

A Block Criterion for Extremal Jointly Square-Zero Tuples

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Abstract

Let $T = (T_1, \dots, T_d)$ be a tuple of contractions on a complex Hilbert space such that $T_i T_j = 0$ for every i, j . With $F = \bigcap_i \ker T_i$ and $E = F^\perp$, write $T_i = \begin{pmatrix} 0 & 0 \\ W_i & 0 \end{pmatrix}$ on $E \oplus F$. We prove that T is extremal among all commuting contractive extensions if and only if every $W_i : E \rightarrow F$ is a coisometry and the map $\mathcal{L}_W(x_1, \dots, x_d) = (W_i x_j - W_j x_i)_{i < j}$ is injective. Every failure admits a nontrivial extension of codimension one. On a nonzero finite-dimensional space, extremality implies $2 \leq q \leq p \leq (d-1)q/2$, where $p = \dim E$, $q = \dim F$. An explicit four-tuple of 5×5 matrices with entries in $\{0, 1, -1\}$ attains the upper bound and has the smallest possible odd dimension within this class. The criterion extends the unitary-block criterion of Timko; the general classification without the square-zero hypothesis is not addressed.

1 Extensions and the square-zero class

Write $\mathcal{C}_d$ for the family of d -tuples of commuting contractions on complex Hilbert spaces. An extension of $T \in \mathcal{C}_d$ on H is a tuple $R \in \mathcal{C}_d$ on $H \oplus K$ for which H is invariant and $R_i|_H = T_i$. The tuple T is *extremal* if every such extension has the form $R_i = T_i \oplus C_i$. Thus extensions here preserve an invariant subspace; merely compressing a dilation to H is a different condition.

The question of identifying the extremals of $\mathcal{C}_d$ for $d > 2$ appears in Richter's Oberwolfach contribution with Sundberg [1, p. 1067]. Richter and Sundberg treat related extension questions for spherical and row contractions [2]. Timko proves extremality criteria for several classical examples [3]. In particular, his Proposition 3.1 treats block tuples whose nonzero blocks are unitaries, for any number of coordinates. His Section 5 also uses a homogeneous linear system for the Varopoulos family. We use the same block-extension method, allowing rectangular contractive blocks and deriving the coisometry condition.

We call T *jointly square-zero* when

$$T_i T_j = 0 \quad (1 \leq i, j \leq d). \quad (1)$$

This is stronger than requiring $T_i^2 = 0$ separately. Let $F = \bigcap_i \ker T_i$, a closed subspace, and $E = F^\perp$. Equation (1) implies $\text{ran } T_i \subset F$; hence

$$H = E \oplus F, \quad T_i = \begin{pmatrix} 0 & 0 \\ W_i & 0 \end{pmatrix}, \quad W_i : E \longrightarrow F, \quad \|W_i\| \leq 1. \quad (2)$$

Conversely, any tuple of the form (2) is jointly square-zero. The criterion below holds for any such decomposition, whether canonical or not. All direct sums carry their Hilbert-space norms.

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2 The extremality criterion

For $M = \binom{d}{2}$, define the bounded operator

$$\mathcal{L}_W : E^d \longrightarrow F^M, \quad \mathcal{L}_W(x_1, \dots, x_d) = (W_i x_j - W_j x_i)_{1 \leq i < j \leq d}. \quad (3)$$

Theorem 1. *For the tuple (2), the following are equivalent:*

1. T is extremal in $\mathcal{C}_d$.
2. T has no nontrivial extension in $\mathcal{C}_d$ of codimension one.
3. $W_i W_i^* = I_F$ for every i , and $\ker \mathcal{L}_W = \{0\}$.

The extension spaces in the first assertion may be infinite dimensional. They are not required to satisfy (1).

Proof. The first assertion implies the second. Suppose first that $W_k W_k^* \neq I_F$, and set $D = (I_F - W_k W_k^*)^{1/2} \neq 0$. Choose a unit vector $y \in F$ with $v = Dy \neq 0$. Identifying v with a map $\mathbb{C} \rightarrow F$, we have $vv^* \leq D^2$. On $E \oplus F \oplus \mathbb{C}$, set

$$R_k = \begin{pmatrix} 0 & 0 & 0 \\ W_k & 0 & v \\ 0 & 0 & 0 \end{pmatrix}, \quad R_i = \begin{pmatrix} 0 & 0 & 0 \\ W_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (i \neq k).$$

The row $[W_k \ v]$ is contractive because $W_k W_k^* + vv^* \leq I_F$. Every other R_i is a contraction, and all products $R_i R_j$ vanish. The extension is nontrivial since $v \neq 0$.

Next suppose that $x = (x_i)_{i=1}^d \neq 0$ belongs to $\ker \mathcal{L}_W$. After a common scaling we may assume $\|x_i\| \leq 1$ for every i . Define

$$R_i = \begin{pmatrix} 0 & 0 & x_i \\ W_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{on } E \oplus F \oplus \mathbb{C}. \quad (4)$$

For $e \in E, f \in F, c \in \mathbb{C}$,

$$\|R_i(e, f, c)\|^2 = \|W_i e\|^2 + |c|^2 \|x_i\|^2 \leq \|e\|^2 + \|f\|^2 + |c|^2.$$

Thus R_i is contractive. The only possibly nonzero block of $R_i R_j$ is its $F, \mathbb{C}$ block, equal to $W_i x_j$. These blocks are symmetric in i, j by $\mathcal{L}_W x = 0$. At least one new column is nonzero, so this is again a nontrivial codimension-one extension. This proves that assertion 2 implies 3.

Assume assertion 3 and let R be any extension in $\mathcal{C}_d$. Invariance of $E \oplus F$ gives the representation

$$R_i = \begin{pmatrix} 0 & 0 & A_i \\ W_i & 0 & B_i \\ 0 & 0 & C_i \end{pmatrix} \quad \text{on } E \oplus F \oplus K. \quad (5)$$

Compressing $R_i R_i^* \leq I$ to F yields

$$W_i W_i^* + B_i B_i^* \leq I_F.$$

Coisometry forces $B_i = 0$. The F, K block of $R_i R_j = R_j R_i$ now gives

$$W_i A_j = W_j A_i.$$

For every $u \in K$, the vector $(A_1 u, \dots, A_d u)$ belongs to $\ker \mathcal{L}_W$, so every A_i vanishes. Equation (5) is therefore a direct-sum extension. This proves assertion 1. $\square$

Remark 2. The extension (4) need not be jointly square-zero. This distinction is essential. For $d = 1$ and $W_1 = [1]$, the two-dimensional nilpotent shift has such a three-dimensional shift extension, with $R_1^2 \neq 0$. In contrast, a contractive extension of this T_1 required to have square zero has $B_1 = 0$ and then $W_1 A_1 = 0$, hence $A_1 = 0$. Extremality inside the square-zero subfamily is thus weaker than the extremality used in Theorem 1.

3 Dimension restrictions and the unitary case

Proposition 3. *Suppose $H \neq \{0\}$ is finite dimensional, T is jointly square-zero and extremal in $\mathcal{C}_d$, and E, F are the canonical spaces in (2). Put $p = \dim E$, $q = \dim F$. Then*

$$2 \leq q \leq p \leq \frac{d-1}{2} q. \quad (6)$$

For $d = 1, 2$ no such tuple exists. For $d = 3$, necessarily $p = q$, so H has even dimension and all the W_i are unitary.

Proof. First $q > 0$: if $F = \{0\}$, every range $T_i H$ is zero, whence $F = H$, contrary to $H \neq \{0\}$. Coisometry gives $p \geq q$. Injectivity of the map (3) gives $dp \leq Mq$, which is the upper bound in (6).

To exclude $q = 1$, identify F with $\mathbb{C}$ and choose an orthonormal basis of E . Write $W_i = (a_{i1}, \dots, a_{ip})$ and set $x_i = (a_{i1}, \dots, a_{ip})^\top$, without conjugating the coefficients. Then

$$W_i x_j = \sum_{k=1}^p a_{ik} a_{jk} = \sum_{k=1}^p a_{jk} a_{ik} = W_j x_i.$$

The row norms are one by coisometry, so this gives a nonzero kernel vector, a contradiction. The assertions about $d \leq 3$ follow from the inequalities, and a square coisometry in finite dimensions is unitary. $\square$

For unitary blocks W_i between equal spaces, choose W_d and put $V_i = W_d^{-1} W_i$. The equations involving d in $\mathcal{L}_W x = 0$ determine $x_i = V_i x_d$. The remaining equations are equivalent to

$$[V_i, V_j] x_d = 0 \quad (1 \leq i, j \leq d).$$

Consequently the criterion reduces to $\bigcap_{i,j} \ker[V_i, V_j] = \{0\}$, exactly Timko's Proposition 3.1 [3]. This recovers a known result, not a new unitary-block classification. The rectangular case $p > q$ can first occur when $d \geq 4$.

4 A sharp four-tuple in dimension five

Set $E = \mathbb{C}^3$, $F = \mathbb{C}^2$, and

$$\begin{aligned} W_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & W_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ W_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & W_4 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned} \quad (7)$$

Theorem 4. *The four matrices T_i obtained from (2) and (7) form an extremal tuple in $\mathcal{C}_4$ on $\mathbb{C}^5$. They are jointly square-zero partial isometries with entries in $\{0, 1, -1\}$. They attain equality in the upper bound of (6). Five is the smallest possible odd dimension of a nonzero jointly square-zero extremal tuple, and four is the fewest number of coordinates attaining an odd finite dimension.*

Proof. Every $W_i W_i^* = I_2$, so each T_i is a partial isometry. For injectivity, write $x_i = (a_i, b_i, c_i)$. The equations $\mathcal{L}_W x = 0$, in the pair order 12, 13, 14, 23, 24, 34, are

$$\begin{aligned} a_2 &= b_1, & b_2 &= c_1, \\ a_3 &= c_1, & b_3 &= a_1, \\ a_4 &= a_1, & b_4 &= -b_1, \\ b_3 &= c_2, & c_3 &= a_2, \\ b_4 &= a_2, & c_4 &= -b_2, \\ c_4 &= a_3, & a_4 &= -b_3. \end{aligned}$$

Combining the two expressions for a_4, b_4, c_4 gives $2a_1 = 2b_1 = 2c_1 = 0$, and then every coordinate is zero. Theorem 1 proves extremality. Here $(d, p, q) = (4, 3, 2)$, giving equality in $p \leq (d-1)q/2$. By Proposition 3, any nonzero finite-dimensional example has $p \geq q \geq 2$, excluding dimensions one and three; and examples with at most three coordinates cannot have odd dimension. $\square$

Corollary 5. *For every integer $r \geq 1$, the blocks $W_i \otimes I_r$ in (7) give an extremal four-tuple on $\mathbb{C}^{5r}$ with $(p, q) = (3r, 2r)$, attaining the dimension bound.*

Proof. Coisometry is preserved by this amplification. After permuting coordinates, its kernel map is a direct sum of r copies of the injective map $\mathcal{L}_W$. Apply Theorem 1. $\square$

In the coordinate order (x_1, x_2, x_3, x_4) and the pair order used above, the 12×12 matrix of $\mathcal{L}_W$ has determinant -8 . The accompanying standard-library Python program verifies this using both rational Gaussian elimination and an independent subset recurrence for the permutation expansion of the determinant. It also verifies the coisometry identities, all products $T_i T_j = 0$, and the amplifications for $r = 1, 2, 3$. With the first three blocks fixed, all 24 signed coordinate coisometries for W_4 give ranks 9, 11, 12 with multiplicities 6, 6, 12, respectively. Kernel vectors in the singular cases yield explicitly checked contractive commuting extensions. These finite checks support, but do not replace, the general proof.

5 A quantitative extension bound

The same proof bounds how far an extension can fail to be a direct sum when the blocks are almost coisometric.

Proposition 6. *Let the $W_i : E \rightarrow F$ be contractions, and assume $\|\mathcal{L}_W x\| \geq \gamma \|x\|$ for every $x \in E^d$, for some $\gamma > 0$. Put $\delta = \max_i \|I_F - W_i W_i^*\|$ and $M = \binom{d}{2}$. For any commuting contractive extension in (5), its new columns satisfy*

$$\sum_{i=1}^d (\|A_i u\|^2 + \|B_i u\|^2) \leq \left(d + \frac{4M}{\gamma^2}\right) \delta \|u\|^2 \quad (u \in K). \quad (8)$$

Proof. The F -compression of $R_i R_i^* \leq I$ gives $B_i B_i^* \leq I_F - W_i W_i^*$, hence $\|B_i\| \leq \sqrt{\delta}$. Also $\|C_i\| \leq 1$ by compression. The F, K block of commutativity is

$$W_i A_j - W_j A_i = B_j C_i - B_i C_j.$$

For each pair, the right side applied to u has norm at most $2\sqrt{\delta}\|u\|$. Summing over pairs and using the lower bound for $\mathcal{L}_W$ gives $\sum_i \|A_i u\|^2 \leq 4M\delta\|u\|^2/\gamma^2$. Adding $\sum_i \|B_i u\|^2 \leq d\delta\|u\|^2$ proves (8). $\square$

The lower-bound assumption in Proposition 6 is automatic for an injective $\mathcal{L}_W$ in finite dimensions, but need not follow from injectivity in infinite dimensions. Theorem 1 requires only injectivity.

The results concern the jointly square-zero class and its extensions in $\mathcal{C}_d$; they do not identify the extremals outside that class. The block method and the unitary special case have the precedents described above. No claim of absolute priority is made. AI assistance was used for literature reconnaissance, proof development, exact-checker preparation, and exposition. The manuscript is self-audited and unrefereed, not independently reviewed or formally verified.

References

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