

Ambient Dependence and Extreme Extension for Exchangeable Free-Product States

Alper Ferudun*

20 October 2026

Abstract

A proposed independence relation declares two subalgebras independent when their joint state is realized in two different copies of an extreme exchangeable state on the infinite unital free product of the ambient algebra. We show that this relation depends on the ambient algebra. The diagonal inclusion $\mathbb{C}^2 \subset M_2(\mathbb{C})$ gives an explicit example; a four-point version has distinct proper subalgebras with scalar intersection. Downstairs, the joint law forces a nontrivial mixture of two invariant characters. Upstairs, a pure vector state provides a pure folded witness. In contrast, every extreme invariant state has an extreme invariant extension along a unital C*-algebra inclusion. The latter standard compact-face argument rules out the particular forward-extension obstruction suggested in the original problem, without implying ambient invariance. The examples distinguish extremality among all invariant states from extremality at a fixed marginal.

1 The relation and the two ambient questions

For a unital C*-algebra A , write

$$\mathcal{F}(A) = \bigast_{i \in \mathbb{N}, \mathbb{C}} A$$

for its full unital free product and $j_i : A \rightarrow \mathcal{F}(A)$ for the canonical embeddings. Finite permutations act by $\alpha_\sigma(j_i(a)) = j_{\sigma(i)}(a)$. Let $\text{SS}(A)$ be the weak-* compact convex set of states invariant under this action.

Let $B, C \subseteq A$ be unital subalgebras and let ϕ be a state of A . Denote by $m : B *_\mathbb{C} C \rightarrow A$ the homomorphism induced by their inclusions, and by $k : B *_\mathbb{C} C \rightarrow \mathcal{F}(A)$ the homomorphism that uses j_1 on B and j_2 on C . The relation proposed by Liebscher [1, p. 540] is

$$\text{there exists } \psi \in \text{ext SS}(A) \text{ such that } \psi \circ k = \phi \circ m. \quad (1)$$

We call (1) the *extremal-exchangeability relation*. This terminology refers only to that proposed definition.

The ambient-dependence question must be distinguished from an extension question. Given a state-preserving unital inclusion $(A, \phi) \subseteq (A', \phi')$, one can ask whether (1) has the same truth value in both algebras. The source also suggests a possible obstruction: an extreme invariant state downstairs might fail to extend to an extreme invariant state upstairs. We show that ambient invariance fails, whereas that extension obstruction does not occur for unital C*-algebras.

Extremality in (1) is taken among *all* invariant states. The source does not impose a fixed-marginal convex set, faithfulness of states, or restrictions on the intersection of B and C . Our upstairs witnesses also have the prescribed single-copy marginal. Section 6 explains why fixed-marginal extremality is nevertheless a different condition.

*Mercury Software GmbH. alper@mercurycodelab.com; <https://github.com/AlperTheKing>.

2 Folding and a two-dimensional counterexample

The universal property gives a surjective homomorphism

$$q_A : \mathcal{F}(A) \longrightarrow A, \quad q_A j_i = \text{id}_A \quad (i \in \mathbb{N}).$$

We call $\phi \circ q_A$ the folded state. This is the maximal-amalgamation construction considered by Dykema, Köstler and Williams [2, Proposition 8.7] in the quantum-symmetric setting. We use the following elementary observation for ordinary permutation invariance.

Lemma 1. *The folded state $\phi \circ q_A$ is invariant and is extreme in $\text{SS}(A)$ if and only if ϕ is pure. If ϕ is pure, its folded state is pure in the full state space.*

Proof. The identity $q_A \alpha_\sigma = q_A$ gives invariance. Pullback through a surjective $*$ -homomorphism preserves purity. Indeed, if ϕq_A has a convex decomposition, positivity and Cauchy–Schwarz force both components to vanish on $\ker q_A$. They therefore descend to states of A , giving a decomposition of ϕ . Conversely, any nontrivial decomposition of ϕ yields, by folding, a nontrivial decomposition into invariant states. $\square$

Theorem 2. *Let $0 < t < 1$, let $D \cong \mathbb{C}^2$ be the diagonal algebra in $M_2(\mathbb{C})$, and put*

$$\phi_t(\text{diag}(a, b)) = ta + (1 - t)b, \quad \xi_t = (\sqrt{t}, \sqrt{1 - t}), \quad \phi'_t(x) = \langle \xi_t, x \xi_t \rangle.$$

For $B = C = D$, relation (1) fails in (D, ϕ_t) and holds in $(M_2(\mathbb{C}), \phi'_t)$. The inclusion preserves the state.

Proof. Set $p = \text{diag}(1, 0)$. Suppose that $\psi \in \text{SS}(D)$ has the required joint state. In its GNS representation (π, H, Ω) put $P_i = \pi(j_i(p))$. The joint-state identity and permutation invariance imply

$$\psi(j_i(p)) = t, \quad \psi(j_i(p)j_j(p)) = \psi(j_j(p)j_i(p)) = t \quad (i \neq j).$$

Consequently

$$\|(P_i - P_j)\Omega\|^2 = t + t - t - t = 0.$$

Write $v = P_i\Omega$, independently of i . Since $P_i^2 = P_i$, we have $P_i v = v$ for every i . Thus every nonempty word in the P_i sends Ω to v and has expectation t . These words and the identity span a dense $*$ -subalgebra of $\mathcal{F}(D)$. It follows that

$$\psi = t\chi_1 + (1 - t)\chi_0, \quad \chi_e(j_i(p)) = e \quad (e \in \{0, 1\}, i \in \mathbb{N}). \quad (2)$$

The χ_e are distinct invariant characters. Since $0 < t < 1$, (2) is a nontrivial decomposition, so no matching state is extreme. Notice that projections from different free copies were not assumed to commute.

The state ϕ'_t is pure and restricts to ϕ_t on D . By Lemma 1, $\Psi_t = \phi'_t q_{M_2(\mathbb{C})}$ is pure and invariant. Folding identifies both copies with the same ambient matrices, so $\Psi_t k = \phi'_t m$ on all of $B *_C C$. This proves the relation upstairs. Its single-copy marginals are ϕ'_t , and its restriction to $\mathcal{F}(D)$ is exactly (2). $\square$

In particular, restriction of an extreme invariant state need not be extreme. A general extension theorem in the other direction will not remove this counterexample.

3 Distinct proper subalgebras

Theorem 3. *Ambient invariance also fails when B and C are distinct proper commuting subalgebras that generate the smaller ambient algebra and satisfy $B \cap C = \mathbb{C}1$.*

Proof. Use $A = \mathbb{C}^4$, with coordinates ordered as 00, 01, 10, 11, and put

$$p = (1, 1, 0, 0), \quad r = (1, 0, 1, 0), \quad B = C^*(1, p), \quad C = C^*(1, r),$$

$$\phi_t(a) = ta_{00} + (1 - t)a_{11}, \quad 0 < t < 1.$$

The algebras are distinct and proper. Comparing four coordinates shows that $\alpha 1 + \beta p = \gamma 1 + \delta r$ forces $\beta = \delta = 0$ and $\alpha = \gamma$. Hence $B \cap C = \mathbb{C}1$. The four products of $p, 1 - p$ with $r, 1 - r$ are the coordinate projections, so B and C generate A .

Suppose an invariant state ψ realizes the joint law. In its GNS representation put $P_i = \pi(j_i(p))$ and $R_i = \pi(j_i(r))$. The two means and both ordered cross moments are t , whence

$$P_i \Omega = R_j \Omega \quad \text{whenever } i \neq j.$$

For any i, k , choose ℓ distinct from both. Then $P_i \Omega = R_\ell \Omega = P_k \Omega$. Thus every $P_i \Omega$ and $R_i \Omega$ has the same value v , and each P_i, R_i fixes v . Every nonempty word in these generators therefore has expectation t . Since they generate $\mathcal{F}(A)$, necessarily

$$\psi = t\chi_{00} + (1 - t)\chi_{11},$$

where χ_{ab} evaluates every copy of A at coordinate ab . This is a nontrivial decomposition into invariant characters.

Embed A diagonally in $M_4(\mathbb{C})$. The vector state of $\sqrt{t}e_{00} + \sqrt{1 - t}e_{11}$ is pure and restricts to ϕ_t . Its folded state gives an extreme invariant witness for the same two subalgebras upstairs. $\square$

Remark 4. The downstairs state in Theorem 2 is faithful. The four-point state and both upstairs vector states are not faithful. The theorems do not address a variant requiring faithfulness of the larger ambient state.

4 Self-independence and pure-state enlargement

Proposition 5. *For a unital C^* -probability space (A, ϕ) , relation (1) with $B = C = A$ holds if and only if ϕ is pure.*

Proof. Let ψ be any invariant state with the required joint law. For every $a \in A$ and every $i \neq j$, the four terms in the squared GNS norm give

$$\|(\pi(j_i(a)) - \pi(j_j(a)))\Omega\|^2 = 0 :$$

each square term and each cross term has value $\phi(a^*a)$. For an arbitrary word acting on Ω , replace its last factor by the same element in the preceding copy and multiply those two factors. Repeating from right to left shows

$$\psi(j_{i_1}(a_1) \cdots j_{i_s}(a_s)) = \phi(a_1 \cdots a_s).$$

Thus the only possible matching state is ϕq_A . Lemma 1 proves the assertion. $\square$

Every pair of unital subalgebras of a pure-state ambient algebra satisfies (1), since its folded state is a witness. Moreover, every unital C^* -probability space embeds state-preservingly into such an ambient algebra. To see this, take its GNS representation $(\pi_\phi, H_\phi, \Omega_\phi)$ and any faithful unital representation $\pi_f : A \rightarrow B(H_f)$. The representation $\pi_\phi \oplus \pi_f$ is faithful, and the vector state of $\Omega_\phi \oplus 0$ on $B(H_\phi \oplus H_f)$ is pure and restricts to ϕ . This is the standard GNS enlargement argument. It shows how broadly the ambient dependence can occur under the unrestricted definition.

5 Extreme invariant extensions

Theorem 6. *A unital inclusion $A \subseteq A'$ induces a surjective restriction map*

$$r : \text{SS}(A') \longrightarrow \text{SS}(A).$$

Every extreme point downstairs has an extreme preimage upstairs:

$$\text{ext SS}(A) \subseteq r(\text{ext SS}(A')).$$

Proof. The induced map $\mathcal{F}(A) \rightarrow \mathcal{F}(A')$ is injective. For finite products this follows by induction from the full amalgamated free-product embedding theorem with the amalgamating algebra unchanged at $\mathbb{C}$; see [3, Proposition 2.2], which credits Pedersen for the original theorem. Passing to the norm closures of the increasing finite products proves the countable case. The inclusion is equivariant.

Let $\psi \in \text{SS}(A)$. By the C*-state extension theorem, it extends to a state η of $\mathcal{F}(A')$. Average over the finite symmetric groups:

$$\eta_n = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \eta \circ \alpha_\sigma.$$

Each η_n still restricts to ψ . The state space is weak-* compact, so there is a convergent subnet. Every fixed finite permutation belongs to all sufficiently late S_n , and the limit is therefore invariant. This proves surjectivity.

If ψ is extreme, the fiber $K = r^{-1}(\{\psi\})$ is a nonempty compact convex face of $\text{SS}(A')$. Indeed, a nontrivial convex decomposition of an element of K restricts to a decomposition of ψ , forcing both restrictions to be ψ . By the Krein–Milman theorem, K has an extreme point. An extreme point of a face is extreme in the containing convex set, as required. $\square$

This is the standard averaging and compact-face argument, not a new extension principle. It rules out the particular obstruction suggested on p. 540 of [1] in the unital C*-category. For the literal definition (1), it also proves that independence is preserved on enlarging the ambient algebra: extend a downstairs extreme witness and use the unchanged joint law. The converse fails by Theorems 2 and 3. The extension construction does not prescribe the full A' marginal.

6 Scope and alternative extremality

The finite-dimensional counterexamples also apply to algebraic unital free products. The downstairs proof uses only positivity, GNS and projection identities. Upstairs, any positive component of a decomposition of the folded vector state vanishes on the folding kernel by Cauchy–Schwarz and descends to a state on $M_2(\mathbb{C})$ or $M_4(\mathbb{C})$; purity there gives a contradiction. The general extension theorem, in contrast, has only been asserted for unital C*-algebras. No corresponding assertion about arbitrary unbounded *-algebras or normal extensions of arbitrary von Neumann states is needed.

Adding prescribed single-copy marginals to the witnesses does not remove our examples, provided extremality is still tested in all of $\text{SS}(A)$. Replacing that extremality by extremality inside the fixed-marginal set is a genuine change. For example, every folded state is extreme in that smaller set: positivity forces the zero-square equalities in any decomposition, and Proposition 5’s word argument then forces both components to fold to the same marginal. The decomposition (2) changes the marginals. For the corresponding distinction in quantum symmetric states, see [2, Theorem 8.9].

Thus the literal ambient-dependence question, also recorded as OWR-3392-009 in the UnsolvedMath dataset, has an explicit affirmative answer: ambient dependence is necessary. This

is not a simultaneous resolution of the further questions about natural products, associativity, trivial tails and categories in [1]. The folding mechanism, purification and compactness arguments are standard and have been attributed; no absolute priority is claimed for their application here. This unrefereed note was prepared with AI assistance in research, drafting and exact finite regression checks. The written proofs, rather than those finite checks, establish the assertions; no independent review or formal certification is claimed.

References

- [1] V. Liebscher, On Quantum Independence, in *Mini-Workshop: Product Systems and Independence in Quantum Dynamics*, Oberwolfach Reports **6** (2009), 493–548, contribution pp. 540–541. doi:10.4171/OWR/2009/09.
- [2] K. Dykema, C. Köstler and J. D. Williams, *Quantum symmetric states on free product C^* -algebras*, arXiv:1305.7293v3, 9 August 2021, Proposition 8.7 and Theorem 8.9.
- [3] S. Armstrong, K. Dykema, R. Exel and H. Li, *On embeddings of full amalgamated free product C^* -algebras*, arXiv:math/0210448v3, 12 February 2003, Proposition 2.2.