

Orbit Generation Does Not Ensure GNS Descent of Markov Dilations

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Abstract

We construct a general dilation of the Ornstein–Uhlenbeck Markov semigroup whose ambient commutative von Neumann algebra is generated by the initial algebra and its nonnegative time shifts, but whose GNS representation kernel is not invariant under those shifts. The dilation endomorphisms are normal, injective and point-strongly continuous, and its expectation is normal. Thus orbit generation alone does not permit passage of a given dilation to the GNS module of its expectation. The construction changes the stationary path measure by a bounded density built from independent Gaussian bridge signs. For each prescribed integer m , this density leaves conditional observations at any m times unchanged while producing the obstruction. We also give an exact support criterion for kernel invariance in this commutative setting. The result concerns orbit-generation minimality; it does not contradict stronger weak or Markov dilation theorems, or preclude a different module dilation of the same semigroup.

1 The given-dilation question

Let B be a unital von Neumann algebra and $T = (T_t)_{t \geq 0}$ a semigroup of normal unital completely positive maps. A *general dilation* consists of a von Neumann algebra A , an embedding $i : B \rightarrow A$, a completely positive retraction $p : A \rightarrow B$, and an endomorphism semigroup θ such that

$$pi = \text{id}_B, \quad p\theta_t i = T_t. \quad (1)$$

Our embeddings and endomorphisms are unital and normal, and ip is a normal conditional expectation. We specify continuity below.

Let (E, ξ, π) be the GNS correspondence of p , so that $p(a) = \langle \xi, \pi(a)\xi \rangle$ and $\pi : A \rightarrow \mathcal{B}^a(E)$ is the left representation. In order to transfer the *given* semigroup to $\pi(A)$ by

$$\beta_t(\pi(a)) = \pi(\theta_t(a)), \quad (2)$$

one must first have $\theta_t(\ker \pi) \subseteq \ker \pi$. Only then can one ask about extending β_t to all of $\mathcal{B}^a(E)$.

Skeide asks about these two steps in the 2009 Oberwolfach report [1, pp. 542–543], suggesting that generation of A by $i(B)$ and its time shifts might suffice. The definition of general dilation there is the one-time compression identity (1); see [1, p. 498]. A related GNS transfer question appears in [2, Section 5, manuscript pp. 34–35]. We disprove sufficiency of this suggested orbit-generation condition in the von Neumann setting. We do not interpret the open-ended phrase *not too big* as ruling out all stronger useful hypotheses.

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Write $\gamma = N(0, 1)$, $B = L^\infty(\mathbb{R}, \gamma)$, and

$$T_t f(x) = \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \gamma(dy). \quad (3)$$

Theorem 1. *For every integer $m \geq 1$, the semigroup (3) has a dilation (A, θ, i, p_m) with the following properties.*

1. *A is commutative with separable predual. The maps θ_t are normal injective unital endomorphisms, point-strongly continuous in a faithful representation.*
2. *$A = W^*(\theta_t i(B) : t \geq 0)$; nonnegative rational times suffice.*
3. *$i p_m$ is a normal conditional expectation and $p_m \theta_t i = T_t$ for all $t \geq 0$.*
4. *For the GNS left representation π_m of p_m , there is a projection $a_m \in A$ such that*

$$\pi_m(a_m) = 0, \quad p_m(\theta_{2(m+1)}(a_m)) = \frac{1}{2}1_B. \quad (4)$$

Consequently (2) is not well-defined.

Moreover, on the stationary path space used in the construction,

$$p_m(F(X_{t_1}, \dots, X_{t_k})) = \mathbb{E}[F(X_{t_1}, \dots, X_{t_k}) \mid X_0] \quad (5)$$

for every $k \leq m$, every $t_1, \dots, t_k \geq 0$, and every bounded Borel function $F : \mathbb{R}^k \rightarrow \mathbb{C}$.

The GNS kernel and its failure of invariance are the same if one uses the normal, selfdual GNS completion. Already $m = 1$ gives the counterexample, with failure at time 4. The higher- m assertion shows that no fixed finite level of coordinate-distribution matching repairs the obstruction.

2 The stationary path algebra

Let W be standard Brownian motion and put

$$X_t = e^{-t}W_{e^{2t}}, \quad t \geq 0.$$

This continuous centered Gaussian process has covariance

$$\mathbb{E}[X_s X_t] = e^{-|s-t|}. \quad (6)$$

Let P be its law on $\Omega = C([0, \infty), \mathbb{R})$ and retain X_t for the coordinate maps. Mean and covariance determine the Gaussian finite-dimensional distributions, so these distributions are stationary. Since coordinates generate the Borel sigma-field, the shifts

$$S_t \omega(u) = \omega(u + t)$$

preserve P . Set

$$A = L^\infty(\Omega, P), \quad \theta_t F = F \circ S_t, \quad i(f) = f(X_0).$$

Measure preservation makes θ_t well-defined on equivalence classes, isometric in essential norm, normal, and injective. Pullback is a unital $*$ -homomorphism and obeys the semigroup law.

Lemma 2. *In the multiplication representation of A on $L^2(P)$, $t \mapsto \theta_t F$ is strongly continuous for every $F \in A$. The algebra A has separable predual and is generated by $\theta_t i(B)$ for nonnegative rational t .*

Proof. For a bounded continuous cylinder function F , path continuity and dominated convergence give $\|\theta_t F - F\|_2 \rightarrow 0$ as $t \downarrow 0$. These functions are dense in $L^2(P)$: rational coordinate maps generate the Borel sigma-field of Ω , and bounded continuous functions are dense for each finite-dimensional Gaussian law. The pullbacks are L^2 isometries, so this continuity extends to every $F \in L^2(P)$. The semigroup law and isometry give continuity at all nonnegative times, including left continuity at positive times.

For $F \in A$ and bounded $g \in L^2(P)$,

$$\|(\theta_t F - F)g\|_2 \leq \|g\|_\infty \|\theta_t F - F\|_2 \rightarrow 0.$$

Truncating a general $g \in L^2(P)$ and using $\|\theta_t F - F\|_\infty \leq 2\|F\|_\infty$ proves strong-operator continuity at zero, and the same argument applies at every time. The coordinate sigma-field statement proves the asserted von Neumann generation. The sigma-field is countably generated modulo completion, so $L^1(P)$ is separable. $\square$

Gaussian conditioning in (6) gives $\mathbb{E}[f(X_t) \mid X_0] = T_t f(X_0)$. Each T_t is normal, unital and completely positive. Combining independent Gaussian innovations verifies $T_s T_t = T_{s+t}$: the resulting mean is $e^{-(s+t)x}$ and the variance is $1 - e^{-2(s+t)}$. Conditional expectation contraction and the preceding L^2 continuity imply strong continuity on $L^2(\gamma)$. Uniform boundedness and density of $L^2(\gamma)$ in $L^1(\gamma)$ then imply point-ultraweak continuity on B .

3 Bridge signs and a hidden support defect

Put $r = e^{-1/2}$ and $c = r/(1 + r^2)$. For $a \geq 0$ define

$$R_a = X_{a+1/2} - c(X_a + X_{a+1}). \quad (7)$$

Lemma 3. *The variable R_a is a nondegenerate centered Gaussian, independent of the sigma-field generated by X_t for $t \notin (a, a+1)$. Residuals associated with disjoint intervals are jointly independent, and their signs are independent fair signs.*

Proof. For $t \leq a$, (6) gives

$$\mathbb{E}[R_a X_t] = e^{-(a-t)} \{r - c(1 + r^2)\} = 0.$$

For $t \geq a + 1$, the same calculation has the factor $e^{-(t-a-1)}$ and also vanishes. Directly,

$$\mathbb{E}[R_a^2] = 1 + 2c^2(1 + r^2) - 4cr = \frac{1 - r^2}{1 + r^2} > 0. \quad (8)$$

Every finite vector of residuals and coordinates is jointly Gaussian. Zero cross-covariances therefore give independence from each finite vector of outside coordinates. A monotone-class argument gives independence from the whole outside sigma-field. The other residuals are functions of outside coordinates when the intervals are disjoint. Joint Gaussian independence and symmetry prove the remaining claims. Endpoints are included among the outside coordinates. $\square$

Fix $m \geq 1$, let $n = m + 1$, and set

$$b_j = 2j - 1, \quad U_j = \operatorname{sgn} R_{b_j}, \quad V = \prod_{j=1}^n U_j, \quad h = 1 + V, \quad q = \frac{1 + V}{2}. \quad (9)$$

The value of the sign at zero is immaterial. Thus q is a projection and $h = 2q$ is a bounded nonnegative density of mean one. All the residuals are independent of X_0 .

Lemma 4. *For any $k \leq m$ and any nonnegative times $t_1, \dots, t_k$,*

$$\mathbb{E}[h \mid X_0, X_{t_1}, \dots, X_{t_k}] = 1. \quad (10)$$

Proof. At least one of the n disjoint open intervals $(b_j, b_j + 1)$ contains none of the k observation times. By Lemma 3, its residual is independent jointly of X_0 , all the observed coordinates, and all the other residuals in (9). Averaging its centered sign first gives conditional mean zero for V . This proves (10). Observation times at endpoints do not affect the argument. $\square$

Define, with the usual identification of functions of X_0 with B ,

$$p_m(F) = \mathbb{E}[hF \mid X_0]. \quad (11)$$

Multiplication by the positive central element h is completely positive, and conditional expectation is normal and completely positive. Lemma 4 gives $p_m(1) = 1$. The map is B -bimodular and fixes $i(B)$, hence ip_m is a normal conditional expectation onto $i(B)$. Multiplying (10) by a bounded function of the observed coordinates proves (5). In particular, $k = 1$ proves $p_m\theta_t i = T_t$ for every real $t \geq 0$.

4 Failure of the first GNS step

The GNS correspondence of (11) is the completion of A , modulo zero-length vectors, with right action through $i(B)$ and inner product

$$\langle F, G \rangle = p_m(\overline{F}G).$$

Indeed $F\xi b = Fi(b)\xi$, so this description is the usual GNS correspondence, not merely a cyclic Hilbert-space representation. The left action is multiplication.

Lemma 5. *The kernel of the left representation is*

$$\ker \pi_m = \{F : h|F|^2 = 0 \text{ almost surely}\} = (1 - q)A. \quad (12)$$

Proof. If $\pi_m(F) = 0$, evaluation at ξ gives $p_m(|F|^2) = 0$. Integrating this nonnegative conditional expectation yields $h|F|^2 = 0$ almost surely. Conversely this identity implies $p_m(|FG|^2) = 0$ for every $G \in A$. Multiplication by F therefore vanishes on a dense generating set. The same kernel holds on the selfdual completion: an operator zero on the original module is zero on its selfdual closure, whereas an operator nonzero there remains nonzero. $\square$

Completion of the proof of Theorem 1. The first three assertions and (5) were proved above. Set $a_m = 1 - q = (1 - V)/2$. Lemma 5 gives $\pi_m(a_m) = 0$. Let $s = 2n$. The shifted residual intervals are $[2n + 2j - 1, 2n + 2j]$, $1 \leq j \leq n$, disjoint from the original intervals. All $2n$ residuals are jointly

independent and independent of X_0 . Writing $V' = \theta_s(V)$, the two parities V, V' are independent fair signs, also independent of X_0 . Therefore

$$p_m(\theta_s(a_m)) = \mathbb{E} \left[(1 + V) \frac{1 - V'}{2} \middle| X_0 \right] = \frac{1}{2}.$$

Since $\theta_s(a_m)$ is a projection, this also computes $\langle \pi_m(\theta_s(a_m))\xi, \pi_m(\theta_s(a_m))\xi \rangle$. Thus its left action is nonzero. Equation (2) would send the zero operator both to zero and to this nonzero operator, which is impossible. $\square$

Remark 6. The reference algebra is $L^\infty(P)$, not $L^\infty(hP)$. The bounded density makes the expectation normal on the former. Passing to the latter would discard the null ideal which the shifts fail to preserve; it would not give a well-defined shifted dilation. There is also no auxiliary unobserved factor: every sign in the construction is a function of three coordinate observables.

5 The exact commutative support condition

The obstruction has a simple general form. Let $A = L^\infty(\Omega, P)$, let $\mathcal{G}$ be a sub-sigma-field, and let

$$p_h(F) = \mathbb{E}[hF \mid \mathcal{G}], \quad h \geq 0, \quad \mathbb{E}[h \mid \mathcal{G}] = 1.$$

An integrable density suffices here. Let θ_s be pullback by a nonsingular measurable map S_s , and put $D = \{h > 0\}$.

Proposition 7. *For the GNS left representation π_h ,*

$$\theta_s(\ker \pi_h) \subseteq \ker \pi_h \iff P(D \cap S_s^{-1}(D^c)) = 0. \quad (13)$$

Proof. The proof of Lemma 5 gives $\ker \pi_h = 1_{D^c}A$. Necessity follows by applying kernel invariance to 1_{D^c} . For sufficiency, a function in the kernel vanishes on D , so its pullback vanishes there if S_s sends almost every point of D into D . Nonsingularity ensures that pullbacks of null exceptional sets remain null, proving the assertion on equivalence classes. $\square$

Consequently forward invariance of the expectation's support is exactly what is needed for the first descent step in this commutative setting. Orbit generation does not imply it.

By contrast, the stronger *coextension* identity

$$p\theta_s = T_s p \quad \text{on all of } A \quad (14)$$

does ensure kernel invariance here. If $p(|F|^2) = 0$, then

$$p(|\theta_s F|^2) = T_s p(|F|^2) = 0,$$

and commutativity identifies this null space with the left kernel. Our dilation does not satisfy (14): it has $p_m(a_m) = 0$ but $p_m\theta_s(a_m) = 1/2$. This distinction between general, weak and strong dilations is explicit in [3, Definition 2.3, Proposition 2.8 and (2.3)]. In particular, the automatic strongness of weak corner Markov dilations is inapplicable: our embedding is proper and unital, and thus is not a corner inclusion.

6 Scope, variants and reproducibility

For each fixed m , a different density is used. No single density here preserves all finite-dimensional distributions simultaneously; that would determine the original path law. The construction also has an automorphic variant on $C(\mathbb{R}, \mathbb{R})$ with the two-sided stationary Ornstein–Uhlenbeck law and the full shift group. The same positive-time density and kernel calculation apply. In that variant the algebra is generated by *all real* time shifts. The one-sided construction above is the version with positive-orbit generation.

The result refutes the explicit orbit-generation suggestion in [1]; it does not classify stronger sufficient minimality conditions, decide the extension problem once descent is available, or disprove existence of a different module dilation of T . The principal weak-dilation results of [3], whose scope is discussed in its Appendix R(i), do not assert the descent of every general dilation.

The covariance, Gaussian conditioning, conditional-expectation and GNS ingredients are classical. The contribution claimed here is the explicit continuous-time orbit-generated counterexample and its finite-observation strengthening. A bounded literature search did not locate this bridge-sign construction; no exhaustive novelty certificate or absolute priority claim is made.

The accompanying standard-library checker verifies the denominator-cleared covariance and variance identities, finite-grid cross-covariances, and the sign calculations for $1 \leq m \leq 6$. It gives identical results in ordinary and optimized Python modes. These finite checks supplement, rather than replace, the analytic proofs for every m and all real times. This manuscript has undergone the originating researcher’s self-audit; it is not independently refereed or formally verified. AI tools assisted with the derivation, source comparison, exact diagnostics and preparation of the manuscript; the author is responsible for its contents.

References

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