\}^N$. The 2×2 windows of a tridiagonal matrix contain all its entries that may be non-zero, so two tridiagonal matrices with the same window word are equal, and $M = M_s$. $\square$

Lemma 3.5 (kernel). *Let M be a counter matrix, and put $A_i = a_0 a_1 \cdots a_{i-1}$ and $B_i = b_0 b_1 \cdots b_{i-1}$ ($A_0 = B_0 = 1$). Then $\dim \ker M = 1$ if $A_N = B_N$, and $\dim \ker M = 0$ otherwise; in the first case the kernel is spanned by $(A_0, B_0, A_1, B_1, \dots, A_N, B_N)$. In particular, for $s \in \{U, D, I\}^N$,*

$$\dim \ker M_s = \begin{cases} 1 & \text{if } \#U(s) = \#D(s), \\ 0 & \text{otherwise,} \end{cases}$$

where $\#U(s)$ and $\#D(s)$ are the numbers of letters U and D in s .

Proof. $Mz = 0$ means $u_0 = v_0$, $u_{i+1} = a_i u_i$ and $v_{i+1} = b_i v_i$ for $0 \leq i \leq N-1$, and $u_N = v_N$. The first $2N+1$ equations hold if and only if $u_i = A_i u_0$ and $v_i = B_i v_0$ for all i ; the last one then reads $(A_N - B_N)u_0 = 0$. If $A_N \neq B_N$, then $u_0 = 0$ and $z = 0$; if $A_N = B_N$, the solutions are the multiples of $(A_0, B_0, \dots, A_N, B_N)$, which is non-zero. For M_s one has $A_N = 2^{\#U(s)}$ and $B_N = 2^{\#D(s)}$. $\square$

3.3 Proof of Theorem 1

By Lemmas 3.4 and 3.5,

$$F_{A_2}(x) = \sum_{N \geq 0} \sum_{s \in \{U, D\}^N} \dim \ker M_s x^{2N+2} = \sum_{N \geq 0} \# \{s \in \{U, D\}^N : \#U(s) = \#D(s)\} x^{2N+2}.$$

The count is 0 for odd N and $\binom{2k}{k}$ for $N = 2k$, so $F_{A_2}(x) = \sum_{k \geq 0} \binom{2k}{k} x^{4k+2}$. Since $\binom{2k}{k} = (-4)^k \binom{-1/2}{k}$, the binomial series gives $\sum_{k \geq 0} \binom{2k}{k} y^k = (1 - 4y)^{-1/2}$ for $|y| < \frac{1}{4}$, and with $y = x^4$ we obtain $F_{A_2}(x) = x^2(1 - 4x^4)^{-1/2}$ for $|x| < 2^{-1/2}$, where the series converges absolutely. At $x = \frac{1}{2}$ we have $y = \frac{1}{16}$ and

$$F_{A_2}\left(\frac{1}{2}\right) = \frac{1}{4} \left(\frac{3}{4}\right)^{-1/2} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

This number satisfies $12t^2 = 1$. If it were rational, $3 = 36t^2$ would be the square of a rational number, which it is not. So $12t^2 - 1$ has no rational root, is irreducible over $\mathbb{Q}$, and $\sqrt{3}/6$ is irrational and algebraic. $\square$

3.4 Remarks

Remark 3.6 (other window sizes). For every $n \geq 2$ there is an automaton $A^{(n)}$ over $\Sigma_{n,2}$ whose recognized matrices are exactly the M_s with $s \in \{U, D\}^N$ and $2N + 2 \geq n$. Hence

$$F_{A^{(n)}}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{6} - \sum_{k \geq 0, 4k+2 < n} \binom{2k}{k} 2^{-4k-2},$$

again irrational and algebraic, so the answer does not depend on the window size. To see this, note that for a tridiagonal $d \times d$ matrix with $d \geq n$ the $n \times n$ window $W_r^{(n)}$ contains the 2×2 windows $W_r, \dots, W_{r+n-2}$, and that every entry of the band $|j - k| \leq n - 1$ lies in some $n \times n$ window. Let $A^{(n)}$ go to a dead state as soon as it reads a letter with a non-zero entry at distance at least 2 from the diagonal. Otherwise it simulates A_2 : on the first letter $W_1^{(n)}$ it feeds $W_1, \dots, W_{n-1}$ to A_2 , on each later letter $W_r^{(n)}$ the new window W_{r+n-2} , and it accepts when A_2 accepts. This is a finite automaton (the states of A_2 and a start state). A matrix M satisfying (i) for n is recognized by $A^{(n)}$ if and only if it is tridiagonal and $w(M) \in \mathcal{L}(A_2)$, that is (by Lemma 3.4) if and only if $M = M_s$ with $s \in \{U, D\}^N$ and $2N + 2 = d \geq n$.

Remark 3.7 (exponent, other points). If one used the length $2N + 1$ of the window word as the exponent instead of the size, the series would be $x^{-1}F_{A_2}(x)$, with value $\sqrt{3}/3$ at $\frac{1}{2}$, again irrational and algebraic. At other rational points $|q| < 2^{-1/2}$ one gets $F_{A_2}(q) = q^2(1 - 4q^4)^{-1/2}$, for instance $F_{A_2}(\frac{1}{3}) = 1/\sqrt{77}$.

4 Possible values of the series

Let V be the set of values $F_A(q)$, over all n , C , automata A and rational q at which the series converges. V is countable, and $F_A(q) \geq 0$ for $q \geq 0$. We do not determine V ; the following statements describe parts of it.

Proposition 4.1 (rational values). *Every non-negative rational number is a value: for integers $p \geq 0$ and $s \geq 1$ there is an automaton A over $\Sigma_{2,C}$ with $C = p(s + 1)$ such that $F_A\left(\frac{1}{s+1}\right) = \frac{p}{s}$.*

Proof. For $p = 0$ take an automaton that accepts nothing. Let $p \geq 1$, let A have an initial state, accepting states a_λ for $1 \leq \lambda \leq C$ and a dead state, with the transitions $\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$ from the initial state to a_λ and $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ from a_λ to itself, all others leading to the dead state. The window word of $\text{diag}(0, \lambda, \dots, \lambda)$ of size $d \geq 2$ is $\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^{d-2}$, so these matrices are recognized. Conversely, a recognized matrix is tridiagonal with such a window word; its off-diagonal entries vanish and its diagonal is $(0, \lambda, \dots, \lambda)$. Each of these matrices has a one-dimensional kernel, and there are C of them in every size $d \geq 2$. Hence $F_A(x) = \sum_{d \geq 2} Cx^d = Cx^2/(1-x)$ for $|x| < 1$, and $F_A(\frac{1}{s+1}) = \frac{p(s+1)}{(s+1)^2} \cdot \frac{s+1}{s} = \frac{p}{s}$. $\square$

The next statement generalizes the counter of Theorem 1. Let Λ be a finite alphabet, $\mathcal{L} \subseteq \Lambda^*$ a regular language and $c: \Lambda \rightarrow \mathbb{Z}$ a weight, extended additively to words, $c(s) = \sum_i c(s_i)$. Put

$$G_{\mathcal{L},c}(y) = \sum_{s \in \mathcal{L}, c(s)=0} y^{|s|} \in \mathbb{Z}[[y]].$$

Proposition 4.2 (one letter-count constraint). *Choose positive integers a_ℓ, b_ℓ ($\ell \in \Lambda$) with $a_\ell/b_\ell = 2^{c(\ell)}$ such that $\ell \mapsto (a_\ell, b_\ell)$ is injective; for instance $a_\ell = \lambda_\ell 2^{\max(c(\ell), 0)}$ and $b_\ell = \lambda_\ell 2^{\max(-c(\ell), 0)}$ with distinct odd $\lambda_\ell \geq 1$. For $s \in \Lambda^N$ let M_s^c be the counter matrix with $(a_i, b_i) = (a_{s_i}, b_{s_i})$, and let $C = \max_\ell \max(a_\ell, b_\ell)$. Then there is an automaton $A_{\mathcal{L},c}$ over $\Sigma_{2,C}$ whose recognized matrices are exactly the M_s^c with $s \in \mathcal{L}$, each once, and*

$$F_{A_{\mathcal{L},c}}(x) = x^2 G_{\mathcal{L},c}(x^2).$$

Proof. By Lemma 3.3, the window word of M_s^c is Z for s empty and $X_{a_{s_0}} P_{b_{s_0}} P_{a_{s_1}} P_{b_{s_1}} \cdots P_{b_{s_{N-1}}} E$ otherwise. Let $\mathcal{D}$ be a deterministic automaton accepting $\mathcal{L}$. The automaton $A_{\mathcal{L},c}$ reads the letters in pairs: the first pair must be $X_a P_b$ and every later pair $P_a P_b$ with $(a, b) = (a_\ell, b_\ell)$ for some $\ell \in \Lambda$; it remembers a between the two letters of a pair, decodes ℓ (by injectivity) and moves $\mathcal{D}$ by ℓ . After at least one pair, the letter E leads to acceptance if $\mathcal{D}$ is in an accepting state; the one-letter word Z is accepted if the empty word lies in $\mathcal{L}$; everything else leads to a dead state. Its states are triples (phase, state of $\mathcal{D}$, remembered a), so it is a finite automaton, and it accepts exactly the words $w(M_s^c)$ with $s \in \mathcal{L}$. As in the proof of Lemma 3.4, the recognized matrices are exactly these M_s^c , and s is determined by $w(M_s^c)$. All their entries lie in $\{-C, \dots, C\}$. By Lemma 3.5, $\dim \ker M_s^c = 1$ if $\prod_i a_{s_i} = \prod_i b_{s_i}$, that is, if $2^{c(s)} = 1$, and 0 otherwise. Summing $x^{2|s|+2}$ over the $s \in \mathcal{L}$ with $c(s) = 0$ gives the formula. $\square$

Corollary 4.3 (algebraic values). *In the situation of Proposition 4.2 the power series $G_{\mathcal{L},c}$ is algebraic over $\mathbb{Q}(y)$. Consequently $F_{A_{\mathcal{L},c}}(q)$ is an algebraic number for every rational q such that q^2 lies in the open disc of convergence of $G_{\mathcal{L},c}$.*

Proof. Let $\mathcal{D}$ have initial vector e , final vector f and transition matrices E_ℓ ($\ell \in \Lambda$). If $c \equiv 0$, then $G_{\mathcal{L},c}(y) = \sum_{s \in \mathcal{L}} y^{|s|} = e^\top (I - y \sum_\ell E_\ell)^{-1} f$ is rational, and its values at rational points are rational. Let $c \not\equiv 0$ and $K = \max_\ell |c(\ell)| \geq 1$. Put $\alpha(s) = \sum_i (K + c(s_i))$ and $\beta(s) = \sum_i (K - c(s_i))$, both non-negative, and

$$H(a, b) = \sum_{s \in \mathcal{L}} a^{\alpha(s)} b^{\beta(s)} = e^\top \left(I - \sum_\ell a^{K+c(\ell)} b^{K-c(\ell)} E_\ell \right)^{-1} f \in \mathbb{Z}[[a, b]],$$

a rational power series (every monomial $a^{K+c(\ell)} b^{K-c(\ell)}$ has total degree $2K \geq 2$). Since $\alpha(s) = \beta(s)$ if and only if $c(s) = 0$, and then $\alpha(s) = K|s|$, the diagonal $\sum_m h_{m,m} y^m$ of H equals $G_{\mathcal{L},c}(y^K)$. By Furstenberg's theorem [Fur67], the diagonal of a rational power series in two variables over $\mathbb{Q}$ is

algebraic over $\mathbb{Q}(y)$. So $G_{\mathcal{L},c}(y^K)$ is algebraic over $\mathbb{Q}(y)$, hence over its subfield $\mathbb{Q}(y^K)$, of which $\mathbb{Q}(y)$ is a finite extension; substituting y for y^K shows that $G_{\mathcal{L},c}(y)$ is algebraic over $\mathbb{Q}(y)$. Let $P \in \mathbb{Q}[y, w]$ be irreducible with $P(y, G_{\mathcal{L},c}(y)) = 0$; it has positive degree in w . For q^2 in the disc of convergence the identity $P(q^2, G_{\mathcal{L},c}(q^2)) = 0$ holds, and $P(q^2, w)$ is not the zero polynomial, since otherwise $y - q^2$ would divide P . So $G_{\mathcal{L},c}(q^2)$ is algebraic, and so is $F_{A_{\mathcal{L},c}}(q) = q^2 G_{\mathcal{L},c}(q^2)$. $\square$

We use Furstenberg's theorem as a citation and do not prove it; Theorems 1 and 2 do not depend on it.

Example 4.4 (central trinomial coefficients). Let $\Lambda = \{U, D, I\}$, $c(U) = 1$, $c(D) = -1$, $c(I) = 0$ and $\mathcal{L} = \Lambda^*$, with the injective choice $(a, b) = (2, 1), (1, 2), (1, 1)$, so that $C = 2$ and $M_s^c = M_s$ (Definition 3.2). Then $G_{\mathcal{L},c}(y) = \sum_N T_N y^N = (1 - 2y - 3y^2)^{-1/2}$ for $0 \leq y < \frac{1}{3}$, with the central trinomial coefficients T_N of Theorem 2 (see (6) below), and

$$F_{A_{\mathcal{L},c}}\left(\frac{1}{2}\right) = \frac{1}{4} \left(1 - \frac{1}{2} - \frac{3}{16}\right)^{-1/2} = \frac{1}{\sqrt{5}}.$$

Remark 4.5 (what is open). We do not know whether V contains transcendental numbers, and we do not know a description of V . In Grabowski's transcendental example the kernel dimension of a component depends on whether $l = 2^{k-1} - 1$ [Gra16, Lemma 18], a condition that is not a letter count; whether band matrices recognized by an automaton can realize such conditions, which would lead to lacunary series, is not settled here. A related question in the framework of rational series is [AC21, Question 4.15].

5 Proof of Theorem 2

5.1 Conventions

Let $\mathbf{A} = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}/2$, written additively as the group of finitely supported sequences $g = (g_j)_{j \in \mathbb{Z}}$ with $g_j \in \{0, 1\}$; e_j is the sequence with a single 1 at j . Let $G = \mathbb{Z}/2 \wr \mathbb{Z} = \mathbf{A} \rtimes \mathbb{Z}$, where the generator t of $\mathbb{Z}$ acts by $te_j t^{-1} = e_{j+1}$. Every element of G is uniquely of the form gt^k with $g \in \mathbf{A}$ and $k \in \mathbb{Z}$, and $(gt^k)(ht^l) = (g + \sigma^k h)t^{k+l}$, where $(\sigma h)_j = h_{j-1}$.

The group ring acts on $\ell^2(G)$ by left multiplication, $\lambda(g)\delta_h = \delta_{gh}$, and $M_m(\mathbb{Q}G)$ acts on column vectors in $\ell^2(G)^m$. Let $L(G)$ be the von Neumann algebra generated by $\lambda(G)$, with the trace $\tau(\theta) = \langle \theta \delta_e, \delta_e \rangle$, extended to $M_m(L(G))$ by $\tau_m(\theta) = \sum_i \tau(\theta_{ii})$. For $a \in M_m(\mathbb{Q}G)$ we put

$$\dim_{\text{vN}} \ker a = \tau_m(P_{\ker \lambda(a)}) \in [0, m],$$

where $P_{\ker \lambda(a)} \in M_m(L(G))$ is the orthogonal projection onto the kernel (see [Lüc02] for background). This is the dimension in the definition of $\mathcal{C}(G)$ [Gra10, p. 2366]. In [Gra14, §1] the kernel dimension of a matrix that is not self-adjoint is defined through a^*a ; this gives the same number, since $\ker a^*a = \ker a$. The paper [Gra16] uses matrices over $\mathbb{Z}G$, and $4T$ is such a matrix. With the normalized trace $\frac{1}{m}\tau_m$ the value in Theorem 2 would be $\sqrt{5}/40$.

Lemma 5.1 (left and right). *For a square matrix $a \in M_m(\mathbb{Q}G)$, the kernel of right multiplication $\eta \mapsto \eta a$ on row vectors in $\ell^2(G)^m$ has the same von Neumann dimension as $\ker \lambda(a)$. The same holds for a finite group in place of G , with $\dim_{\mathbb{C}}$ of kernels.*

Proof. Let $J\delta_g = \delta_{g^{-1}}$, a unitary with $J\delta_e = \delta_e$. For $b = \sum b_g g \in \mathbb{Q}G$ one has $J(\xi b) = \lambda(b^*)J\xi$, where $b^* = \sum b_g g^{-1}$, because $J\delta_{hg} = \delta_{g^{-1}h^{-1}}$. Applied componentwise, J conjugates right multiplication by a to $\lambda(a^*)$, where $(a^*)_{ji} = (a_{ij})^*$, and it carries the trace vector to itself. So the right kernel has

dimension $\tau_m(P_{\ker \lambda(a)^*})$. Since $\ker \lambda(a)^* = (\text{ran } \lambda(a))^\perp$ and, by the polar decomposition in $M_m(L(G))$, the projections onto $(\ker \lambda(a))^\perp$ and onto the closure of the range of $\lambda(a)$ are equivalent, hence have the same trace, $\tau_m(P_{\ker \lambda(a)^*}) = m - \tau_m(P_{\overline{\text{ran } \lambda(a)}}) = \tau_m(P_{\ker \lambda(a)})$. For a finite group the same argument is ordinary linear algebra. $\square$

So the number in Theorem 2 does not depend on the side on which T acts.

5.2 The operator

In $\mathbb{Q}G$ put $\pi_j = \frac{1}{2}(1 - e_j)$ and $\pi'_j = \frac{1}{2}(1 + e_j)$ (complementary idempotents, $\pi_j \pi'_j = 0$), and

$$\chi = \pi_0 \pi_1, \quad \alpha = 1 + \pi'_0 \pi_1, \quad \beta = 1 + \pi_0 \pi'_1, \quad S = t^2.$$

The matrix of Theorem 2 is

$$T = \begin{pmatrix} \chi - (1 - \chi)\alpha + (1 - \chi)S & -\chi \\ \chi S & (1 - \chi)S - \chi S - (1 - \chi)\beta \end{pmatrix} \in M_2(\mathbb{Q}G), \quad (2)$$

with products taken in $\mathbb{Q}G$. Since $\chi^2 = \chi$ and $\chi\alpha = \chi\beta = \chi$, its entries are

$$\begin{aligned} T_{11} &= -\frac{3}{4} - \frac{3}{4}e_0 - \frac{1}{4}e_1 + \frac{3}{4}e_0e_1 + \frac{1}{4}(3 + e_0 + e_1 - e_0e_1)t^2, \\ T_{12} &= -\frac{1}{4}(1 - e_0 - e_1 + e_0e_1), \quad T_{21} = \frac{1}{4}(1 - e_0 - e_1 + e_0e_1)t^2, \\ T_{22} &= -1 - \frac{1}{2}e_1 + \frac{1}{2}e_0e_1 + \frac{1}{2}(1 + e_0 + e_1 - e_0e_1)t^2. \end{aligned} \quad (3)$$

In particular $4T \in M_2(\mathbb{Z}G)$, and $\ker(4T) = \ker T$.

5.3 Pontryagin duality

Let $X = \{0, 1\}^\mathbb{Z}$ with the product μ of the uniform measures on $\{0, 1\}$ (the Haar measure of the compact dual group of $\mathbb{A}$), and $(\tau x)_j = x_{j+1}$.

Lemma 5.2 (fibre operators). *There is a unitary $W: \ell^2(G) \rightarrow L^2(X \times \mathbb{Z}, \mu \otimes \text{counting})$ with $W\delta_e = 1_X \otimes \delta_0$ such that, for $g \in \mathbb{A}$ and $l \in \mathbb{Z}$,*

$$(W\lambda(gt^l)W^{-1}\Phi)(x, k) = (-1)^{\sum_j g_j x_{j-k}} \Phi(x, k - l). \quad (4)$$

Consequently, for $a \in M_m(\mathbb{Q}G)$ the operator $W\lambda(a)W^{-1}$ (applied componentwise) acts fibrewise: $(W\lambda(a)W^{-1}\Phi)(x, \cdot) = \theta_x(a)\Phi(x, \cdot)$, where $\theta_x(a)$ is the bounded operator on $\ell^2(\mathbb{Z})^m$ obtained by replacing every group element gt^l in the entries of a by the operator $f \mapsto (k \mapsto (-1)^{\sum_j g_j x_{j-k}} f(k - l))$.

Proof. The characters $\chi_g(x) = (-1)^{\sum_j g_j x_j}$, $g \in \mathbb{A}$, form an orthonormal basis of $L^2(X, \mu)$ (Pontryagin duality for the compact group X). Hence $W_1: \delta_{gt^k} \mapsto \chi_g \otimes \delta_k$ is a unitary $\ell^2(G) \rightarrow L^2(X \times \mathbb{Z})$. Since $e_j \cdot gt^k = (g + e_j)t^k$ and $\chi_{g+e_j} = (-1)^{x_j} \chi_g$, we get $(W_1\lambda(e_j)W_1^{-1}\Psi)(x, k) = (-1)^{x_j} \Psi(x, k)$. Since $t \cdot gt^k = (\sigma g)t^{k+1}$ and $\chi_{\sigma g}(x) = \chi_g(\tau x)$, we get $(W_1\lambda(t)W_1^{-1}\Psi)(x, k) = \Psi(\tau x, k - 1)$. Let $(V\Psi)(x, k) = \Psi(\tau^{-k}x, k)$, a unitary because τ preserves μ , and $W = VW_1$. Then

$$(W\lambda(t)W^{-1}\Phi)(x, k) = \Phi(x, k - 1), \quad (W\lambda(e_j)W^{-1}\Phi)(x, k) = (-1)^{x_{j-k}} \Phi(x, k),$$

because $(\tau^{-k}x)_j = x_{j-k}$; for instance $(W\lambda(t)W^{-1}\Phi)(x, k) = (W_1\lambda(t)W_1^{-1}V^{-1}\Phi)(\tau^{-k}x, k) = (V^{-1}\Phi)(\tau^{1-k}x, k - 1) = \Phi(x, k - 1)$. Writing $gt^l = (\prod_{j: g_j=1} e_j)t^l$ gives (4), and $W\delta_e = V(1_X \otimes \delta_0) = 1_X \otimes \delta_0$. The statement on a follows by linearity and composition. $\square$

Lemma 5.3 (trace formula). *Let $a \in M_m(\mathbb{Q}G)$. Suppose that for μ -almost every x an orthogonal projection P_x on $\ell^2(\mathbb{Z})^m$ with range $\ker \theta_x(a)$ is given, such that $x \mapsto \langle P_x \xi, \eta \rangle$ is measurable for all standard basis vectors ξ, η . Then*

$$\dim_{\text{vN}} \ker a = \sum_{i=1}^m \int_X \langle P_x(\delta_0 \otimes \varepsilon_i), \delta_0 \otimes \varepsilon_i \rangle d\mu(x), \quad (5)$$

where $\varepsilon_1, \dots, \varepsilon_m$ is the standard basis of $\mathbb{C}^m$.

Proof. Let $D = W\lambda(a)W^{-1}$ and let P be the operator $(P\Phi)(x) = P_x\Phi(x)$ on $L^2(X; \ell^2(\mathbb{Z})^m)$, an orthogonal projection. Its range consists of the Φ with $\Phi(x) \in \ker \theta_x(a)$ for almost every x , which is exactly $\ker D$, since $(D\Phi)(x) = \theta_x(a)\Phi(x)$. So $W^{-1}PW = P_{\ker \lambda(a)}$, and by $W\delta_e = 1_X \otimes \delta_0$,

$$\tau_m(P_{\ker \lambda(a)}) = \sum_i \langle P(1_X \otimes \delta_0 \otimes \varepsilon_i), 1_X \otimes \delta_0 \otimes \varepsilon_i \rangle = \sum_i \int_X \langle P_x(\delta_0 \otimes \varepsilon_i), \delta_0 \otimes \varepsilon_i \rangle d\mu(x). \quad \square$$

5.4 The fibres of T

Fix $x \in X$. The entries of T involve only t^0 and t^2 , so $\theta_x(T)$ preserves the even part $\ell^2(2\mathbb{Z})^2$ and the odd part $\ell^2(2\mathbb{Z} + 1)^2$, and so does every projection onto its kernel; only the even part enters (5). We call the index $k = -2m$ of the even part the *position* $m \in \mathbb{Z}$ and write $f_u(m) = f_u(-2m)$, $f_v(m) = f_v(-2m)$ for $f = (f_u, f_v)$. By (4), at position m the lamps e_0 and e_1 act by $(-1)^{x_{2m}}$ and $(-1)^{x_{2m+1}}$, and $(Sf)(-2m) = f(-2m - 2)$ is the value at position $m + 1$. The *symbol* at position m is $\sigma_m(x) = (x_{2m}, x_{2m+1})$; it is a *marker* if $\sigma_m = (1, 1)$, and the *letter* U , D or I if $\sigma_m = (0, 1)$, $(1, 0)$ or $(0, 0)$. Thus at position m the element χ acts by 1 at a marker and by 0 at a letter, α by 2 at U and by 1 otherwise, and β by 2 at D and by 1 otherwise.

Lemma 5.4 (rows). *The two coordinates of $\theta_x(T)f$ at position m are*

$$\begin{aligned} f_u(m) - f_v(m) \quad \text{and} \quad f_u(m+1) - f_v(m+1) \quad & \text{if } \sigma_m \text{ is a marker,} \\ f_u(m+1) - a f_u(m) \quad \text{and} \quad f_v(m+1) - b f_v(m) \quad & \text{if } \sigma_m \text{ is a letter,} \end{aligned}$$

with $(a, b) = (2, 1)$, $(1, 2)$, $(1, 1)$ for U , D , I .

Proof. By (2), the first coordinate is $(\chi - (1 - \chi)\alpha)f_u + (1 - \chi)Sf_u - \chi f_v$ and the second is $\chi Sf_u + (1 - \chi)Sf_v - \chi Sf_v - (1 - \chi)\beta f_v$, where the elements χ, α, β act at the current position and S moves the value from position $m + 1$. Insert $\chi = 1$ at a marker and $\chi = 0$, $\alpha = a$, $\beta = b$ at a letter. $\square$

Lemma 5.5 (segments). *Let $m < m'$ be positions of markers with only letters in between, forming the word $s \in \{U, D, I\}^N$, $N = m' - m - 1$. The rows “second coordinate at m ”, “both coordinates at $m + 1, \dots, m' - 1$ ” and “first coordinate at m' ” involve only the columns (j, u) , (j, v) with $m + 1 \leq j \leq m'$, and no other row involves these columns. In the order of the rows just listed and the columns $(m + 1, u)$, $(m + 1, v)$, $\dots$, (m', u) , (m', v) this block is the counter matrix M_s of Definition 3.2. If x has markers at infinitely many negative and infinitely many positive positions, the even part of $\theta_x(T)$ is the direct sum of these finite blocks, and the orthogonal projection onto its kernel is the direct sum of the projections $P_{\ker M_s}$ onto the kernels of the blocks.*

Proof. By Lemma 5.4, the second coordinate at the marker m involves position $m + 1$, the coordinates at a letter j involve j and $j + 1$, and the first coordinate at the marker m' involves m' . A position j with $m + 1 \leq j \leq m'$ is involved only in rows at j and at $j - 1$, and those of them that involve j belong to the list. With $u_i = f_u(m + 1 + i)$ and $v_i = f_v(m + 1 + i)$ ($0 \leq i \leq N$) the listed rows are

$u_0 - v_0$, then $u_{i+1} - a_i u_i$ and $v_{i+1} - b_i v_i$ for the letters $s_0, \dots, s_{N-1}$, then $u_N - v_N$: this is M_s . Under the hypothesis on x , every row belongs to exactly one segment (the first coordinate at a marker to the segment ending there, the second to the segment starting there, the coordinates at a letter to the segment containing it), and every position to exactly one segment. So the even part of $\theta_x(T)$ maps the ℓ^2 -direct sum of the column spaces of the segments into the direct sum of their row spaces, block by block; a vector lies in its kernel if and only if each component lies in the kernel of its block. $\square$

5.5 Proof of Theorem 2

Under μ the symbols $\sigma_m(x)$, $m \in \mathbb{Z}$, are independent and uniformly distributed on $\{0, 1\}^2$, since they involve disjoint pairs of coordinates. Hence almost every x has markers at infinitely many negative and at infinitely many positive positions, in the even and in the odd part, and for such x we let P_x be the projection of Lemma 5.5 on the even part (and the analogous projection on the odd part, where the symbols are (x_{2m-1}, x_{2m})). Each matrix coefficient of P_x vanishes unless the two positions involved lie in one segment, and then it depends only on the word of this segment and on the places of the two positions in it; so it is constant on the cells of a countable partition of a set of full measure into cylinder sets, hence measurable, and Lemma 5.3 applies.

For $N \geq 0$, $s \in \{U, D, I\}^N$ and $1 \leq j \leq N+1$, let $E_{N,s,j}$ be the set of x with a marker at $-j$, the letters s at $-j+1, \dots, -j+N$, and a marker at $-j+N+1$; that is, position 0 is the j -th column position of a segment with word s . These sets are disjoint (the markers at $-j$ and $-j+N+1$ are the last marker before position 0 and the first marker at or after it), their union has full measure, and $\mu(E_{N,s,j}) = 4^{-(N+2)}$, since $N+2$ symbols are prescribed. On $E_{N,s,j}$ position 0 carries the variables u_{j-1}, v_{j-1} of the block M_s , so the integrand of (5) equals $(P_{\ker M_s})_{2j-1, 2j-1} + (P_{\ker M_s})_{2j, 2j}$, and summing over j gives $\text{Tr } P_{\ker M_s} = \dim \ker M_s$. By Lemmas 3.5 and 5.3,

$$\dim_{\text{vN}} \ker T = \sum_{N \geq 0} \sum_{s \in \{U, D, I\}^N} 4^{-(N+2)} \dim \ker M_s = \sum_{N \geq 0} \frac{T_N}{4^{N+2}},$$

where $T_N = \#\{s \in \{U, D, I\}^N : \#U(s) = \#D(s)\}$. Choosing the $2k$ positions of the letters U and D and then the k positions of U among them gives $T_N = \sum_k \binom{N}{2k} \binom{2k}{k}$. With $\sum_{N \geq 0} \binom{N}{2k} y^N = y^{2k} (1-y)^{-2k-1}$ for $|y| < 1$ and the binomial series, for $0 \leq y < \frac{1}{3}$ (all terms are non-negative)

$$\sum_{N \geq 0} T_N y^N = \frac{1}{1-y} \sum_{k \geq 0} \binom{2k}{k} \left(\frac{y}{1-y}\right)^{2k} = \frac{1}{1-y} \left(1 - \frac{4y^2}{(1-y)^2}\right)^{-1/2} = (1 - 2y - 3y^2)^{-1/2}, \quad (6)$$

where $4y^2 < (1-y)^2$ because $y < \frac{1}{3}$. At $y = \frac{1}{4}$ we get $1 - \frac{1}{2} - \frac{3}{16} = \frac{5}{16}$, so $\sum_N T_N 4^{-N} = 4/\sqrt{5}$ and

$$\dim_{\text{vN}} \ker T = \frac{1}{16} \cdot \frac{4}{\sqrt{5}} = \frac{1}{4\sqrt{5}} = \frac{\sqrt{5}}{20}.$$

Finally $80(\sqrt{5}/20)^2 = 1$, and $\sqrt{5}/20$ is irrational because $\sqrt{5}$ is. $\square$

Remark 5.6. Grabowski's formula [Gra16, Proposition 13] integrates $\dim \ker T(x)/|G(x)|$ over the points x , where $G(x)$ is the component of x in the graph whose edges are the non-zero matrix coefficients, and it requires almost every component to be finite and simply connected. For T itself the components are infinite: the coefficients of the row at the vertex (m, v) (the second coordinate at a marker m) lie in the columns of the segment starting at m , while (m, v) is itself a column of the segment ending at m , so consecutive segments are linked. The formula applies to T^*T , which has the same kernel and whose fibre operators are the direct sums of the matrices $M_s^* M_s$ on the column positions of the segments (the components are simply connected since the shift acts essentially freely on X).

Read with the vertices (x, u) , (x, v) it gives the same series: such a vertex lies in a segment with word s (with $2N + 2$ vertices) with probability $(N + 1)4^{-N-2}$, and $2(N + 1)4^{-N-2} \dim \ker M_s / (2N + 2) = 4^{-N-2} \dim \ker M_s$.

6 A second proof through finite quotients

For $K \geq 1$ let $G_K = \mathbb{Z}/2 \wr \mathbb{Z}/K = (\bigoplus_{r \in \mathbb{Z}/K} \mathbb{Z}/2) \rtimes \mathbb{Z}/K$, of order $K2^K$, and let $\varphi_K: G \rightarrow G_K$ reduce the lamp positions and the exponent of t modulo K : $\varphi_K(gt^k) = \bar{g} t^{k \bmod K}$ with $\bar{g}_r = \sum_{j \equiv r} g_j \bmod 2$. This is a surjective homomorphism, since reduction modulo K commutes with the shift σ . For $a \in M_m(\mathbb{Q}G)$ let a_K be its image in $M_m(\mathbb{Q}G_K)$, acting by left multiplication on $\mathbb{C}[G_K]^m$, and put

$$d_K(a) = \frac{\dim_{\mathbb{C}} \ker a_K}{|G_K|}.$$

Lemma 6.1 (finite Fourier transform). *For $x \in X_K = \{0, 1\}^{\mathbb{Z}/K}$ let $\theta_x^{(K)}(a)$ be the operator on $\mathbb{C}^{\mathbb{Z}/K} \otimes \mathbb{C}^m$ defined as in Lemma 5.2, with all indices taken modulo K . Then $\dim_{\mathbb{C}} \ker a_K = \sum_{x \in X_K} \dim \ker \theta_x^{(K)}(a)$.*

Proof. The proof of Lemma 5.2 applies verbatim to the finite group $(\mathbb{Z}/2)^{\mathbb{Z}/K}$, whose characters $x \in X_K$ form an orthogonal basis of $\mathbb{C}^{X_K}$; the map $(V\Psi)(x, k) = \Psi(\tau^{-k}x, k)$ is well defined since τ^K is the identity. So left multiplication by a_K is conjugate to the direct sum of the $\theta_x^{(K)}(a)$. $\square$

Proposition 6.2 (exact formula). *For every $P \geq 1$,*

$$d_{2P}(T) = \sum_{N=0}^{P-2} \frac{T_N}{4^{N+2}} + \frac{T_{P-1}}{4^P} + \frac{2^{1-P}}{P}. \quad (7)$$

Proof. Let $K = 2P$. As in Section 5, $\theta_x^{(K)}(T)$ preserves the even and the odd indices modulo K . On the even indices we use the positions $m \in \mathbb{Z}/P$ (index $-2m$) with the symbols $\sigma_m = (x_{2m}, x_{2m+1})$, indices modulo K ; these P symbols use every coordinate of x exactly once, so $x \mapsto (\sigma_m)_{m \in \mathbb{Z}/P}$ is a bijection from X_K onto $(\{0, 1\}^2)^P$. The odd indices give the same model with the symbols (x_{2m-1}, x_{2m}) , again a bijection. Lemma 5.4 holds verbatim with positions modulo P . So by Lemma 6.1, $\dim_{\mathbb{C}} \ker \bar{T} = 2 \sum_{\sigma} \kappa(\sigma)$, where $\bar{T}$ is the image of T in $M_2(\mathbb{Q}G_K)$ and the sum is over the 4^P cyclic symbol sequences σ , where $\kappa(\sigma)$ is the kernel dimension of the cyclic operator given by the rows of Lemma 5.4.

If σ contains a marker, the argument of Lemma 5.5 applies on the cycle; when there is only one marker, its segment consists of the $P - 1$ letters and starts and ends at this marker (for $P = 1$ the block is $M_{\emptyset} = Z$). So $\kappa(\sigma)$ is the number of segments whose word s has $\#U(s) = \#D(s)$. We count the pairs $(\sigma, \text{marker } m \text{ of } \sigma)$ such that the segment starting at m is balanced. For fixed m and a segment of $N \leq P - 2$ letters with word s , the symbols at $m, m + 1, \dots, m + N + 1$ are prescribed and 4^{P-N-2} choices remain; for $N = P - 1$ the sequence σ is determined by s . This gives $P(\sum_{N=0}^{P-2} T_N 4^{P-N-2} + T_{P-1})$. If σ contains no marker, the two tracks decouple into $f_u(m + 1) = a_m f_u(m)$ and $f_v(m + 1) = b_m f_v(m)$ around the cycle, which have a non-zero solution if and only if $\prod_m a_m = 1$, resp. $\prod_m b_m = 1$; so $\kappa(\sigma)$ is the number of the letters U, D that do not occur in σ , and these σ contribute $2^P + 2^P$. Dividing $2(P \sum_{N=0}^{P-2} T_N 4^{P-N-2} + P T_{P-1} + 2^{P+1})$ by $|G_{2P}| = 2P \cdot 4^P$ gives (7). $\square$

Second proof of Theorem 2. The kernels $N_i = \ker \varphi_{2^i}$, $i \geq 1$, are normal subgroups of finite index. They are nested, since φ_{2^i} factors through $\varphi_{2^{i+1}}$, and their intersection is trivial: if gt^k lies in all of

them, then 2^i divides k for every i , so $k = 0$, and every residue class modulo 2^i contains an even number of the lamps of g ; once 2^i exceeds the diameter of the support of g , each class contains at most one of them, so $g = 0$. Lück's approximation theorem [Lüc94], in its form for matrices over $\mathbb{Z}G$ and a nested chain of normal subgroups of finite index with trivial intersection (see [KKL25, Theorem 6.2 and Section 7.1]), applied to the square matrix $4T$ and this chain, gives $\dim_{\mathbb{V}\mathbb{N}} \ker T = \lim_{i \rightarrow \infty} d_{2^i}(T)$; by Lemma 5.1 it does not matter whether the theorem is formulated with left or with right multiplication, on G and on the finite quotients. In (7) the last two terms tend to 0, because $T_N \leq 3^N$, and the sum tends to $\sum_{N \geq 0} T_N 4^{-N-2}$, which equals $\sqrt{5}/20$ by (6). $\square$

This proof uses the finite Fourier transform and elementary counting instead of the direct integral of Lemma 5.3, and Lück's theorem as a citation.

Remark 6.3 (the error on the finite quotients). Subtracting $\sqrt{5}/20 = \sum_{N \geq 0} T_N 4^{-N-2}$ from (7) gives

$$d_{2P}(T) - \frac{\sqrt{5}}{20} = \frac{3}{4} \cdot \frac{T_{P-1}}{4^P} - \sum_{N \geq P} \frac{T_N}{4^{N+2}} + \frac{2^{1-P}}{P}.$$

The first two terms come from the configurations with a single marker, whose segment of $P-1$ letters gets the weight 4^{-P} instead of 4^{-P-1} , and from the segments that are too long to fit on the cycle; since $T_N \leq 3^N$, each of them is at most $\frac{1}{4}(3/4)^P$. The marker-free configurations, although they make up a share $(3/4)^P$ of all configurations, contribute only $2^{1-P}/P$, because most of them have a trivial kernel. Table 2 lists some values.

$K = 2P$	$d_K(T)$	decimal	$d_K(T) - \sqrt{5}/20$
2	5/4	1.2500000	$1.1 \cdot 10^0$
4	3/8	0.3750000	$2.6 \cdot 10^{-1}$
6	5/24	0.2083333	$9.7 \cdot 10^{-2}$
8	19/128	0.1484375	$3.7 \cdot 10^{-2}$
10	327/2560	0.1277344	$1.6 \cdot 10^{-2}$
12	731/6144	0.1189779	$7.2 \cdot 10^{-3}$
14	3305/28672	0.1152693	$3.5 \cdot 10^{-3}$
16	3721/32768	0.1135559	$1.8 \cdot 10^{-3}$
20	294457/2621440	0.1123264	$5.2 \cdot 10^{-4}$
60		0.1118036	$2.4 \cdot 10^{-7}$

Table 2: The exact values (7); $\sqrt{5}/20 = 0.1118034\dots$ For $K \leq 14$ they were also obtained from Lemma 6.1 by computing all fibre operators, and for $K \leq 5$ directly in the group ring (Section 8).

7 Relation to the questions on $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$

Grabowski's question of 2014. In [Gra14, §1c] Grabowski asks what $\mathcal{C}(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z})$ is (Question 3 of the arXiv version v4, cited as Question 1.7 in [GS21, AC22]) and, after recalling the irrational algebraic numbers of [LW13] for free lamplighters and the transcendental numbers in $\mathcal{C}(\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z})$, writes that this “raises the question whether $\mathcal{C}(\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z})$ contains irrational algebraic numbers”. Theorem 2 answers this for $p = 2$. Ara and Claramunt note that showing that their virtual number $\frac{1}{4}\sqrt{2/7}$ is itself an ℓ^2 -Betti number would solve this question [AC21, Remark 4.14]. In the same paragraph of [Gra14] Grabowski announces algebraic numbers of degree k in $\mathcal{C}((\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z})^k)$, to be shown in a future version of [Gra16]; these would concern the direct powers. Neither the arXiv version v4 (2015) nor the journal version of [Gra16] contains such a statement.

Remark 7.1 ($p = 2$ only). Theorem 2 is a statement about $p = 2$. For even p , the group $\mathbb{Z}/2 \wr \mathbb{Z}$ embeds into $\mathbb{Z}/p \wr \mathbb{Z}$ (lamps in the subgroup of order 2 of $\mathbb{Z}/p$), and $\mathcal{C}(H) \subseteq \mathcal{C}(G)$ for every subgroup $H \leq G$ [Gra10, pp. 2366–2367]: for $a \in M_m(\mathbb{Q}H)$ the space $\ell^2(G)^m$ is the orthogonal sum of the spaces $\ell^2(Hg)^m$ over the right cosets, each invariant under $\lambda(a)$ and unitarily equivalent to $\ell^2(H)^m$ by right translation, and δ_e lies in the summand of the coset H . So $\sqrt{5}/20$ also lies in $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ for every even p . For odd p the group $\mathbb{Z}/p \wr \mathbb{Z}$ has no element of order 2 (its elements of finite order lie in the lamp group, which has exponent p), so it does not contain $\mathbb{Z}/2 \wr \mathbb{Z}$; we claim nothing for odd p .

Problem 4 of [Gra16]. Grabowski’s lamplighter paper asks for a description of $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ substantially different from the definition and, more concretely, “is $\sqrt{2} \in \mathcal{C}(\mathbb{Z}_2 \wr \mathbb{Z})$?” [Gra16, Problem 4]. Theorem 2 does not answer this question. Since $\mathcal{C}(G)$ is closed under addition (block-diagonal sums of matrices), Theorem 2 gives the numbers $k\sqrt{5}/20$, $k \geq 1$; these are different from $\sqrt{2}$.

The question of Grigorchuk and Simanek. In [GS21, §7, Question 6(a)] Grigorchuk and Simanek ask “What is the set of values of $\mu_m(0)$ for self-adjoint $m \in \mathbb{Q}[L]$?”, where $L = \mathbb{Z}/2 \wr \mathbb{Z}$ and μ_m is the spectral measure of a *single self-adjoint element*; they state that this part is Grabowski’s question from [Gra14]. The matrix T of Theorem 2 is a 2×2 matrix and not self-adjoint. One has $\ker T = \ker T^*T$ with $T^*T \in M_2(\mathbb{Q}G)$ self-adjoint, but we do not reduce the matrix to a single element of $\mathbb{Q}G$. So Theorem 2 answers Grabowski’s version of the question, in which $\mathcal{C}(G)$ is defined through matrices, and not literally Question 6(a) of [GS21].

Virtual and genuine values. Ara and Claramunt work with the subgroup $\mathcal{G}(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Q})$ of $\mathbb{R}$ generated by $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$. Their numbers $\frac{1}{4}\sqrt{2/7}$ and $\frac{1}{8}\sqrt{2^t/(2^t - 1)}$ ($t \geq 3$) lie in this group [AC21, Theorem 4.13 and the paragraph after it]; whether $\frac{1}{4}\sqrt{2/7}$ lies in $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ is left open in [AC21, Remark 4.14], where it is also said that no group G with $\mathcal{C}(G, \mathbb{Q}) \neq \mathcal{G}(G, \mathbb{Q}) \cap \mathbb{R}_+$ is known to the authors. Theorem 2 does not decide this either: $\sqrt{5}/20$ is a different number (it is not of the form $\frac{1}{8}\sqrt{2^t/(2^t - 1)}$, since $(\sqrt{5}/20)^2 = \frac{1}{80} < \frac{1}{64}$). What Theorem 2 provides is a single matrix whose kernel has an irrational algebraic dimension.

The link with Question 6. The blocks of the fibre operators of T (Lemma 5.5) are exactly the matrices recognized by the automaton of Example 4.4, and

$$\dim_{\text{vN}} \ker T = \sum_{N \geq 0} \frac{T_N}{4^{N+2}} = \frac{1}{4} F_{A_{\mathcal{L},c}}\left(\frac{1}{2}\right) = \frac{1}{4} \cdot \frac{1}{\sqrt{5}}.$$

This illustrates the connection between Questions 5 and 6 described in [Gra10, p. 2368]; we do not claim a general statement of this kind.

8 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation. The computations test the definitions and the statements on finite instances. They do not test the statements about the infinite group in Section 5 (the direct integral and Lemma 5.3) or the cited theorems of Lück and Furstenberg, and they do not use them. The reproducibility package has three parts: the exact program of this note, the programs with which the results were first obtained, and those of the independent verification runs.

The exact program of this note (`check_note.py`; standard library, exact arithmetic throughout, about one minute). Part A (Theorem 1): the transition table over all 625 letters; a depth-first search

over all 625 letters, pruned only at the absorbing dead state, finds exactly the 2047 accepted words of length at most 21 of Lemma 3.1; a brute force over all tridiagonal matrices with entries in $\{-2, \dots, 2\}$ of size at most 4 (9,765,625 of size 4) finds exactly the matrices M_s ; every recognized matrix of size at most 22 is reconstructed from its word, equals M_s , and has kernel dimension $[\#U = \#D]$ over $\mathbb{Q}$; the coefficients agree with those of $x^2(1 - 4x^4)^{-1/2}$, and $\sqrt{3}/6$ is enclosed between rational bounds. Part B (Section 4): the automaton of Proposition 4.1 for $p/s = 1$ ($C = 2, q = \frac{1}{2}$) and the automaton of Example 4.4, each run over all 625 letters (recognized matrices, kernels, the coefficients T_N , the value $1/\sqrt{5}$ enclosed); the identity $Cq^2/(1 - q) = p/s$ for six rationals; the kernel rule of Proposition 4.2 with odd labels on all 2047 words of length at most 10 over two letters. Part C (Theorem 2): T is built in $\mathbb{Q}[\mathbb{Z}/2 \wr \mathbb{Z}]$ from $\pi_j, \chi, \alpha, \beta$ and S and compared with (3), and $4T$ is integral; the kernels of left and of right multiplication by $4T$ on $\mathbb{Q}[G_K]^2$ are computed directly in the group ring, without Fourier transform, for $K \leq 5$; the fibre operators $\theta_x^{(K)}(T)$, built from the group-ring coefficients of T by the rule of Lemma 6.1, are computed for all x and all $K \leq 14$, and their kernel dimensions agree with the direct values for $K \leq 5$ and with Proposition 6.2 for even K ; for all 21,844 configurations with $P \leq 7$ positions, the even part is checked to split into segment blocks (36,409 of them) equal entry by entry to $4M_s$, with kernels $[\#U = \#D]$, and without markers the kernel is $[\text{no } U] + [\text{no } D]$; finally T_N , (6), an enclosure of $\sqrt{5}/20$ and the identity of Remark 6.3 for $P \leq 60$ are checked.

The programs with which the results were first obtained (original/) test Theorem 1 (with a brute force over all tridiagonal matrices of size at most 4), Proposition 4.1, Example 4.4, and the block structure and the values $d_K(T)$ for even $K \leq 14$.

Three independent verification runs, with their programs and recorded outputs in `verification_run_A/`, `verification_run_B/` and `independent_run_2/`; see the next paragraph. When the note was written, every program of the package with a recorded output was run again from a copy, except the two longest runs of the second verification run ($K = 9$ in the group ring and $K \leq 22$ for the fibre operators); the outputs agree with the recorded ones up to running times (`RERUN_LOG.txt`). The run of the second round of verification ran all of them again from the archive, the two longest runs included, with the same result (`independent_run_2/rerun_package_v1.txt`).

Verification. The results were first obtained, with proofs, together with the programs in `original/`. Two rounds of independent verification, all AI-assisted, followed; each run wrote its own programs. In the first round two runs examined the first written version; each wrote its programs from the definitions of the source, and neither saw the other's report.

The first run (A) compared Question 6 and the definition of $\mathcal{C}(G)$ with the source. It re-derived Lemmas 3.1 and 3.3 to 3.5, Theorem 1 and Propositions 4.1 and 4.2, and for Theorem 2 the Fourier picture of Lemma 5.2, the rows of Lemma 5.4, the decomposition into segments, the bookkeeping of measures, the generating function (also through Grabowski's formula, as in Remark 5.6), the independence of the side of multiplication and the integrality of $4T$; and it compared the statements quoted from [Gra14, Gra16, AC21, AC22, GS21] with these papers. Its programs enumerate the recognized matrices of A_2 up to size 26 by a depth-first search over all 625 letters, with exact kernels; compute the kernels of left and of right multiplication by $4T$ on $\mathbb{Q}[G_K]^2$ directly in the group ring for $K \leq 8$ (exactly over $\mathbb{Q}$ for $K \leq 6$, modulo two primes otherwise); compute a Fourier model of their own for $K \leq 17$ (exactly for $K \leq 10$), which agrees with the group-ring values for all $K \leq 8$, odd and even; and check the block structure configuration by configuration for all 87,380 configurations with $P \leq 8$ (167,481 segment blocks). This run derived the exact formula of Proposition 6.2 and the second proof of Theorem 2. It ran two of the original programs again, with identical outputs.

The second run (B) wrote its programs before opening the original ones: a brute force over all tridiagonal matrices with entries in $\{-2, \dots, 2\}$ of size at most 4 and an exhaustive enumeration of the recognized matrices of A_2 up to size 30, with exact kernels; left multiplication by $4T$ on $\mathbb{Q}[G_K]^2$

for $K \leq 9$ (exactly over $\mathbb{Q}$ for $K \leq 5$, modulo two primes otherwise); and the fibre operators for $K \leq 22$ (all configurations for $K \leq 12$, one configuration per rotation class beyond), without any mismatch with the prediction from the segments. It derived Proposition 6.2 independently; the formula reproduces every computed value for even $K \leq 22$. Its programs detected two deliberately wrong automata and a deliberately wrong operator. It searched the literature, checked the bibliographic data with Crossref, and ran all three original programs again, with identical outputs.

Neither run of the first round found an error in a theorem, proposition or lemma. The corrections they required are all made here: the block-diagonal reading (Section 2.2); the status of the records (Section 1.1); Problem 4 of [Gra16] (Section 7); the restriction of the question of [GS21] to single self-adjoint elements; the wording on the works citing [Gra16]; the case $c \equiv 0$ in Corollary 4.3; no unchecked theorem number of [Fur67]; the source of the error on the finite quotients (Remark 6.3); a novelty statement naming the sources and the virtual numbers of Ara and Claramunt; $p = 2$ (Remark 7.1); the version of [Gra16] behind the remark in Section 7; and the second proof (Section 6).

When the note was written, Lemma 5.3 was proved directly for the explicit field of projections, instead of quoting the general fact that the kernel projection of a decomposable operator is decomposable; Remark 3.6 (other window sizes), which both runs had checked by reading only, was written out; Lemma 5.1 was written out; Remark 7.1, suggested by the first run, was added; and the program `check_note.py` was written.

In the second round a further run examined the final text line by line, these parts first. It read the source and the cited texts [Gra14, Gra16, AC21, AC22, GS21] again, [Gra16] and [AC21] also in their journal versions, compared the form in which Lück's theorem is used with its statement in [KKL25], re-derived all proofs, and read `check_note.py` completely. It found no error in a theorem, proposition or lemma. The corrections it required are made here: the irrational algebraic values known for other groups (Sections 1.1 and 1.3); Remark 5.6, where Grabowski's formula applies to T^*T and not to T ; the journal version of [Gra16] (Section 7); a reference for the form of Lück's theorem; and smaller points of wording. Its three programs, written for this run, enumerate all tridiagonal matrices with entries in $\{-2, \dots, 2\}$ that A_2 accepts up to size 28 (16,383 matrices) and, for Remark 3.6, all band matrices recognized by $A^{(3)}$ up to size 16 and by $A^{(4)}$ up to size 12, every letter being passed to the transition function (for $A^{(4)}$ every letter without non-zero entries at distance at least 2 from the diagonal; the others are rejected by its first rule); check Proposition 4.1, Proposition 4.2 for the language of the words over $\{a, b, c\}$ without two consecutive letters a , with the weights $(2, -1, 0)$, and the diagonal identity in the proof of Corollary 4.3; compute the kernels of left and of right multiplication by $4T$ on $\mathbb{Q}[G_K]^2$ in the group ring for $K \leq 8$ (exactly over $\mathbb{Q}$ for $K \leq 6$, modulo two primes otherwise), and on $\mathbb{Q}[\mathbb{Z}/4 \wr \mathbb{Z}/K]^2$ for $K \leq 4$ (Remark 7.1); build the change of basis of Lemma 6.1 explicitly for $K \leq 4$ and find the matrix of $4T$ block diagonal, with the blocks given by (4); compute all fibre operators for $K \leq 20$ (exactly for $K \leq 10$), whose kernel dimensions agree with the group-ring values for $K \leq 8$ and with Proposition 6.2 for every even $K \leq 20$; and check the trace formula (5) on the finite quotients with the exact kernel projections for $K \leq 10$. Its programs detected a deliberately wrong sign rule and a deliberately wrong operator.

Scope and priority. The note answers the “in particular” part of Question 6 of [Gra10] (Theorem 1) and gives partial information on the values of the series F_A (Section 4). It shows that $\sqrt{5}/20 \in \mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ (Theorem 2), which answers, for $p = 2$, the question of [Gra14, §1c] whether $\mathcal{C}(\mathbb{Z}/p \wr \mathbb{Z})$ contains irrational algebraic numbers. It does not answer Question 4 of [Gra10], Problem 4 of [Gra16], or Question 6(a) of [GS21] for single self-adjoint elements, and it does not describe $\mathcal{C}(\mathbb{Z}/2 \wr \mathbb{Z})$ or the set of values of F_A ; both corpus records, OWR-4428-011 and OWR-4428-012, remain partially solved. The questions, the automaton framework and the remark that families of components are described by regular languages are Grabowski's [Gra10, Gra16]. The method of the first proof of Theorem 2

is the one used by Dicks–Schick, Austin and Grabowski, among others, and the letter-count idea is the one Ara and Claramunt used for their irrational algebraic virtual ℓ^2 -Betti numbers [AC21, Theorem 4.13 and Remark 4.14], [AC22]. Lück’s approximation theorem [Lüc94], Pontryagin duality and Furstenberg’s theorem [Fur67] are standard. Only Theorems 1 and 2 are, to our knowledge, new; no priority is claimed, and both proofs are short once the counter is seen.

What was read: the report [Gra10] in the publisher’s open PDF (the title page and pp. 2366–2369); [Gra16] in its arXiv version v4 (introduction, Sections 1, 3 and 4) and, by the run of the second round, in the journal version, where Problem 4, Proposition 13, Remark 14 and Lemma 18 have the same numbers; [Gra14] in its arXiv version v4 (Section 1), whose numbering we use; [AC21] in its arXiv version v2 (Sections 1–4, the long proof of Theorem 4.5 only in outline; Section 5, on the odometer, not read) and, by the run of the second round, in the journal version (open access), where Theorem 4.13, Remark 4.14 and Question 4.15 have the same numbers; [AC22] in its arXiv version v2 (introduction); [GS21] in its arXiv version v2 (Section 7); and [KKL25] in its arXiv version v1 (Sections 6.1–6.4 and 7.1), whose numbering we use. Of [LW13] we used the abstract and the summary in [Gra10]. The references [Lüc94, Lüc02, Fur67] were not read; they are cited for standard results, and since the publisher’s site refused automated access to [Fur67], no theorem number of it is given. The references [DS02, GŻ01, Aus13, PSŻ15] were not read either; what we say about them is taken from [Gra10, Gra14, Gra16].

The searches were made on 14 October 2026, when the results were first obtained, by the verification runs of the first round, in part again when the note was written, and again by the run of the second round: the arXiv API (keyword searches combining “lamplighter” with Betti numbers, the Atiyah problem, von Neumann dimension or spectral measures, “irrational algebraic” or “virtual” with Betti numbers, and regular languages or automata with Betti numbers and kernels; author searches; all titles containing “lamplighter”), Crossref, zbMATH Open, OpenAlex and Semantic Scholar (the works citing [Gra10, Gra14, Gra16, AC21, AC22, LW13, PSŻ15, GS21], by title and abstract), and seven web searches. Question 6 was found only in [Gra10]; [Gra16] mentions regular languages only in its Remark 14(ii); none of the citing works found treats the series F_A . In these sources we found no earlier non-virtual irrational algebraic ℓ^2 -Betti number of a lamplighter group $\mathbb{Z}/p \wr \mathbb{Z}$: the values known there are transcendental and other irrational numbers [Gra16, AC21], irrational algebraic numbers for the free lamplighters $\mathbb{Z}/p \wr F_d$ [LW13], and irrational algebraic virtual ℓ^2 -Betti numbers of $\mathbb{Z}/2 \wr \mathbb{Z}$ [AC21, AC22]; for other groups see Section 1.3. An unpublished manuscript of Grabowski and Virág on random walks on lamplighters, cited in [GS21], could not be inspected. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AC21] Pere Ara and Joan Claramunt. L^2 -Betti numbers arising from the lamplighter group. *Journal of Algebraic Combinatorics*, 54(4):1201–1245, 2021. [arXiv:2005.12814](#), [doi:10.1007/s10801-021-01044-8](#).
- [AC22] Pere Ara and Joan Claramunt. Approximating the group algebra of the lamplighter by infinite matrix products. *Forum Mathematicum*, 34(4):851–891, 2022. [arXiv:2005.12374](#), [doi:10.1515/forum-2020-0141](#).

- [Aus13] Tim Austin. Rational group ring elements with kernels having irrational dimension. *Proceedings of the London Mathematical Society*, 107(6):1424–1448, 2013. [arXiv:0909.2360](#), [doi:10.1112/plms/pdt029](#).
- [DS02] Warren Dicks and Thomas Schick. The spectral measure of certain elements of the complex group ring of a wreath product. *Geometriae Dedicata*, 93(1):121–137, 2002. [doi:10.1023/A:1020381532489](#).
- [Fur67] Harry Furstenberg. Algebraic functions over finite fields. *Journal of Algebra*, 7(2):271–277, 1967. [doi:10.1016/0021-8693\(67\)90061-0](#).
- [Gra10] Łukasz Grabowski. On the Atiyah problem for the lamplighter groups. *Oberwolfach Reports*, 7(3):2366–2369, 2010. Abstract in Report 40/2010, *Actions and Invariants of Residually Finite Groups: Asymptotic Methods*, workshop of 5–11 September 2010, pp. 2335–2391. [doi:10.4171/OWR/2010/40](#).
- [Gra14] Łukasz Grabowski. On Turing dynamical systems and the Atiyah problem. *Inventiones Mathematicae*, 198(1):27–69, 2014. [arXiv:1004.2030](#), [doi:10.1007/s00222-013-0497-5](#).
- [Gra16] Łukasz Grabowski. Irrational l^2 invariants arising from the lamplighter group. *Groups, Geometry, and Dynamics*, 10(2):795–817, 2016. [arXiv:1009.0229](#), [doi:10.4171/GGD/366](#).
- [GS21] Rostislav Grigorchuk and Brian Simanek. Spectra of Cayley graphs of the lamplighter group and random Schrödinger operators. *Transactions of the American Mathematical Society*, 374(4):2421–2445, 2021. [arXiv:1902.10129](#), [doi:10.1090/tran/8156](#).
- [GŻ01] Rostislav I. Grigorchuk and Andrzej Żuk. The lamplighter group as a group generated by a 2-state automaton, and its spectrum. *Geometriae Dedicata*, 87(1–3):209–244, 2001. [doi:10.1023/A:1012061801279](#).
- [KKL25] Dominik Kirstein, Christian Kremer, and Wolfgang Lück. Some problems and conjectures about L^2 -invariants. In *Geometry and Topology of Aspherical Manifolds*, volume 816 of *Contemporary Mathematics*, pages 3–43. American Mathematical Society, Providence, RI, 2025. [arXiv:2311.17830](#), [doi:10.1090/conm/816/16349](#).
- [Lüc94] Wolfgang Lück. Approximating L^2 -invariants by their finite-dimensional analogues. *Geometric and Functional Analysis*, 4(4):455–481, 1994. [doi:10.1007/BF01896404](#).
- [Lüc02] Wolfgang Lück. L^2 -Invariants: Theory and Applications to Geometry and K-Theory, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*. Springer, Berlin, 2002. [doi:10.1007/978-3-662-04687-6](#).
- [LW13] Franz Lehner and Stephan Wagner. Free lamplighter groups and a question of Atiyah. *American Journal of Mathematics*, 135(3):835–849, 2013. [arXiv:1005.2347](#), [doi:10.1353/ajm.2013.0029](#).
- [PSŻ15] Mikaël Pichot, Thomas Schick, and Andrzej Żuk. Closed manifolds with transcendental L^2 -Betti numbers. *Journal of the London Mathematical Society*, 92(2):371–392, 2015. [arXiv:1005.1147](#), [doi:10.1112/jlms/jdv026](#).
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; records OWR-4428-011 and OWR-4428-012.