

A Four-Color-Theorem Proof of the Duplicate-Leaf Parse-Word Conjecture

Alper Ferudun*

24 October 2026

Abstract

Cooper, Rowland and Zeilberger conjectured that two ordered full binary trees sharing a pair of consecutive uncle–nephew leaves have a common parse word with equal letters at that prescribed pair. We prove this existence assertion using the four-color theorem. In the planar graph obtained by gluing the two corresponding polygon triangulations, the specified leaves meet at a vertex of degree at most four. Contracting a length-two path, coloring the minor, and restoring that vertex forces the required equality. A padding argument also shows that the duplicate-leaf assertion implies unrestricted common parsing; consequently, by the known parsing equivalence, it has the strength of the four-color theorem. The proof uses that established theorem and does not supply a new proof of it or the proposed reduction to a particular smaller tree pair.

1 The assertion and its scope

All trees in this note are finite, rooted, ordered, and full binary: each vertex has zero or two children. Number the leaves from left to right. The grammar considered by Cooper, Rowland and Zeilberger has alphabet $\{0, 1, 2\}$ and productions

$$0 \longrightarrow 12, 21, \quad 1 \longrightarrow 02, 20, \quad 2 \longrightarrow 01, 10.$$

A tree *parses* a word if its vertices can be labeled consistently with these productions, with the word read along its leaves.

Two leaves are an *uncle–nephew pair* if one is a sibling of the other’s parent. Rowland’s workshop contribution [2, p. 2228] asks for equal labels at a consecutive pair having this relation in both trees. The indexed formulation is Conjecture 19 of Cooper–Rowland–Zeilberger [1], following their leaf-duplication operation.

Theorem 1. *Let T_1, T_2 have n leaves, and fix $1 \leq i < n$. If leaves $i, i + 1$ are an uncle–nephew pair in each tree, then the trees parse a common word $w_1 \cdots w_n$ with $w_i = w_{i+1}$.*

The orientations in the two trees may differ. The word may depend on the chosen i . Our proof invokes the four-color theorem [4]. Thus it settles the stated existential assertion while retaining the distinction from the authors’ program of obtaining a new language-theoretic proof of that theorem. Related signed-reassociation and special-class results appear in Madonia–Scollo [3].

*Mercury Software GmbH. Email: alper@mercurycodelab.com.

2 Forcing equal colors at two neighbors

Lemma 2. *Let G be a finite simple planar graph, v a vertex of degree at most four, and u, w distinct nonadjacent neighbors of v . Then G has a proper four-coloring in which u and w receive the same color.*

Proof. Contract the edges uv and vw , merging u, v, w into one vertex x . Delete loops and suppress parallel edges. The resulting simple graph H is planar, so the four-color theorem gives a proper coloring of H .

Pull this coloring back to $G - v$, assigning the color of x to both u and w . This pullback is proper: uw is not an edge, and every other edge of $G - v$ maps to an edge joining distinct vertices of H . There are at most four neighbors of v , two already equally colored, so at most three colors occur among them. A missing color can be assigned to v . $\square$

This is the elementary degree-four contraction argument; the substantive external input is four-colorability of the minor. A loopless planar multigraph may first have its parallel edges suppressed.

3 Trees, triangulations, and labels

Use a convex polygon with vertices $0, 1, \dots, n$ in cyclic order. For a tree T , associate an edge $\{a, b\}$ to each tree vertex whose descendant leaves are $a + 1, \dots, b$. Leaf j corresponds to $\{j - 1, j\}$, and the root corresponds to $\{0, n\}$. If an internal vertex splits its interval at k , the parent edge and the two child edges form the triangle

$$(a, k, b).$$

Induction on the two subtrees shows that these triangles triangulate the polygon. Denote this triangulation by $\Delta(T)$.

Place $\Delta(T_1)$ and $\Delta(T_2)$ in the two hemispheres of a sphere and identify their polygon boundaries. This gives a loopless planar multigraph. A shared diagonal may appear twice; suppressing its parallel copies gives a simple planar graph G containing all edges of every triangle of either triangulation.

Regard four colors as elements of $\mathbb{F}_2^2$. A proper vertex coloring c induces the nonzero edge label

$$\ell(\{a, b\}) = c(a) + c(b).$$

On any triangle the three vertex colors are distinct, so its edge labels are exactly the three distinct nonzero elements of $\mathbb{F}_2^2$. Choose a bijection from those elements to the grammar alphabet. Each triangle then supplies a valid parent-and-children production. In particular, the boundary labels

$$\ell(\{0, 1\}), \ell(\{1, 2\}), \dots, \ell(\{n - 1, n\})$$

give a common parse word for the two trees. This explicit construction uses the standard four-color/tree-labeling correspondence underlying the equivalence recalled in [1]; the grammar letter 0 is not the zero vector.

Lemma 3. *Suppose leaves $i, i + 1$ are an uncle–nephew pair in T . In $\Delta(T)$, the polygon vertex i belongs to exactly two triangles and has exactly one incident diagonal. Moreover, $\{i - 1, i + 1\}$ is not an edge.*

Proof. If leaf i is the uncle, consecutiveness forces the relevant subtree to have the form

$$(a, (b, S)),$$

where a, b are the specified leaves and S is a nonempty subtree. Write r for the final leaf of this subtree. The two triangles meeting polygon vertex i are

$$(i-1, i, r), \quad (i, i+1, r).$$

They fill the polygon's interior angle at i , so no other triangle of the triangulation meets that vertex. The only incident diagonal is $\{i, r\}$.

If leaf $i+1$ is the uncle, the subtree has the form $((S, a), b)$. Writing $l+1$ for its first leaf, the two triangles are

$$(l, i-1, i), \quad (l, i, i+1),$$

and the only incident diagonal is $\{l, i\}$.

Finally, the edge $\{i-1, i+1\}$ would represent a tree vertex with precisely the two descendant leaves $i, i+1$, making them siblings. This contradicts the uncle–nephew relation. The hypotheses imply $n \geq 3$, so this edge cannot be the polygon's root boundary edge in a triangle. $\square$

Proof of Theorem 1. In the glued graph set

$$u = i-1, \quad v = i, \quad w = i+1.$$

By Lemma 3, v has the two boundary neighbors u, w and at most one additional neighbor from each hemisphere. Hence $\deg_G(v) \leq 4$. The same lemma excludes the edge uw on both sides.

Lemma 2 supplies a proper four-coloring with $c(u) = c(w)$. The associated common parse word has

$$w_i = c(u) + c(v) = c(v) + c(w) = w_{i+1},$$

where the displayed equality is understood through the fixed bijection with the nonzero vectors. This proves the required assertion. $\square$

4 The logical strength of the assertion

Corollary 4. *Using the established equivalence between unrestricted common parsing and the four-color theorem, the assertion of Theorem 1 is equivalent to the four-color theorem.*

Proof. The forward implication from four-colorability is Theorem 1. For the converse, start with arbitrary equally sized trees R_1, R_2 . Adjoin two initial leaves and form

$$T_j = (a, (b, R_j)), \quad j = 1, 2.$$

Their first two leaves are an uncle–nephew pair. They arise by duplicating the first leaf of (a, R_j) in the terminology of [1]. The duplicate-leaf assertion gives a common parse word for T_1, T_2 . Its suffix on R_j is a common parse word for R_1, R_2 , since these are actual subtrees. Thus every equally sized tree pair parses a common word. The known equivalence, recalled as Theorem 1 in [1], implies the four-color theorem. $\square$

The proof does not show how to extend an arbitrary common word for the particular pair before duplication. Nor does it provide the proposed four-color-free reduction by deleting one of the duplicate leaves. It proves equality at any one prescribed eligible pair, with no claim of simultaneous equalities.

As finite diagnostics, the accompanying Python script enumerates all 102,039 unordered pairs with repetition of full binary trees with $3 \leq n \leq 8$. The 67,460 eligible specified positions all admit an equal-letter common word. For $n \leq 7$, it separately constructs and lifts a minor coloring in 6,014 instances. Direct recursive parsing and the interval-edge representation are checked independently. These computations check implementation and indexing; the general proof is the argument above.

The standard coloring ingredients are credited explicitly, and no absolute priority claim is made for this application. AI tools assisted with literature searches, drafting, and exact finite checks. The author is responsible for the manuscript. This preprint is unrefereed; the argument has not been formalized in a proof assistant.

References

- [1] B. Cooper, E. Rowland, and D. Zeilberger, *Toward a language theoretic proof of the four color theorem*, author version dated November 22, 2011, Conjecture 19 and Theorem 1. [arXiv:1006.1324](https://arxiv.org/abs/1006.1324).
- [2] E. Rowland (joint work with B. Cooper and D. Zeilberger), *Ambiguity in a certain context-free grammar*, in Mini-Workshop: Combinatorics on Words, Oberwolfach Report 37/2010, pp. 2226–2229. [doi:10.4171/owr/2010/37](https://doi.org/10.4171/owr/2010/37).
- [3] M. Madonia and G. Scollo, *Reducibility and solvability of some classes of Kryuchkov binary tree pairs*, Electronic Journal of Combinatorics **18**(2) (2012), P32. [doi:10.37236/2028](https://doi.org/10.37236/2028).
- [4] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, *The Four-Colour Theorem*, Journal of Combinatorial Theory, Series B **70**(1) (1997), 2–44. [doi:10.1006/jctb.1997.1750](https://doi.org/10.1006/jctb.1997.1750).