

A Centering Counterexample for Prékopa–Leindler Stability

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26 October 2026

Abstract

A conjecture printed in the 2009 Oberwolfach report on convex geometry proposes a quantitative Prékopa–Leindler inequality in the unshifted L^1 distance of probability densities with zero mean. We give an explicit counterexample in every dimension. Positive real-analytic densities with mean zero and covariance I_n have L^1 distance at least $11/8$, while an admissible midpoint majorant has integral $1 + 2k/(1 + k^2)$, for every integer $k \geq 3$. A related family works for each interpolation parameter in $(0, 1)$, uniformly away from the endpoints. Small remote Gaussian components cancel the barycenters without aligning the main mass. An explicit translation makes the distance at most $2/(1 + k^2)$, so the construction is consistent with the established stability theorems modulo translation. The result concerns the literal centered formulation; no contradiction to those theorems or historical priority claim is intended.

1 The printed question

For $0 < \lambda < 1$, suppose that nonnegative integrable functions f, g, m on $\mathbb{R}^n$ satisfy

$$m(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda} \quad (x, y \in \mathbb{R}^n), \quad (1)$$

and that f, g are probability densities with zero mean. Writing

$$\delta_1(f, g) = \min\{1, \|f - g\|_{L^1(\mathbb{R}^n)}\},$$

Conjecture 1 in Böröczky’s report [1, p. 2863] asks for

$$\int_{\mathbb{R}^n} m \geq 1 + \gamma_n \lambda \delta_1(f, g)^2 \quad (2)$$

with a positive constant depending only on n . The paragraph specifying this conjecture does not require log-concavity; the following Theorem 5 explicitly adds a log-concavity hypothesis on m . We address the assumptions as printed, without inferring additional ones. The record OWR-4136-004 in UnsolvedMath v1.6.0 points to this question.

The role of translation must be distinguished from the sharp exponent. Ball–Böröczky [2, Theorem 1.2] permit translations in their stability conclusion under a log-concavity hypothesis. The general sharp result of Figalli–van Hintum–Tiba [3, Corollary 1.7] also permits translation. Our example shows that zero barycenters alone do not remove that freedom, even after a common normalization of the covariance.

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2 An explicit Gaussian construction

Set $\varphi_n(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)$ and let e be the first coordinate vector. All integrals below are over $\mathbb{R}^n$ unless specified.

Lemma 1 (Mixture majorants). *Let (a_i) and (b_j) be finite positive probability vectors, and let $u_i, v_j \in \mathbb{R}^n$. Define*

$$f(x) = \sum_i a_i \varphi_n(x - u_i), \quad g(x) = \sum_j b_j \varphi_n(x - v_j).$$

For any $0 < \lambda < 1$, the function

$$m_\lambda(z) = \sum_{i,j} a_i^\lambda b_j^{1-\lambda} \varphi_n(z - \lambda u_i - (1-\lambda)v_j)$$

satisfies (1), and

$$\int m_\lambda = \left(\sum_i a_i^\lambda \right) \left(\sum_j b_j^{1-\lambda} \right).$$

Proof. For $A_i \geq 0$ and $0 < \alpha < 1$, one has $(\sum_i A_i)^\alpha \leq \sum_i A_i^\alpha$. For two terms this follows by differentiating $1 + t^\alpha - (1+t)^\alpha$, whose derivative is nonnegative; induction gives the finite version. Apply this to both factors in $f(x)^\lambda g(y)^{1-\lambda}$. Each resulting term is bounded using the log-concavity of φ_n :

$$\varphi_n(x - u_i)^\lambda \varphi_n(y - v_j)^{1-\lambda} \leq \varphi_n(\lambda x + (1-\lambda)y - \lambda u_i - (1-\lambda)v_j).$$

Integration of the finite sum proves the identity. $\square$

Theorem 2. *For every $n \geq 1$ and integer $k \geq 3$ there are strictly positive, real-analytic functions F_k, G_k, M_k satisfying (1) at $\lambda = 1/2$, such that F_k, G_k are probability densities with mean zero and covariance I_n , and*

$$\|F_k - G_k\|_1 \geq \frac{11}{8}, \quad \int M_k = 1 + \frac{2k}{1+k^2}. \quad (3)$$

All three functions have finite moments of every order.

Proof. Write $d = 1 + k^2$, $p = k^2/d$, and $q = 1/d$. Before normalizing covariance, put

$$f_k(x) = p\varphi_n(x - 4e) + q\varphi_n(x + 4k^2e), \quad (4)$$

$$g_k(x) = p\varphi_n(x + 4e) + q\varphi_n(x - 4k^2e),$$

$$m_k(z) = \varphi_n(z) + \frac{k}{d} [\varphi_n(z - 2de) + \varphi_n(z + 2de)]. \quad (5)$$

Lemma 1 at $\lambda = 1/2$ gives the pointwise inequality. Indeed, the two diagonal pairs have midpoint zero and weights p and q ; the other two have midpoints $\pm 2de$ and weight $\sqrt{pq} = k/d$. Thus (5) is exactly the majorant from that lemma, and $\int m_k = 1 + 2k/d$.

The functions f_k, g_k have mass one and mean zero, since $4p - 4k^2q = 0$. Their covariance matrices coincide and equal

$$\Sigma_k = \text{diag}(1 + 16k^2, 1, \dots, 1). \quad (6)$$

For the first diagonal entry, compute $1 + 16p + 16k^4q = 1 + 16k^2$; all other entries follow from the standard Gaussian covariance and the centers lying on the first axis.

For the L^1 bound, let $H = \{x : x_1 > 0\}$ and $Z \sim N(0, 1)$. Markov's inequality gives $\mathbb{P}(Z \leq -4) \leq 1/16$ and $\mathbb{P}(Z > 4) \leq 1/16$. Consequently, the difference of the H -masses of

the two main components in (4) is at least $7/8$. The corresponding difference for the remote components is at least -1 . Therefore

$$\int_H (f_k - g_k) \geq \frac{7}{8}p - q \geq \frac{11}{16},$$

where the last bound uses $p \geq 9/10$ and $q \leq 1/10$. Since $\int (f_k - g_k) = 0$, this implies $\|f_k - g_k\|_1 \geq 2 \int_H (f_k - g_k) \geq 11/8$.

Finally, let $A_k = \Sigma_k^{1/2}$, $a_k = \det A_k$, and define

$$F_k(x) = a_k f_k(A_k x), \quad G_k(x) = a_k g_k(A_k x), \quad M_k(x) = a_k m_k(A_k x).$$

The common change of variables preserves masses, the L^1 distance and the integral of the majorant. It preserves (1) because $a_k^\lambda a_k^{1-\lambda} = a_k$. It sends the two covariance matrices to I_n and leaves their means zero. Finite Gaussian mixtures and their invertible linear transforms are strictly positive and real analytic, with all moments finite. $\square$

Corollary 3. *The centered estimate (2) is false, even for positive analytic isotropic densities and the fixed parameter $\lambda = 1/2$.*

Proof. Theorem 2 gives $\delta_1(F_k, G_k) = 1$. For any proposed $\gamma_n > 0$, choose an integer $k > \max\{3, 4/\gamma_n\}$. Then

$$\int M_k - 1 = \frac{2k}{1+k^2} < \frac{2}{k} < \frac{\gamma_n}{2},$$

contradicting (2). $\square$

In particular, no positive lower bound depending only on the dimension and this unshifted truncated distance can hold for the stated class. The suggested one-dimensional linear estimate on the same source path also fails for this sequence.

3 Other parameters and the aligning translation

The same densities F_k, G_k also work for each fixed $0 < \lambda < 1$. Apply Lemma 1 to the weights (p, q) and centers in (4), followed by the same linear normalization. The resulting majorant $M_{k,\lambda}$ satisfies

$$\int M_{k,\lambda} - 1 = p^\lambda q^{1-\lambda} + q^\lambda p^{1-\lambda}. \quad (7)$$

For $0 < \tau \leq 1/2$ and $\lambda \in [\tau, 1 - \tau]$, the right side is at most $2q^\tau$, since $0 < p, q < 1$. It tends to zero uniformly on that parameter interval. The failure therefore occurs at each fixed interior parameter; it does not exploit an endpoint degeneration.

There is no conflict with stability modulo translation. Directly from (4), translating g_k by $8e$ aligns its main component with that of f_k . The two remaining components each have mass q , so

$$\|f_k - g_k(\cdot - 8e)\|_1 \leq 2q, \quad \|F_k - G_k(\cdot - 8A_k^{-1}e)\|_1 \leq \frac{2}{1+k^2}. \quad (8)$$

The small remote component cancels the barycenter but not the location of the main mass. Isotropic normalization retains the obstruction and does not impose log-concavity or uniform bounds on higher moments. Neither log-concave nor symmetry-restricted stability results are refuted. Nor do we assert a new sharp quantitative stability theorem: the purpose is to distinguish the literal centered formulation from the translated conclusions in the literature.

The construction and proof use elementary Gaussian calculations and standard inequalities. A bounded primary-source search did not identify this exact explicit family, but does not establish historical priority. This preprint was prepared with AI assistance and checked by the author. The accompanying standard-library Python script checks exact rational weights, moments, mid-point centers and bounds; it supplements the written argument and is not independent peer review or formal verification.

References

- [1] K. J. Böröczky, Stability of some inequalities related to the Prékopa–Leindler inequality, in *Convex Geometry and its Applications*, Oberwolfach Reports **6** (2009), no. 4, pp. 2862–2864. Conjecture 1 on p. 2863. doi:10.4171/OWR/2009/53.
- [2] K. M. Ball and K. J. Böröczky, Stability of the Prékopa–Leindler inequality, *Mathematika* **56** (2010), 339–356. doi:10.1112/S002557931000063X; arXiv:0909.3738.
- [3] A. Figalli, P. van Hintum and M. Tiba, Sharp Quantitative Stability for the Prékopa–Leindler and Borell–Brascamp–Lieb Inequalities, arXiv:2501.04656v2 (2025; revised 6 October 2026), Corollary 1.7.