

Rademacher's Half-Ball and Cone Counterexample in Three Dimensions

Alper Ferudun*

13 October 2026

Abstract

Rademacher proposed a half-ball joined to a cone, with a small tip truncation, as a counterexample to monotonicity of the expected volume of a random tetrahedron under inclusion. We prove that this specific construction works in dimension three, with cone height 10^{-5} and truncation height 10^{-10} . The proof reduces the half-ball expectation to a two-dimensional plane-section integral. A finite lower Darboux sum, evaluated using integer-only outward interval arithmetic, certifies that the expectation exceeds its origin-anchored counterpart $9\pi/1024$ by more than $1/6000$. A quantitative mixture argument transfers this gap to the explicit nested bodies. The certificate source and complete row sums are included for reproduction. General three-dimensional nonmonotonicity was already proved by Kunis, Reichenwallner and Reitzner using a tetrahedron; the result here verifies Rademacher's particular half-ball construction, without evaluating its expectation in closed form.

1 The construction

For a convex body $B \subset \mathbb{R}^3$ with nonempty interior, define

$$F(B) = \mathbb{E} \operatorname{vol}_3 \operatorname{conv}(X_0, X_1, X_2, X_3),$$

where the four points are independent and uniform in B . The volume is the ordinary, unnormalized Euclidean volume. Let

$$H = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, z \geq 0\}, \quad V = \operatorname{vol}_3(H) = \frac{2\pi}{3}.$$

Theorem 1. *Put $\varepsilon = 10^{-5}$, $\delta = 10^{-10}$ and*

$$L = \operatorname{conv}(H \cup \{-\varepsilon e_3\}), \quad K = L \cap \{z \geq -\varepsilon + \delta\}.$$

Then $K \subsetneq L$ are convex bodies and $F(K) > F(L)$.

Rademacher described this construction in an Oberwolfach talk [2, p. 2868] and in [1, p. 78], proving the higher-dimensional analogue and reporting numerical evidence in dimension three. Kunis, Reichenwallner and Reitzner subsequently settled the general three-dimensional monotonicity question using a tetrahedron and a polynomial approximation argument [3]. We do not claim that general result as new. Our purpose is to verify the particular half-ball construction with explicit

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

parameters. The standard plane change of variables and determinant-moment identity are derived below to fix their normalizations. No absolute priority claim is made.

Write

$$G_B(y) = \mathbb{E} \operatorname{vol}_3 \operatorname{conv}(y, X_1, X_2, X_3),$$

where the three random points are uniform in B . The central estimate is the following computer-assisted bound.

Proposition 2. *For the unit half-ball,*

$$G_H(0) = \frac{9\pi}{1024}, \quad F(H) - G_H(0) > \frac{1}{6000}.$$

The equality is elementary. If Y is uniform in the full unit ball, then $\operatorname{sign}(Y_3)Y$ is uniform in H , up to a null set. Applying this independently to three vectors preserves the absolute determinant. In the polar decomposition $Y = RU$, $\mathbb{E}R = 3/4$. For three independent uniform unit directions,

$$\mathbb{E}|\det(U_1, U_2, U_3)| = \mathbb{E} \sin \angle(U_1, U_2) \mathbb{E}|U_3 \cdot e_3| = \frac{\pi}{4} \cdot \frac{1}{2}.$$

Thus $G_H(0) = \frac{1}{6}(\frac{3}{4})^3 \frac{\pi}{8} = 9\pi/1024$. Sections 2–4 prove the strict inequality. Section 5 completes Theorem 1.

2 Reduction to plane sections

An unoriented affine plane is represented uniquely, apart from a null set, by $u \cdot x = h$ with u in the upper unit hemisphere and $h \in \mathbb{R}$. In orthonormal plane coordinates (s, t) set

$$D(x_1, x_2, x_3) = \det \begin{pmatrix} s_1 & t_1 & 1 \\ s_2 & t_2 & 1 \\ s_3 & t_3 & 1 \end{pmatrix}.$$

The affine plane change of variables for three points is

$$dx_1 dx_2 dx_3 = |D| d\sigma(u) dh d^2 x_1 d^2 x_2 d^2 x_3. \quad (1)$$

Indeed, take a local normal chart at $u = e_3$ and a tangent frame whose derivatives there are normal. After eliminating the six in-plane coordinate columns, the remaining differential has rows $(-s_i, -t_i, 1)$ for the two normal parameters and h . Its absolute determinant is $|D|$. At that normal the chart area agrees with spherical area, and rotation proves the formula everywhere. The hemisphere convention counts each plane once. Degenerate triples are a null set.

The tetrahedron volume with fourth point x_0 is $|D| |u \cdot x_0 - h|/6$. For a section $A = H \cap \{u \cdot x = h\}$, choose s so that the section is symmetric in t , and write

$$A_0 = \int_A 1, \quad m = \int_A s, \quad S = \int_A s^2, \quad T = \int_A t^2.$$

The first t -moment and the mixed moment vanish. Expanding the squared determinant gives

$$\int_{A^3} D^2 = 3! \det \begin{pmatrix} S & 0 & m \\ 0 & T & 0 \\ m & 0 & A_0 \end{pmatrix} = 6T(A_0 S - m^2).$$

Consequently, with $W = T(A_0S - m^2)$ and $J(u, h) = \int_H |u \cdot x - h| dx$, we obtain

$$F(H) = \frac{1}{V^4} \int_{u_3 \geq 0} \int_{-1}^1 W(u, h) J(u, h) dh d\sigma(u). \quad (2)$$

Put $\mu = u_3 \in [0, 1]$. Azimuthal integration gives $2\pi d\mu$. For $h \geq 0$ set $r = \sqrt{1 - h^2}$ and

$$q = \min \left(1, \frac{\mu h}{\sqrt{(1 - \mu^2)(1 - h^2)}} \right). \quad (3)$$

Choose the in-plane coordinate so that $z = \mu h + \sqrt{1 - \mu^2} s$. The normalized sections at offsets h and $-h$ are the unit disk intersected with $s \geq -q$ and $s \geq q$, respectively. Denote their W values by $w_+(q)$ and $w_-(q)$. Since A_0, m, S, T scale as r^2, r^3, r^4, r^4 , the section weight scales as r^{10} . Thus

$$F(H) = \frac{2\pi}{V^4} \int_0^1 \int_0^1 (1 - h^2)^5 [w_+(q) J_+(\mu, h) + w_-(q) J_-(\mu, h)] dh d\mu, \quad (4)$$

where $J_+(\mu, h) = J(u, h)$ and $J_-(\mu, h) = J(u, -h)$. Boundary values are understood by continuity wherever unique; exceptional corners do not affect the integral or the cell bounds used below.

3 Elementary formulas for the integrand

For $q \in [0, 1]$, put

$$b = \frac{\pi}{2} + \arcsin q, \quad t = q\sqrt{1 - q^2}, \quad f = \frac{4}{9}(1 - q^2)^3.$$

Integration of the vertical chords of the disk gives

$$\begin{aligned} A_+ &= b + t, & S_+ &= \frac{b - t(1 - 2q^2)}{4}, & T_+ &= \frac{b}{4} + \frac{t(5 - 2q^2)}{12}, \\ A_- &= \pi - A_+, & S_- &= \frac{\pi}{4} - S_+, & T_- &= \frac{\pi}{4} - T_+, \\ w_{\pm} &= T_{\pm}(A_{\pm}S_{\pm} - f). \end{aligned} \quad (5)$$

The first s -moment of each positive-side cap is $2(1 - q^2)^{3/2}/3$. The complementary cap has the opposite first moment, and both squares equal f .

Lemma 3. *The functions w_+ and w_- are nonnegative. As q increases, w_+ is nondecreasing and w_- is nonincreasing.*

Proof. For any section,

$$A_0S - m^2 = \frac{1}{2} \int_A \int_A (s - s')^2 dx dx'.$$

This and $T = \int_A t^2$ are nonnegative and nondecreasing under inclusion. The positive-offset cap increases and the negative-offset cap decreases with q . $\square$

We next give closed formulas for $J_{\pm}$. Put $s_0 = \sqrt{1 - \mu^2}$ and $r^2 = 1 - h^2$. Let $C(\mu, h)$ be the volume of $\{x \in H : u \cdot x \geq h\}$ and $M(\mu, h)$ its first moment in the u direction. Since $\int_H u \cdot x dx = \mu\pi/4$,

$$\begin{aligned} J_+ &= Vh - \frac{\mu\pi}{4} + 2(M - hC), \\ J_- &= \pi \left(\frac{1}{2} + h^2 - \frac{h^4}{6} \right) - J_+. \end{aligned} \quad (6)$$

The second identity follows by joining H to $-H$ and integrating the full ball's section areas $\pi(1-t^2)$.

If $h \geq s_0$, the cap is wholly above the equatorial plane, and

$$C = \pi(2/3 - h + h^3/3), \quad M = \pi r^4/4. \quad (7)$$

For $0 \leq h < s_0$, use the unclipped expression for q in (3), and define

$$\begin{aligned} A &= r^2(\pi/2 + \arcsin q + q\sqrt{1-q^2}), \\ B &= \pi + 2\arcsin(\mu/r) - h(\pi + 2\arcsin q), \\ C &= (B - hA)/3, \\ M &= \frac{r^2 A}{4} + \frac{\mu}{4s_0^2} \left[s_0^2(\pi/2 - \arcsin(h/s_0)) - h\sqrt{s_0^2 - h^2} \right]. \end{aligned} \quad (8)$$

Lemma 4. *Equations (7) and (8) give the cap volume and first moment used in (6).*

Proof. The full-cap formulas are direct integrals. For the cut cap, A is the section area. Put $k = \sqrt{s_0^2 - h^2}$ and $b = \pi/2 + \arcsin q$. For fixed $0 < \mu < 1$,

$$q' = \frac{\mu}{s_0 r^3}, \quad b' = \frac{\mu}{r^2 k}, \quad (\arcsin(\mu/r))' = \frac{\mu h}{r^2 k}.$$

It follows that $B' = -2b$ and

$$A' = -2h(b + q\sqrt{1-q^2}) + \frac{2\mu k}{s_0^2 r^2}.$$

Substitution gives $(B - hA)' = -3A$. The derivative of $r^2 A/4$ is $-hA + \mu k/(2s_0^2)$, while that of the expression in square brackets in (8) is $-2k$. Thus $C' = -A$ and $M' = -hA$. At $h = s_0$, these formulas continuously agree with (7), proving the claim by integration from that endpoint. At $\mu = 0$ the identities follow directly or by continuity. At $\mu = 1$ and $h > 0$, (7) applies. $\square$

4 The integer interval certificate

We bound the integral by a finite lower Darboux sum. Set $N = 1024$. For the cell with indices $0 \leq i, j < N$, use endpoints

$$\mu_l = i/N, \quad \mu_r = (i+1)/N, \quad h_l = j/N, \quad h_r = (j+1)/N,$$

and midpoints μ_c, h_c . Define

$$\begin{aligned} c_i &= \sqrt{(\mu_r - \mu_l)^2 + \left(\sqrt{1 - \mu_l^2} - \sqrt{1 - \mu_r^2} \right)^2}, \\ e_i &= \frac{V}{2N} + \frac{\pi}{4} c_i. \end{aligned} \quad (9)$$

For every vector v , $\int_H |v \cdot x| dx = \pi \|v\|/4$: the integrand is even under $x \mapsto -x$, and the full-ball integral is $\pi \|v\|/2$. The reverse triangle inequality therefore gives

$$|J(u, h) - J(u', h')| \leq V|h - h'| + \frac{\pi}{4} \|u - u'\|.$$

Both coordinates of $u(\mu) = (\sqrt{1-\mu^2}, 0, \mu)$ vary monotonically, so c_i bounds $\|u(\mu) - u(\mu_c)\|$ on the cell. The error e_i consequently bounds the variation of either J_+ or J_- from its center value.

The function q is nondecreasing in each variable. By Lemma 3, the integrand in (4), including its factor $(1-h^2)^5$, is at least

$$B_{ij} = (1-h_r^2)^5 [w_+(q(\mu_l, h_l)) \max(0, J_+(\mu_c, h_c) - e_i) + w_-(q(\mu_r, h_r)) \max(0, J_-(\mu_c, h_c) - e_i)]. \quad (10)$$

Thus

$$F(H) \geq \frac{2\pi}{V^4 N^2} \sum_{i,j} B_{ij}. \quad (11)$$

This inequality controls the entire cells. It does not assume that a midpoint quadrature approximates the integral to a prescribed accuracy.

The supplied program `certify_halfball.cpp` evaluates (5)–(11) using intervals whose endpoints are integers divided by $S = 2^{48}$. Its arithmetic has the following specification.

1. Addition and subtraction are exact on the fixed-denominator endpoints. Multiplication and division use signed 128-bit intermediates and round the minimum down and the maximum up. All conversions to 64-bit endpoints are range-checked.
2. Square roots use integer square roots of the scaled arguments, rounding the lower endpoint down and upper endpoint up.
3. For $0 \leq x \leq 1$, use

$$\arcsin x = 4 \arctan \frac{t}{1 + \sqrt{1+t^2}}, \quad t = \frac{x}{1 + \sqrt{1-x^2}}.$$

The arctangent argument is at most $\sqrt{2} - 1 < 1/2$. Its 32-term alternating series, ending in $-x^{63}/63$, is a lower bound, and the next term $x^{65}/65$ bounds the remainder from above. Every term is evaluated with outward intervals. Monotonicity gives the arcsine enclosure from its endpoint evaluations.

4. The same series and Machin's identity $\pi = 16 \arctan(1/5) - 4 \arctan(1/239)$ enclose π . Intersections with nonnegative or unit intervals are used only for quantities whose ranges follow from the preceding geometric formulas.

There is no floating-point arithmetic in this certificate. The recorded full-grid output is

$$\begin{aligned} S &= 281474976710656, \\ S\pi &\in [884279719003228, 884279719003888], \\ \sum_{i,j} \underline{SB}_{ij} &= 25114698084089425520, \\ F(H) &\geq \frac{7821201271174}{S}, \quad G_H(0) \leq \frac{7771989717808}{S}. \end{aligned} \quad (12)$$

Here $\underline{SB}_{ij}$ denotes the stored integer lower endpoint. The program divides their sum by N^2 with another downward rounding, then multiplies by an outward interval for $2\pi/V^4$. This preserves a valid lower bound. Finally,

$$\frac{7821201271174 - 7771989717808}{281474976710656} = \frac{49211553366}{281474976710656} > \frac{1}{6000}.$$

This completes the computer-assisted proof of Proposition 2.

Reproducibility and diagnostic checks

The supplementary source archive contains the C++ program, all 1024 row sums, a Python interval replay and build instructions. Ordinary and undefined-behavior-instrumented complete C++ runs give identical 20767-byte output. Its SHA-256 digest is

218a49b96d8a91c0a8e531241d00d731427
cdc811ad922f0b34109faf0b2de81

The separate Python implementation replays eight entire production rows and all 256 cells of a 16×16 grid, for 8448 cells total, using arbitrary-precision integers. It tests the interval primitives against exact rational endpoints and recomputes the final scalar bound. The coarse grid does not establish the desired inequality, as expected. Normal and optimized Python runs agree.

Further diagnostics compare the distance formula with independent one-dimensional section quadrature in 30 cases and check that the plane reduction gives the known full-ball expectation $12\pi/715$. Ordinary quadrature gives approximately 0.0280956 for $F(H)$, consistent with a two-million-sample Monte Carlo estimate. None of these floating-point diagnostics is used to prove the strict inequality. The certification rests on the written reduction, the Darboux bound and the full directed-integer computation.

5 From the gap to explicit nested bodies

We now prove Theorem 1 using only the certified gap and elementary finite estimates. The union of H and the circular cone of base radius 1 at $z = 0$ and apex $-\varepsilon e_3$ is convex. Indeed, its section radius is

$$\rho(z) = \begin{cases} 1 + z/\varepsilon, & -\varepsilon \leq z \leq 0, \\ \sqrt{1 - z^2}, & 0 \leq z \leq 1, \end{cases}$$

a nonnegative concave function. This union is therefore L . We have $H \subset K \subsetneq L$, and L lies in the unit ball because its generators do.

Any tetrahedron whose vertices lie in the unit ball has volume at most $M_0 = 4/3$: each of three edges from a vertex has length at most 2. Replacing a vertex 0 by y changes the volume by at most $2\|y\|/3$, since the two edge vectors of the opposite face have lengths at most 2.

The uniform measure on K is a mixture of the uniform measure on H and the added region, of mass

$$\alpha = \frac{|K| - |H|}{|K|} \leq \frac{\varepsilon}{2},$$

because the whole added cone has volume $\pi\varepsilon/3$ and $|K| \geq 2\pi/3$. Coupling four samples and using the union bound for at least one added-region sample yields

$$F(K) \geq F(H) - 4M_0\alpha \geq F(H) - \frac{8\varepsilon}{3}. \quad (13)$$

Let $C = L \setminus K$, ignoring its null boundary. Its points have horizontal radius at most $\delta/\varepsilon = \varepsilon$ and $|z| \leq \varepsilon$, so $\|y\| \leq \sqrt{2}\varepsilon < 2\varepsilon$ for $y \in C$. Coupling three samples in K to samples in H , and then replacing y by 0, gives uniformly on C

$$G_K(y) \leq G_H(0) + 3M_0\alpha + \frac{2}{3}\|y\| \leq G_H(0) + \frac{10\varepsilon}{3}. \quad (14)$$

By Proposition 2, (13) and (14),

$$F(K) - G_K(y) > \frac{1}{6000} - 6\varepsilon = \frac{1}{9375} > \frac{1}{10000} =: g. \quad (15)$$

The cap probability in a uniform L sample is exactly

$$p = \frac{|C|}{|L|} = \frac{\delta^3}{\varepsilon^2(2 + \varepsilon)} = \frac{1}{2000010000000000000000} < 10^{-20}.$$

For four independent samples the probability of exactly one point in C is $4p(1-p)^3$, while the probability of at least two is at most $6p^2$. On the first event, the conditional expected volume is at most $F(K) - g$ by (15). On the second it is at most M_0 . On the event of no cap point it is $F(K)$. Since $F(K) \geq 0$, we obtain

$$F(L) \leq F(K) - 4p(1-p)^3g + 6p^2M_0 = F(K) - 4p(1-p)^3g + 8p^2 < F(K).$$

For the last strict inequality, $p < 1/2$ gives $4(1-p)^3g > g/2 = 1/20000 > 8p$. This proves Theorem 1.

AI tools assisted with the derivation, code and exposition. The argument and computation have been self-audited; this unrefereed preprint does not claim independent review or proof-assistant verification.

References

- [1] L. Rademacher, *On the monotonicity of the expected volume of a random simplex*, *Mathematika* **58** (2012), 77–91. [doi:10.1112/S0025579311002063](https://doi.org/10.1112/S0025579311002063).
- [2] L. Rademacher, contribution to *Convex Geometry and its Applications*, Oberwolfach Reports **6** (2009), 2851–2926, at 2867–2869. [doi:10.4171/owr/2009/53](https://doi.org/10.4171/owr/2009/53).
- [3] S. Kunis, B. Reichenwallner and M. Reitzner, *Monotonicity of the Sample Range of 3-D Data: Moments of Volumes of Random Tetrahedra*, 2016. [arXiv:1612.01893](https://arxiv.org/abs/1612.01893).