

A Counterexample to the Weighted Homogeneous Jacobian Jet Closure Conjecture

Alper Ferudun*

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Abstract

Chen and Zuo conjectured that the r th jet closure of the Jacobian ideal of a weighted homogeneous polynomial equals that ideal plus the $(r+1)$ st power of the maximal ideal. For $f = z(x^2 + y^3)$ we exhibit $xy^3 \in J(f)^{4\text{-jc}} \setminus (J(f) + \mathfrak{m}^5)$. Membership follows from a two-term polynomial identity in the universal jet coefficients, and nonmembership from a homomorphism to $k[s]/(s^{10})$. We also give an irreducible four-variable example and a general procedure transferring a principal-ideal jet defect to a Jacobian ideal. The construction uses the cusp example of de Fernex, Ein and Ishii. Our examples have nonisolated singularities; they disprove the conjecture as written, not a version restricted to isolated singularities.

1 The conjecture and the jet convention

Throughout, k is an algebraically closed field of characteristic zero. For $R = k[[u_1, \dots, u_n]]$, let $\mathfrak{m} = (u_1, \dots, u_n)$. Given a positive integer r , introduce coefficient variables $u_{i,j}$ and substitute

$$u_i(t) = \sum_{j=1}^r u_{i,j} t^j \quad \text{in } C_r[t]/(t^{r+1}), \quad C_r = k[u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq r].$$

For an ideal $I \subset R$, let $K_r(I) \subset C_r$ be the ideal generated by all coefficients through degree r of the substituted elements of I . A generating set of I suffices, by the product rule. The local jet closure is

$$I^{r\text{-jc}} = \ker \left(R \longrightarrow (C_r/K_r(I))[t]/(t^{r+1}) \right).$$

This is the scheme-theoretic convention of de Fernex, Ein and Ishii [2], not jet support closure. Formal series can be substituted because every $u_i(t)$ is divisible by the nilpotent element t .

For $f \in R$ write $J(f) = (\partial f / \partial u_1, \dots, \partial f / \partial u_n)$. Chen and Zuo [1, Conjecture 1.1, p. 101] propose the equality

$$J(f)^{r\text{-jc}} = J(f) + \mathfrak{m}^{r+1} \tag{1}$$

for weighted homogeneous f . The same statement is Conjecture 1.3 on page 2 of their arXiv version. The displayed conjecture allows an arbitrary number of variables and does not assume an isolated singularity. The ordinary homogeneous ideal case is their Theorem A; weighted homogeneity is the additional issue here.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing). Unrefereed preprint. Licensed under CC BY 4.0.

Theorem 1. In $R = k[[x, y, z]]$, put

$$f = z(x^2 + y^3), \quad I = J(f) = (2xz, 3y^2z, x^2 + y^3).$$

Then f is weighted homogeneous of degree 7 for weights $(3, 2, 1)$, and

$$xy^3 \in I^{4\text{-jc}} \setminus (I + \mathfrak{m}^5).$$

In particular, (1) is false in its stated generality.

2 Two exact certificates

Proof of Theorem 1. The weights and partial derivatives are immediate. Write $x(t) = x_1t + \dots + x_4t^4$ and similarly for $y(t), z(t)$. Because $g = x^2 + y^3$ is a generator of I , the ideal $K_4(I)$ contains the degree-two and degree-three coefficients of $g(t)$:

$$G_2 = x_1^2, \quad G_3 = 2x_1x_2 + y_1^3.$$

On the other hand,

$$x(t)y(t)^3 \equiv x_1y_1^3t^4 \pmod{t^5}.$$

The identity

$$x_1y_1^3 = x_1G_3 - 2x_2G_2 \tag{2}$$

therefore proves $xy^3 \in I^{4\text{-jc}}$.

For nonmembership, consider the local homomorphism

$$\phi : R \longrightarrow k[s]/(s^{10}), \quad x \longmapsto s^3, \quad y \longmapsto -s^2, \quad z \longmapsto 0. \tag{3}$$

It is well defined on formal series because these images are nilpotent. Both xz and y^2z vanish, while $\phi(x^2 + y^3) = s^6 - s^6 = 0$. Moreover $\phi(\mathfrak{m}) \subseteq (s^2)$, so $\phi(\mathfrak{m}^5) = 0$. But

$$\phi(xy^3) = -s^9 \neq 0.$$

Thus $xy^3 \notin I + \mathfrak{m}^5$, as required. $\square$

The inclusion $I + \mathfrak{m}^5 \subseteq I^{4\text{-jc}}$ holds for every ideal: the defining coefficients kill I , and a local fourth jet sends $\mathfrak{m}^5$ to $(t^5) = 0$. The theorem proves that this inclusion is strict. The map (3) is a witness to ideal nonmembership; it is not asserted to be a fourth jet of R/I .

Remark 2. The cusp input is already in [2, Example 3.11]: $xy^3 \in (x^2 + y^3)^{4\text{-jc}}$ but $xy^3 \notin (x^2 + y^3) + (x, y)^5$. The additional observation is that a multiplier variable makes $x^2 + y^3$ a Jacobian generator while preserving the nonmembership. Thus this is an application of the published cusp defect, not a claim to have discovered that defect.

3 A transfer principle and an irreducible example

Proposition 3. Let $B = k[[u_1, \dots, u_n]]$ with maximal ideal $\mathfrak{q}$, let $g \in k[u_1, \dots, u_n] \cap \mathfrak{q}$, and suppose

$$h \in (g)^{r\text{-jc}} \setminus ((g) + \mathfrak{q}^{r+1}).$$

In $B[[z]]$ set $f = zg$ and $\mathfrak{n} = (\mathfrak{q}, z)$. Then

$$h \in J(f)^{r\text{-jc}} \setminus (J(f) + \mathfrak{n}^{r+1}).$$

If g is positively weighted homogeneous, so is zg after assigning any positive weight to z .

Proof. Since $\partial f/\partial z = g$, the coefficient ideal $K_r(J(f))$ contains the extended coefficient ideal $K_r((g))$. This preserves the coefficient identities proving membership of h . If $h \in J(f) + \mathfrak{n}^{r+1}$, evaluation at $z = 0$ sends all derivatives $z \partial g/\partial u_i$ to zero and sends $\partial f/\partial z$ to g . It would give $h \in (g) + \mathfrak{q}^{r+1}$, a contradiction. The weight assertion follows by adding the weight of z to that of g . $\square$

The three-variable example is reducible. This is not essential.

Corollary 4. *In $S = k[[x, y, z, w]]$, with maximal ideal $\mathfrak{n}$, put*

$$F = z(x^2 + y^3) + w^7.$$

Then F is irreducible and weighted homogeneous of degree 7 with weights $(3, 2, 1, 1)$, and

$$xy^3 \in J(F)^{4-jc} \setminus (J(F) + \mathfrak{n}^5).$$

Proof. Here

$$J(F) = (2xz, 3y^2z, x^2 + y^3, 7w^6).$$

Identity (2) proves jet membership, and extending (3) by $w \mapsto 0$ proves nonmembership.

To prove irreducibility, view F in $k[x, y, w][z]$. Its two coefficients $x^2 + y^3$ and w^7 are coprime, so it is primitive. It is linear and irreducible over $k(x, y, w)[z]$; Gauss's lemma gives irreducibility in the polynomial ring. If it factored into two nonunits in S , the lowest weighted parts of the two factors would have positive weights and multiply to F . This would be a nontrivial polynomial factorization, a contradiction. $\square$

The singular locus of each example contains the z -axis. Consequently these examples do not decide the conjecture restricted to isolated singularities, nor do they decide its two-variable case. No complete description of either fourth-jet closure is needed.

4 Relation to arc closure and verification

A failure at a fixed jet order does not contradict triviality of arc closure. The intersection of all finite jet closures is arc closure [2, Proposition 3.12]; Mallory [3, Theorem 5.1] proves that this intersection equals the original ideal for Noetherian local algebras with separable residue-field extension. In particular, the all-orders question recorded as OWR-4086-008 in the UnsolvedMath dataset, read with Ishii's original algebraically closed ground-field assumption [4, pp. 2455–2456], is already answered by that prior theorem. The present note addresses the separate finite-order conjecture of Chen and Zuo.

The accompanying standard-library Python program expands the jets over the integers and checks both certificates exactly. It also checks nonmembership independently by rational row reduction modulo the fifth maximal-ideal power. In three variables the Jacobian span has rank 19, rising to 20 after adjoining xy^3 , in an ambient space of dimension 35. The corresponding four-variable ranks are 29 and 30 in dimension 70. Normal and optimized execution give the same 104 checks. These computations support, and are not substitutes for, the algebraic proofs above.

AI assistance was used in literature search, developing the argument, exact-checking code and exposition. This note is unrefereed. The cusp calculation and general arc-closure theorem are attributed to their original authors; no absolute bibliographic priority claim is made for the present elementary application.

References

- [1] Y. Chen and H. Zuo, *On jet closures of singularities*, Sbornik: Mathematics **216**:1 (2025), 99–131. [doi:10.4213/sm10010e](https://doi.org/10.4213/sm10010e). Preprint: [arXiv:2507.05796v1](https://arxiv.org/abs/2507.05796v1).
- [2] T. de Fernex, L. Ein and S. Ishii, *Jet closures and the local isomorphism problem*, Journal of Algebra **501** (2018), 166–181. [doi:10.1016/j.jalgebra.2018.01.008](https://doi.org/10.1016/j.jalgebra.2018.01.008).
- [3] D. Mallory, *Triviality of arc closures and the local isomorphism problem*, Journal of Algebra **544** (2020), 47–61. [doi:10.1016/j.jalgebra.2019.09.028](https://doi.org/10.1016/j.jalgebra.2019.09.028).
- [4] S. Ishii, *From jet schemes to the base scheme—Isomorphism problems (global and local)*, in *Singularities*, Oberwolfach Report 43/2009, pp. 2455–2456. [doi:10.4171/OWR/2009/43](https://doi.org/10.4171/OWR/2009/43).