A Proof of the Nichols–Stolz Conjecture on the Almost-Sure Spectrum of the Bernoulli Displacement Model

Alper Ferudun*

September 29, 2026

Abstract

The Bernoulli displacement model is the random Schrödinger operator $h_{\omega,\lambda} = h_0 + V_\omega$ on $\ell^2(\mathbb{Z})$ in which each cell of two neighbouring sites carries one single-site potential $\lambda \neq 0$. The potential sits on the left or on the right site of the cell according to an independent Bernoulli variable ω_k. Nichols and Stolz determined the almost-sure spectrum Σ_λ for $0 < |\lambda| \leq 2$. They conjectured (Oberwolfach Report 55/2009; J. Spectral Theory 1 (2011), Conjecture 6.2) that $\Sigma_\lambda = \sigma(h_{\omega^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda})$ for every $\lambda \neq 0$, where the configurations ω^* and ω^1 give potentials of period four and two. For $|\lambda| > 2$ this means that Σ_λ consists of exactly six bands. We prove the conjecture. In fact, the inclusion

$$\sigma(h_{\omega,\lambda}) \subset \sigma(h_{\omega^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda}) = \left\{E \in \mathbb{R} : \ |E(E-\lambda) - 2| \in [0,2] \cup [|\lambda|, \sqrt{\lambda^2+4}\,]\right\}$$

holds for every configuration ω, not only almost surely. The proof uses the reduction of Nichols and Stolz, under which $h_{\omega,\lambda}(h_{\omega,\lambda} - \lambda) - 2$ becomes a direct sum of two discrete Schrödinger operators with potentials $\pm s$, where $s_k = \lambda(\omega_k - \omega_{k+1})$. It combines this reduction with an elementary Schur-complement argument. The argument works because s never takes the same nonzero value at two neighbouring sites. This is an unrefereed note.

1 Introduction

We follow Nichols and Stolz [NS11, §2.6]. Fix $\lambda \in \mathbb{R} \setminus \{0\}$. For a configuration $\omega = (\omega_k)_{k\in\mathbb{Z}} \in \Omega := \{0,1\}^{\mathbb{Z}}$ let $V_\omega : \mathbb{Z} \to \{0, \lambda\}$ be the potential which, on each cell $C_k = \{2k, 2k+1\}$, equals λ at the site $2k + \omega_k$ and 0 at the other site. Let

$$h_\omega = h_{\omega,\lambda} = h_0 + V_\omega, \qquad (h_0 u)(n) = u(n-1) + u(n+1),$$

on $\ell^2(\mathbb{Z})$. Nichols and Stolz use the cells $\{2k+1, 2k+2\}$; this is a translation and does not change any spectrum. Their kinetic term is $(h_0^{\mathrm{NS}} u)(n) = -\sum_{|m-n|=1} u(m)$ [NS11, p. 125], the negative of our h_0. This change of sign does not matter either, because the unitary $u(n) \mapsto (-1)^n u(n)$ maps h_0 to $-h_0$ and commutes with V_ω, so $h_0 + V_\omega$ and $-h_0 + V_\omega$ have the same spectrum for every ω. We write $\Delta := h_0$ for the same operator when it acts on the auxiliary spaces of Section 3.

In the *Bernoulli displacement model* the ω_k are independent with $P(\omega_k = 0) = p \in (0,1)$. The family is ergodic, so there is a closed set Σ_λ with $\sigma(h_{\omega,\lambda}) = \Sigma_\lambda$ for almost every ω [NS11, §2.4]. Two periodic configurations play a special role. The first is ω^*, with $\omega^*_k = 0$ for even k and $\omega^*_k = 1$ for odd k; its potential is 4-periodic with period $(\lambda, 0, 0, \lambda)$. The second is ω^1, with $\omega^1_k = 1$ for all k; its potential is 2-periodic with period $(0, \lambda)$. Nichols and Stolz proved the following [NS11].

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`.
Unrefereed preprint. CC BY 4.0.

- The extremal configuration ω^* gives the edges of the almost-sure spectrum: $\inf \Sigma_\lambda = \min \sigma(h_{\omega^*,\lambda})$ and $\sup \Sigma_\lambda = \max \sigma(h_{\omega^*,\lambda})$ [NS11, Thm. 2.2, Cor. 2.4].
- The central gap (G_-, G_+) of $\sigma(h_{\omega^*,\lambda})$ is a gap of $\sigma(h_{\omega,\lambda})$ for every ω [NS11, Thm. 2.6].
- If $0 < |\lambda| \le 2$, then $\Sigma_\lambda = [E_-, E_+] \setminus (G_-, G_+) = \sigma(h_{\omega^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda})$ [NS11, Cor. 2.7 and the preceding paragraph]; see also [SV09, p. 2986].

For $|\lambda| > 2$ they wrote that they could not handle this case and stated the following conjecture, first in the Oberwolfach report [SV09, p. 2986] and then in [NS11, p. 151].

Conjecture 1.1 (Nichols–Stolz [NS11, Conj. 6.2])**.** *For every $\lambda \neq 0$, $\Sigma_\lambda = \sigma(h_{\omega^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda})$.*

They note that for $|\lambda| > 2$ this would mean that Σ_λ consists of exactly six bands separated by five nonvanishing gaps, and they report numerical evidence for this computed by J. Baker. The question is also record OWR-4138-001 of the UnsolvedMath dataset [ula26]. The Oberwolfach report describes the model by the cell length $L_1 = 2$ and the single-site potential $q = \lambda\delta_1$; this is the model above. The report and the corpus record write the extremal configuration as ν^*, with $\nu^*_k = \delta_1((-1)^k)$. This is ω^* shifted by one cell, $\nu^*_k = \omega^*_{k+1}$, so $\sigma(h_{\nu^*,\lambda}) = \sigma(h_{\omega^*,\lambda})$. Put

$$x(E) := E(E-\lambda)-2 = \Big(E-\frac{\lambda}{2}\Big)^2-\Big(2+\frac{\lambda^2}{4}\Big), \qquad S_\lambda := \big\{E \in \mathbb{R} :\ |x(E)| \in [0,2]\cup[|\lambda|, \sqrt{\lambda^2+4}\,]\big\}. \tag{1}$$

Theorem 1.2. *Let $\lambda \in \mathbb{R} \setminus \{0\}$.*

(a) *$\sigma(h_{\omega,\lambda}) \subset S_\lambda$ for every $\omega \in \Omega$.*

(b) *$S_\lambda = \sigma(h_{\omega^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda})$.*

(c) *For every $p \in (0,1)$, $\Sigma_\lambda = S_\lambda = \sigma(h_{\omega^*,\lambda}) \cup \sigma(h_{\omega^1,\lambda})$. In particular, Conjecture 1.1 holds.*

(d) *If $|\lambda| > 2$, then Σ_λ is the union of six pairwise disjoint nondegenerate compact intervals, which are listed in Proposition 5.1.*

Part (a) is a deterministic statement: no configuration has spectrum in the gaps of S_λ. The proof has two parts. The first is the reduction that Nichols and Stolz used for the central gap [NS11, proof of Thm. 2.6]. Under this reduction $x(h_\omega)$ is unitarily equivalent to $(\Delta + s) \oplus (\Delta - s)$ with $s_k = \lambda(\omega_k - \omega_{k+1})$, and their positive solution gives the one-sided bound [NS11, (32)] that they need for the central gap. We reproduce both steps in Section 3. In our convention the positive solution bounds $\Delta \pm s$ from above by $\sqrt{\lambda^2+4}$, and the two-sided bound $\sigma(\Delta \pm s) \subset [-\sqrt{\lambda^2+4}, \sqrt{\lambda^2+4}\,]$ follows from the symmetry $u(k) \mapsto (-1)^k u(k)$ (Lemma 3.4). The second part is Lemma 4.2. It is a direct consequence of a standard fact about self-adjoint block operator matrices (Lemma 4.1), and it applies because s never takes the same nonzero value at two neighbouring sites. This excludes the four remaining gaps, which correspond to $|x| \in (2, |\lambda|)$.

Conjecture 6.3 of [NS11], which asks for thin (for example Lifshitz-type) tails of the integrated density of states at the eight band edges of Σ_λ other than $E_\pm$ and $G_\pm$, is not addressed here. Proposition 5.1 identifies these eight edges explicitly. We also mention work on the spectral type of related models, which does not concern the set Σ_λ. Localization for one-dimensional discrete random word models, a class that contains this model, was proved by Damanik, Sims and Stolz [DSS04]. According to its title and available summary, Chulaevsky's paper [Chu23] concerns localization for lattice displacement models. Localization for block-factor potentials is proved in [DGK25].

2 The two periodic spectra

For a potential V, solutions of $u(n+1)+u(n-1)+V(n)u(n)=Eu(n)$ satisfy $(u(n+1),u(n))^\top = T(V(n))\,(u(n),u(n-1))^\top$ with

$$T(v)=\begin{pmatrix}E-v & -1\\ 1 & 0\end{pmatrix},\qquad T(a)T(b)=\begin{pmatrix}(E-a)(E-b)-1 & -(E-a)\\ E-b & -1\end{pmatrix}.$$

If V has period N, Floquet theory gives $\sigma(h_0+V)=\{E\in\mathbb{R}:\ |\operatorname{tr}M(E)|\le 2\}$, where $M(E)$ is the product of the N transfer matrices over one period [Tes00]. The trace does not depend on the choice of the starting site, because it is invariant under cyclic permutations.

Lemma 2.1. *$\sigma(h_{\omega^1})=\{E:\ |x(E)|\le 2\}$ and $\sigma(h_{\omega^*})=\{E:\ \lambda^2\le x(E)^2\le\lambda^2+4\}$. Consequently $S_\lambda=\sigma(h_{\omega^*})\cup\sigma(h_{\omega^1})$.*

Proof. Write $w=E(E-\lambda)=x+2$. For ω^1 the period is $(0,\lambda)$ and $\operatorname{tr}T(\lambda)T(0)=w-2=x$. For ω^* a period, up to cyclic permutation, is $(0,\lambda,\lambda,0)$. With $B=T(\lambda)T(0)$ and $B'=T(0)T(\lambda)$ we get

$$\operatorname{tr}\big(T(0)T(\lambda)T(\lambda)T(0)\big)=\operatorname{tr}(B'B)=(w-1)^2-E^2-(E-\lambda)^2+1=w^2-4w+2-\lambda^2,$$

since $E^2+(E-\lambda)^2=2w+\lambda^2$. This is the discriminant $D(E)=E^4-2\lambda E^3+(\lambda^2-4)E^2+4\lambda E+2-\lambda^2$ computed in [NS11, §2.6.1]. In terms of x it equals $x^2-2-\lambda^2$, and $|x^2-2-\lambda^2|\le 2$ means $\lambda^2\le x^2\le\lambda^2+4$. The union of the two sets is $\{E:\ |x|\le 2 \text{ or } |\lambda|\le|x|\le\sqrt{\lambda^2+4}\}=S_\lambda$. $\square$

If $|\lambda|\le 2$ then $S_\lambda=\{E:\ |x(E)|\le\sqrt{\lambda^2+4}\}$. This is the set $[E_-,E_+]\setminus(G_-,G_+)$ of [NS11, Cor. 2.7], with

$$E_\pm=\frac{\lambda}{2}\pm\sqrt{2+\frac{\lambda^2}{4}+\sqrt{\lambda^2+4}},\qquad G_\pm=\frac{\lambda}{2}\pm\sqrt{2+\frac{\lambda^2}{4}-\sqrt{\lambda^2+4}},\tag{2}$$

the preimages of $x=\sqrt{\lambda^2+4}$ and of $x=-\sqrt{\lambda^2+4}$.

3 The reduction of Nichols and Stolz

For $\omega\in\Omega$ define $s=s(\omega)\colon\mathbb{Z}\to\mathbb{R}$ by

$$s_k=\lambda(\omega_k-\omega_{k+1}),\qquad k\in\mathbb{Z}.\tag{3}$$

Up to the sign of s, which is irrelevant below, this is the potential q_ω of [NS11, §5.1]. We write Δ for the operator $(\Delta f)(k)=f(k-1)+f(k+1)$ on $\ell^2(\mathbb{Z})$, and $\Delta\pm s$ for Δ plus multiplication by $\pm s$. The following is the computation in [NS11, proof of Thm. 2.6].

Proposition 3.1. *There is a unitary $U\colon\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})\oplus\ell^2(\mathbb{Z})$, which depends on neither ω nor λ, such that for every $\omega\in\Omega$*

$$U\big(h_\omega(h_\omega-\lambda)-2\big)U^*=(\Delta+s)\oplus(\Delta-s).$$

Consequently

$$x\big(\sigma(h_\omega)\big)=\sigma(\Delta+s)\cup\sigma(\Delta-s),\tag{4}$$

and in particular $E\in\sigma(h_\omega)$ implies $x(E)\in\sigma(\Delta+s)\cup\sigma(\Delta-s)$.

Proof. Let $V' = V_\omega - \lambda/2$. Then $V'(n) = \pm\lambda/2$, so $V'^2 = \lambda^2/4$, and

$$h_\omega(h_\omega - \lambda) = \Big(h_\omega - \frac{\lambda}{2}\Big)^2 - \frac{\lambda^2}{4} = h_0^2 + h_0 V' + V' h_0.$$

Let $(Su)(n) = u(n-1)$. Then $h_0 = S + S^*$ and $h_0^2 = S^2 + S^{*2} + 2$. The operator $h_0 V' + V' h_0$ has matrix elements $V'(m) + V'(n)$ for $|m-n| = 1$ and 0 otherwise. Hence

$$h_\omega(h_\omega - \lambda) - 2 = S^2 + S^{*2} + J, \qquad J = \sum_{n\in\mathbb{Z}} \tau(n)\big(\langle \delta_{n+1}, \cdot\rangle \delta_n + \langle \delta_n, \cdot\rangle \delta_{n+1}\big),$$

with $\tau(n) = V_\omega(n) + V_\omega(n+1) - \lambda$. Each cell carries the values 0 and λ once, so $\tau(2k) = 0$. Moreover $V_\omega(2k+1) = \lambda\omega_k$ and $V_\omega(2k+2) = \lambda(1-\omega_{k+1})$, so $\tau(2k+1) = s_k$.

The pairs $P_k = \{2k+1, 2k+2\}$ partition $\mathbb{Z}$. Define $Uu = (u_+, u_-)$ by

$$u_\pm(k) = \frac{u(2k+1) \pm u(2k+2)}{\sqrt{2}}, \qquad k \in \mathbb{Z};$$

this is a unitary map. The operator S^2 maps δ_{2k+1} to δ_{2k+3} and δ_{2k+2} to δ_{2k+4}. So $S^2 + S^{*2}$ acts in the same way on the two sequences $(u(2k+1))_k$ and $(u(2k+2))_k$, and $U(S^2 + S^{*2})U^* = \Delta \oplus \Delta$. The operator J acts inside each pair P_k by the matrix $\left(\begin{smallmatrix} 0 & s_k \\ s_k & 0 \end{smallmatrix}\right)$, whose eigenvectors $(1, \pm 1)/\sqrt{2}$ have eigenvalues $\pm s_k$. So UJU^* is multiplication by s on the first summand and by $-s$ on the second. This proves the identity. Since h_ω is bounded and self-adjoint, the spectral mapping theorem gives $\sigma(x(h_\omega)) = x(\sigma(h_\omega))$. Together with $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$ this gives (4). □

Remark 3.2. Identity (4) gives only one direction for a given energy. Every point of $\sigma(\Delta+s)\cup\sigma(\Delta-s)$ is $x(E)$ for *some* $E \in \sigma(h_\omega)$, but in general not for both preimages. Take the domain wall $\omega_k = 0$ for $k < 0$ and $\omega_k = 1$ for $k \geq 0$. Then $s = -\lambda\delta_{-1}$. The operator $\Delta + \mu\delta_0$ has spectrum $[-2, 2]$ together with one simple eigenvalue $\operatorname{sign}(\mu)\sqrt{\mu^2+4}$, with eigenvector $r^{|n|}$, where $r + 1/r = \operatorname{sign}(\mu)\sqrt{\mu^2+4}$ and $|r| < 1$. Hence each of $x_0 = \pm\sqrt{\lambda^2+4}$ is a simple eigenvalue of exactly one of $\Delta \pm s$ and lies outside the spectrum of the other. So $\ker(x(h_\omega) - x_0)$ is one-dimensional, and $x(\sigma(h_\omega)) = [-2, 2] \cup \{\pm\sqrt{\lambda^2+4}\}$. Let E, E' be the two preimages of x_0. Since $|x_0| > 2$, any of E, E' that lies in $\sigma(h_\omega)$ is an isolated point of $\sigma(h_\omega)$, hence an eigenvalue. For self-adjoint h_ω, $\ker(x(h_\omega) - x_0) = \ker(h_\omega - E) \oplus \ker(h_\omega - E')$. By (4) at least one of E, E' lies in $\sigma(h_\omega)$, and by the dimension count at most one does. So exactly one of E_+, E_- and exactly one of G_+, G_- lie in $\sigma(h_\omega)$. Numerically, matching decaying solutions for $\lambda \in \{-3, 1, 5/2, 3, 5\}$ shows that the infinite wall has the eigenvalues E_- and G_- for $\lambda > 0$ and E_+ and G_+ for $\lambda < 0$; on long finite domain walls with $\lambda = 3$ the numerics also pick E_- and G_-. Only the implication stated in Proposition 3.1 is used below.

Lemma 3.3. *For every ω, $s_k \in \{-|\lambda|, 0, |\lambda|\}$, and no two consecutive entries of s are equal and nonzero. (In fact the nonzero entries of s alternate in sign.) Moreover $s(1-\omega) = -s(\omega)$.*

Proof. If $s_k = s_{k+1} \neq 0$, then $\omega_k - \omega_{k+1} = \omega_{k+1} - \omega_{k+2} = \pm 1$. This forces ω_{k+1} to be both 0 and 1, which is impossible. If $s_k \neq 0$ and $j > k$ is the next index with $s_j \neq 0$, then ω is constant on $\{k+1, \dots, j\}$, so $\omega_j = \omega_{k+1}$ and $\omega_{j+1} = \omega_k$; hence $s_j = -s_k$. The last claim is clear from (3). □

The next lemma contains the bound [NS11, (32)], which Nichols and Stolz prove with the positive solution [NS11, (35), (36)]. They state the one-sided bound that they need for the central gap; the two-sided form follows from the symmetry R in the proof below. We include the short argument.

Lemma 3.4. *For every ω, $\sigma(\Delta \pm s) \subset [-\sqrt{\lambda^2+4}, \sqrt{\lambda^2+4}\,]$ and $\sigma(\Delta + s) = -\sigma(\Delta - s)$.*

Proof. Let $z = (\sqrt{\lambda^2+4}+\lambda)/2 > 0$. Then $1/z = (\sqrt{\lambda^2+4}-\lambda)/2$, so $z + 1/z = \sqrt{\lambda^2+4}$ and $z - 1/z = \lambda$. Define $\psi > 0$ on $\mathbb{Z}$ by $\psi(0) = 1$ and $\psi(k+1) = \psi(k)\, z^{2\omega_{k+1}-1}$. Then

$$\frac{\psi(k+1)+\psi(k-1)}{\psi(k)} + s_k = z^{2\omega_{k+1}-1} + z^{1-2\omega_k} + \lambda(\omega_k - \omega_{k+1}).$$

For $(\omega_k, \omega_{k+1}) = (0,0)$ and $(1,1)$ this is $z+1/z$; for $(0,1)$ it is $2z-\lambda$; for $(1,0)$ it is $2/z+\lambda$. In all four cases it equals $\sqrt{\lambda^2+4}$, so $(\Delta+s)\psi = \sqrt{\lambda^2+4}\,\psi$ pointwise. For a finitely supported $f\colon \mathbb{Z} \to \mathbb{C}$ and $u = \psi f$ this gives $((\sqrt{\lambda^2+4}-\Delta-s)u)(k) = \psi(k-1)(f(k)-f(k-1)) + \psi(k+1)(f(k)-f(k+1))$, and summation by parts gives the ground-state identity

$$\langle u, (\sqrt{\lambda^2+4}-\Delta-s)u\rangle = \sum_{k\in\mathbb{Z}} \psi(k)\psi(k+1)\,|f(k+1)-f(k)|^2 \ \geq 0.$$

Such u are dense, so $\Delta + s \leq \sqrt{\lambda^2+4}$. Applied to $1-\omega$, this gives $\Delta - s \leq \sqrt{\lambda^2+4}$ by Lemma 3.3. The unitary $(Ru)(k) = (-1)^k u(k)$ satisfies $R\Delta R^* = -\Delta$ and commutes with s, so $R(\Delta+s)R^* = -(\Delta-s)$. This proves $\sigma(\Delta+s) = -\sigma(\Delta-s)$ and $\Delta \pm s \geq -\sqrt{\lambda^2+4}$. □

4 The four inner gaps

The following fact is standard. For much finer results on self-adjoint block operator matrices whose diagonal parts have separated spectra see [LMMT01, KMM04, Tre08]; the lemma also follows from the spectral inclusion property of the quadratic numerical range [LMMT01, Tre08]. We include the short Schur-complement proof.

Lemma 4.1. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and let $H = \left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)$ be a bounded self-adjoint operator on $\mathcal{H}$. If $A \geq a$ and $D \leq d$ with $d < a$, then $\sigma(H) \cap (d,a) = \emptyset$.*

Proof. Let $t \in (d,a)$. Then $D - t \leq -(t-d) < 0$, so $D-t$ is invertible and $-(D-t)^{-1} \geq 0$. The Schur complement $\Sigma(t) = A - t - B(D-t)^{-1}B^* = (A-t) + B\big(-(D-t)^{-1}\big)B^*$ satisfies $\Sigma(t) \geq a - t > 0$, so it is invertible. The factorization

$$H - t = \begin{pmatrix} I & B(D-t)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma(t) & 0 \\ 0 & D-t \end{pmatrix} \begin{pmatrix} I & 0 \\ (D-t)^{-1}B^* & I \end{pmatrix}$$

exhibits $H - t$ as a product of three invertible operators. □

Lemma 4.2. *Let $L > 2$ and let $q\colon \mathbb{Z} \to \mathbb{R}$ be bounded. Let $S \subset \mathbb{Z}$ contain no two consecutive integers, and suppose that $q \geq L$ on S and $q \leq 0$ on $\mathbb{Z} \setminus S$. Then $\sigma(\Delta+q) \cap (2,L) = \emptyset$.*

Proof. Decompose $\ell^2(\mathbb{Z}) = \ell^2(S) \oplus \ell^2(Q)$ with $Q = \mathbb{Z} \setminus S$. Distinct points of S are not neighbours, so $\langle \delta_m, \Delta\delta_n\rangle = 0$ for $m,n \in S$. Hence the S–S block of $\Delta+q$ is multiplication by $q|_S \geq L$. For $u \in \ell^2(Q)$ we have $\langle u, (\Delta+q)u\rangle \leq \langle u, \Delta u\rangle \leq 2\|u\|^2$, because $q \leq 0$ on Q and $\|\Delta\| = 2$. So the Q–Q block is at most 2. Now apply Lemma 4.1 with $a = L$ and $d = 2$. □

Corollary 4.3. *If $|\lambda| > 2$, then for every ω*

$$\sigma(\Delta \pm s) \cap \big((-|\lambda|, -2) \cup (2, |\lambda|)\big) = \emptyset.$$

Proof. Take $q = s$ and $S = \{k :\ s_k = |\lambda|\}$. By Lemma 3.3, S contains no two consecutive integers, and $s \in \{0, -|\lambda|\}$ off S. Lemma 4.2 with $L = |\lambda|$ gives $\sigma(\Delta+s) \cap (2,|\lambda|) = \emptyset$. The same argument with $q = -s$ and $S = \{k :\ s_k = -|\lambda|\}$ gives $\sigma(\Delta - s) \cap (2,|\lambda|) = \emptyset$. The statement for $(-|\lambda|,-2)$ follows from $\sigma(\Delta \pm s) = -\sigma(\Delta \mp s)$ (Lemma 3.4). □

Remark 4.4. The hypothesis on S in Lemma 4.2 cannot be dropped. For $L > 2$ the operator $\Delta + L(\delta_0 + \delta_1)$ has the eigenvalue $L - 1 + 1/(L-1) \in (2, L)$, with the ℓ^2 eigenvector $u(n) = r^{n-1}$ for $n \geq 1$ and $u(n) = -r^{-n}$ for $n \leq 0$, where $r = 1/(L-1)$. Numerically, random potentials with values in $\{-L, 0, L\}$ that allow neighbouring $+L$ sites have many eigenvalues in $(2, L)$.

Remark 4.5 (a dynamical view; not used). The gap $(2, |\lambda|)$ of $\Delta + s$ can also be seen with an invariant cone field. Let $\lambda > 0$; the case $\lambda < 0$ reduces to this one, because $\omega \mapsto 1 - \omega$ together with $\lambda \mapsto -\lambda$ leaves s unchanged. Let $x \in (2, \lambda)$. For solutions of $(\Delta + s)u = xu$ the ratios $r_k = u(k+1)/u(k) \in \mathbb{R} \cup \{\infty\}$ satisfy $r_k = x - s_k - 1/r_{k-1}$. Let $P = [1, +\infty]$, and let $N = [x - \lambda - 1 - \eta,\ x - \lambda + \eta]$, where $\eta > 0$ is so small that $N \subset (-\infty, 0)$. Assign to the index k the interval $C_k = N$ if $s_k = \lambda$ and $C_k = P$ otherwise. We claim that $r_{k-1} \in C_{k-1}$ implies that r_k lies in the interior of C_k.

- If $C_{k-1} = P$, then $1/r_{k-1} \in [0, 1]$. So r_k lies in $[x-1, x]$ when $s_k = 0$ and in $[x + \lambda - 1, x + \lambda]$ when $s_k = -\lambda$; both lie inside P because $x > 2$. When $s_k = \lambda$, r_k lies in $[x - \lambda - 1, x - \lambda]$, which is inside N.

- If $C_{k-1} = N$, then $s_{k-1} = \lambda$, so $s_k \in \{0, -\lambda\}$ by Lemma 3.3. Since $r_{k-1} < 0$, we get $r_k > x$, and r_k is bounded above; so r_k lies inside $P = C_k$.

Such a locally constant, strictly invariant cone field gives uniform hyperbolicity of the transfer matrices (a standard cone criterion; cf. [Yoc04, ABY10]), and uniform hyperbolicity excludes x from the spectrum (see e.g. [Zha20, DF22]). We only sketch this as a cross-check; the proof above avoids dynamical tools.

5 Proof of the main theorem

Proof of Theorem 1.2. (a) Let $E \in \sigma(h_\omega)$. By Proposition 3.1, $x(E) \in \sigma(\Delta + s) \cup \sigma(\Delta - s)$. By Lemma 3.4, $|x(E)| \leq \sqrt{\lambda^2 + 4}$. If $|\lambda| > 2$, Corollary 4.3 gives $|x(E)| \notin (2, |\lambda|)$. If $|\lambda| \leq 2$ this interval is empty. In both cases $E \in S_\lambda$.

(b) This is Lemma 2.1.

(c) Since $0 < p < 1$, both values 0 and 1 have positive probability. By the periodic support theorem [NS11, Thm. 2.3], the spectrum of every periodic configuration is contained in Σ_λ; in particular $\sigma(h_{\omega^*}) \cup \sigma(h_{\omega^1}) \subset \Sigma_\lambda$. Conversely, $\Sigma_\lambda = \sigma(h_\omega)$ for almost every ω, and $\sigma(h_\omega) \subset S_\lambda$ by (a). By (b), $\Sigma_\lambda = S_\lambda = \sigma(h_{\omega^*}) \cup \sigma(h_{\omega^1})$.

(d) This is Proposition 5.1 below. □

For $|\lambda| \leq 2$ the proof of (a) uses only Proposition 3.1 and Lemma 3.4, so it reproduces the argument of Nichols and Stolz for [NS11, Thm. 2.6, Cor. 2.7]. The new input for $|\lambda| > 2$ is Corollary 4.3.

Proposition 5.1. *Let $|\lambda| > 2$ and $c = 2 + \lambda^2/4$. Consider the x-intervals*

$$J_1 = [-\sqrt{\lambda^2 + 4}, -|\lambda|], \qquad J_2 = [-2, 2], \qquad J_3 = [|\lambda|, \sqrt{\lambda^2 + 4}\,],$$

and for $i = 1, 2, 3$ the E-intervals

$$I_i^\pm = \Big\{ \frac{\lambda}{2} \pm \sqrt{c + X} : \ X \in J_i \Big\}.$$

Then $x^{-1}(J_i) = I_i^- \cup I_i^+$ and $\Sigma_\lambda = \bigcup_i (I_i^- \cup I_i^+)$. The six intervals are nondegenerate, compact and pairwise disjoint, and they are ordered as $I_3^- < I_2^- < I_1^- < I_1^+ < I_2^+ < I_3^+$. Explicitly, with $E_\pm, G_\pm$

as in (2),

$$I_3^- = \left[E_-,\ \tfrac{\lambda}{2} - \sqrt{c+|\lambda|}\right], \qquad I_3^+ = \left[\tfrac{\lambda}{2} + \sqrt{c+|\lambda|},\ E_+\right],$$
$$I_2^- = \left[\tfrac{\lambda}{2} - \sqrt{\tfrac{\lambda^2}{4}+4},\ \min\{0,\lambda\}\right], \qquad I_2^+ = \left[\max\{0,\lambda\},\ \tfrac{\lambda}{2} + \sqrt{\tfrac{\lambda^2}{4}+4}\right],$$
$$I_1^- = \left[\tfrac{\lambda}{2} - \sqrt{c-|\lambda|},\ G_-\right], \qquad I_1^+ = \left[G_+,\ \tfrac{\lambda}{2} + \sqrt{c-|\lambda|}\right].$$

The five gaps are the central gap (G_-, G_+)*, where* $x < -\sqrt{\lambda^2+4}$*; the two preimages of* $(-|\lambda|, -2)$*, between* $I_1^\pm$ *and* $I_2^\pm$*; and the two preimages of* $(2, |\lambda|)$*, between* $I_2^\pm$ *and* $I_3^\pm$*.*

Proof. By (1), $x(E) = (E - \lambda/2)^2 - c$. Since $c^2 - (\lambda^2+4) = \lambda^4/16 > 0$, we have $c > \sqrt{\lambda^2+4}$, so $c + X > 0$ for every $X \in J_1 \cup J_2 \cup J_3$. Hence each such X has exactly the two preimages $\lambda/2 \pm \sqrt{c+X}$, and $x^{-1}(J_i) = I_i^- \cup I_i^+$ with $I_i^- < \lambda/2 < I_i^+$. For $|\lambda| > 2$ the intervals $J_1 < J_2 < J_3$ are nondegenerate and pairwise disjoint, and $X \mapsto \sqrt{c+X}$ is strictly increasing. This gives the ordering and the disjointness. The explicit endpoints follow from $c - 2 = \lambda^2/4$ and $c + 2 = \lambda^2/4 + 4$. By Theorem 1.2(c), $\Sigma_\lambda = S_\lambda = x^{-1}(J_1 \cup J_2 \cup J_3)$. ☐

band	$\lambda = 3$	$\lambda = 5$
I_3^-	$[-1.3028,\ -1.1926]$	$[-1.1926,\ -1.1401]$
I_2^-	$[-1,\ 0]$	$[-0.7016,\ 0]$
I_1^-	$[0.3820,\ 0.6972]$	$[0.6972,\ 0.8074]$
I_1^+	$[2.3028,\ 2.6180]$	$[4.1926,\ 4.3028]$
I_2^+	$[3,\ 4]$	$[5,\ 5.7016]$
I_3^+	$[4.1926,\ 4.3028]$	$[6.1401,\ 6.1926]$

Table 1: The six bands of Σ_λ for $\lambda = 3$ (compare [NS11, Fig. 2]) and $\lambda = 5$, rounded to four decimals.

6 Concluding remarks

Theorem 1.2 shows that Σ_λ does not depend on $p \in (0,1)$, and that every configuration has its spectrum inside the six bands (two bands if $|\lambda| \le 2$). Individual spectra can be smaller, as for ω^1 itself or for the domain wall of Remark 3.2. The eight band edges of Σ_λ other than $E_\pm$ and $G_\pm$, at which Conjecture 6.3 of [NS11] predicts thin tails of the integrated density of states, are

$$\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4}+4}, \qquad 0, \qquad \lambda, \qquad \frac{\lambda}{2} \pm \sqrt{2 + \frac{\lambda^2}{4} + |\lambda|}, \qquad \frac{\lambda}{2} \pm \sqrt{2 + \frac{\lambda^2}{4} - |\lambda|}.$$

That conjecture remains open.

Verification. The proofs above are complete without computer assistance. The claims were also checked as follows. None of these computations is used in the proofs.

1. Programs of the finder, all in exact rational arithmetic unless stated otherwise.
   - `reduction_exact.py` checks the identity of Proposition 3.1 on rings of $K = 3, \ldots, 8$ cells, for all $\omega \in \{0,1\}^K$ and $\lambda \in \{3, 5/2, -7/3, 1, 8/3\}$ (2520 cases).
   - `positive_solution_exact.py` checks the four local cases and the ground-state identity of Lemma 3.4 for $\lambda = t - 1/t$ with $t \in \{2, 3, 4, 7/2, 11/5\}$.

- `periodic_words_exact.py` works in the original model, independently of the proof. It takes each value $\lambda \in \{3, 5, 8/3, 21/10, -3, 12\}$ and each of the 2615 periodic configurations with period at most 14 cells (binary necklaces). For each of them it checks that the Floquet discriminant is a polynomial in $x(E)$ whose modulus exceeds 2 on the open intervals $2 < |x| < |\lambda|$. It certifies this by Descartes' rule of signs with bisection, and negative controls check that the test does detect spectrum when the intervals are widened.
- `random_numeric.py` (floating point) finds no eigenvalue in the gaps for random configurations on rings of 800 sites, and it finds eigenvalues in $(2, L)$ when neighbouring $+L$ sites are allowed.

2. An independent verifier re-derived every step by hand, reran the programs above (the necklace check up to 11 cells) with the same results, and wrote separate code.

   - An exact inertia test of Lemma 4.2 on 3600 random admissible Dirichlet segments. The number of negative pivots always equals $|Q|$, as the Schur-complement proof predicts.
   - Floquet–Bloch numerics in the original model for random periods of 3 to 40 cells and $\lambda \in \{2.2, 3, 4, 7, -5\}$.
   - All words of at most 12 cells for $\lambda = 3$ at three Bloch phases (24,570 matrices).
   - The domain wall of Remark 3.2.

3. The author reran all programs of item 1, including the necklace check up to 14 cells, with identical results. The author's program `paper_checks.py` checks the trace identities of Lemma 2.1 exactly, as polynomial identities. It also checks the band formulas of Proposition 5.1 and Table 1 (in floating point), the example of Remark 4.4 (exactly), and the domain wall of Remark 3.2.

4. A second independent referee re-checked every step by hand and wrote separate programs before reading those above.

   - `ref2_exact.py` (exact rational arithmetic) works in the original model, for all 2615 binary necklaces with at most 14 cells and the seven values $\lambda = 8/3$, $15/4$, $24/5$, $21/10$, $-8/3$, $3/2$, $9/20$, two of which satisfy $|\lambda| \leq 2$. It certifies by Sturm sequences in x that no energy of the periodic spectrum has $x(E)$ in any of the x-intervals $|x| > \sqrt{\lambda^2 + 4}$ (the central gap and the exterior of $[E_-, E_+]$) and $2 < |x| < |\lambda|$ (the four inner gaps, when $|\lambda| > 2$). The test is exact because $\sqrt{\lambda^2 + 4}$ is rational for $\lambda = t - 1/t$. The same program checks the reduction of Proposition 3.1 at the level of Floquet discriminants for every necklace, and the identity of Proposition 3.1 exactly on rings of 3 to 7 cells (1240 cases). Negative controls are detected.
   - `ref2_numeric.py` (floating point) repeats the random-ring and random-period tests, recomputes Table 1 from Floquet–Bloch spectra, and checks the domain wall of Remark 3.2 and the example of Remark 4.4.

Scope and priority. Theorem 1.2 proves Conjecture 6.2 of Nichols and Stolz [NS11], which is the question in the Oberwolfach report [SV09] and in record OWR-4138-001 of [ula26], for every $\lambda \neq 0$ and every $p \in (0, 1)$. The reduction (Proposition 3.1) and the outer bound (Lemma 3.4) are due to Nichols and Stolz. Lemma 4.1 is standard. What we add is the observation that Lemma 4.1 applies to the reduced operators because of Lemma 3.3, which excludes the four remaining gaps and gives the inclusion of Theorem 1.2(a) for every configuration. Experts may well regard this as easy once the reduction is in hand. Conjecture 6.3 of [NS11] is not addressed. We searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar, and ran three independent web searches. A second referee repeated the arXiv, OpenAlex, Semantic Scholar, Crossref and zbMATH checks and ran one more web search; nothing new was found. OpenAlex lists three works citing [NS11]: two papers of H. Krüger on skew-shift operators, which concern a different question, and Chulaevsky's paper [Chu23]. The full text of [Chu23] is behind a paywall, and we have still not read it. Its title and available description indicate a localization result for lattice displacement models with Bernoulli displacements, not a result on the almost-sure spectrum. The thesis of R. Nichols [Nic10] predates [NS11], which still states the conjecture as open; we did not read it. We found no proof or disproof of the conjecture. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

[ABY10] Artur Avila, Jairo Bochi, and Jean-Christophe Yoccoz. Uniformly hyperbolic finite-valued $\mathrm{SL}(2,\mathbb{R})$-cocycles. *Commentarii Mathematici Helvetici*, 85:813–884, 2010. doi:10.4171/CMH/212.

[Chu23] Victor Chulaevsky. Anderson localization in discrete random displacements models. *Journal of Statistical Physics*, 190(1):Paper No. 5, 2023. doi:10.1007/s10955-022-03020-3.

[DF22] David Damanik and Jake Fillman. *One-Dimensional Ergodic Schrödinger Operators. I. General Theory*, volume 221 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2022. doi:10.1090/gsm/221.

[DGK25] David Damanik, Anton Gorodetski, and Victor Kleptsyn. Localization for random Schrödinger operators defined by block factors. Preprint, arXiv:2504.08153, 2025.

[DSS04] David Damanik, Robert Sims, and Günter Stolz. Localization for discrete one-dimensional random word models. *Journal of Functional Analysis*, 208(2):423–445, 2004. doi:10.1016/j.jfa.2003.07.011.

[KMM04] Vadim Kostrykin, Konstantin A. Makarov, and Alexander K. Motovilov. A generalization of the $\tan 2\theta$ theorem. In *Current Trends in Operator Theory and its Applications*, volume 149 of *Operator Theory: Advances and Applications*, pages 349–372. Birkhäuser, Basel, 2004. doi:10.1007/978-3-0348-7881-4_14.

[LMMT01] Heinz Langer, Alexander Markus, Vladimir Matsaev, and Christiane Tretter. A new concept for block operator matrices: the quadratic numerical range. *Linear Algebra and its Applications*, 330:89–112, 2001. doi:10.1016/S0024-3795(01)00230-0.

[Nic10] Roger Nichols. *Spectral Properties of Structurally Disordered Media*. PhD thesis, University of Alabama at Birmingham, 2010.

[NS11] Roger Nichols and Günter Stolz. Spectral properties of the discrete random displacement model. *Journal of Spectral Theory*, 1(2):123–153, 2011. Preprint arXiv:1008.1266. doi:10.4171/JST/6.

[SV09] Günter Stolz and Ivan Veselić. Mini-Workshop: Modeling and Understanding Random Hamiltonians: Beyond Monotonicity, Linearity and Independence. *Oberwolfach Reports*, 6(4):2965–2992, 2009. Report No. 55/2009 (organisers' report); abstract of R. Nichols (joint work with G. Stolz), *Some spectral properties of discrete displacement operators*, pp. 2985–2986. `doi:10.4171/OWR/2009/55`.

[Tes00] Gerald Teschl. *Jacobi Operators and Completely Integrable Nonlinear Lattices*, volume 72 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000. `doi:10.1090/surv/072`.

[Tre08] Christiane Tretter. *Spectral Theory of Block Operator Matrices and Applications*. Imperial College Press, London, 2008. `doi:10.1142/p493`.

[ula26] ulamai. UnsolvedMath. Hugging Face dataset, `https://huggingface.co/datasets/ulamai/UnsolvedMath`, 2026. Version 1.6.0; record OWR-4138-001.

[Yoc04] Jean-Christophe Yoccoz. Some questions and remarks about $\mathrm{SL}(2, \mathbf{R})$ cocycles. In *Modern Dynamical Systems and Applications*, pages 447–458. Cambridge University Press, Cambridge, 2004.

[Zha20] Zhenghe Zhang. Uniform hyperbolicity and its relation with spectral analysis of 1D discrete Schrödinger operators. *Journal of Spectral Theory*, 10(4):1471–1517, 2020. `doi:10.4171/JST/333`.