

Exchangeability and Trivial Tails Do Not Imply the Coordinate Intersection Property

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Abstract

For every prime $p \geq 5$, we construct a minimal exchangeable sequence of trace-preserving embeddings of $L(C_p)$ into a hyperfinite finite von Neumann algebra whose tail algebra is scalar but whose canonical filtration fails the intersection axiom for noncommutative factorizations. The intersection of the algebras on $\{0, 1\}$ and $\{1, 2\}$ has dimension p^2 , while the algebra on $\{1\}$ has dimension p . The example comes from an explicit class-three nilpotent group: a central coordinate is recovered by a triple commutator using any auxiliary index, but not from its single generator. We also identify the least enlargement of the coordinate algebras that does form a factorization. Thus the obstruction concerns retaining the given singleton ranges; it does not assert that the ambient algebra admits no factorization with enlarged coordinates.

1 The coordinate question

A random variable in a finite tracial von Neumann algebra (M, τ) is a trace-preserving injective $*$ -homomorphism $\iota : B \rightarrow M$. Its range is automatically the range of a trace-preserving conditional expectation. For a sequence $(\iota_i)_{i \in \mathbb{N}_0}$ of such maps with a common domain, its *canonical filtration* is

$$A_I = \bigvee_{i \in I} \iota_i(B), \quad I \subseteq \mathbb{N}_0.$$

The sequence is minimal if $A_{\mathbb{N}_0} = M$, and its tail is $\bigcap_{n \geq 0} A_{\{n, n+1, \dots\}}$. Exchangeability means invariance of all mixed moments under finite permutations of the indices. These conventions agree with [2, Definitions 1.1, 1.2, 1.6 and 1.12].

In the discrete case, the definition of noncommutative factorization in [1, Definition 1] requires expected subalgebras $(F_I)_{I \subseteq \mathbb{N}_0}$ satisfying

$$F_{I \cap J} = F_I \cap F_J, \quad F_{I \cup J} = F_I \vee F_J, \quad (1)$$

$$\tau(ab) = \tau(a)\tau(b) \quad (a \in F_I, b \in F_J, I \cap J = \emptyset), \quad (2)$$

as well as $\bigvee_n F_{I_n} = M$ whenever $I_n \uparrow \mathbb{N}_0$. Köstler's Conjecture 5 in that report asks whether every minimal exchangeable sequence with trivial tail gives rise to a factorization. The preceding Theorem 4 uses the coordinate-generated family. We answer negatively the *coordinate-preserving* formulation, in which $F_{\{i\}} = \iota_i(B)$. We keep this specification explicit: our example has a factorization after enlargement of the singleton ranges.

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Theorem 1. *For every prime $p \geq 5$ there exist a hyperfinite finite von Neumann algebra (M, τ) with separable predual and a minimal exchangeable sequence $\iota_i : L(C_p) \rightarrow M$ with scalar tail such that*

$$A_{\{0,1\}} \cap A_{\{1,2\}} \supsetneq A_{\{1\}}.$$

The two algebras in this strict inclusion have complex dimensions p^2 and p , respectively. The canonical filtration satisfies all the factorization axioms above except the intersection identity. No factorization can retain all its singleton ranges.

The theorem does not conflict with the extended noncommutative de Finetti theorem: full independence there concerns disjoint index sets [2, Definition 4.1]. Nor is ordinary exchangeability the stronger quantum-permutation invariance in [3]. Automorphism realizations of exchangeable sequences, including a distinction between chosen ranges and larger fixed-point algebras, are discussed in [4, Theorem 2.7 and Definition 2.8].

2 A permutation-equivariant nilpotent group

Fix $p \geq 5$. Let $\mathfrak{g}$ be the direct-sum vector space over $\mathbb{F}_p$ with independent basis

$$X_i, Y_i \quad (i \in \mathbb{N}_0), \quad C_{ij} \quad (i < j).$$

All vectors have finite support. Write $C_{ji} = -C_{ij}$ and $C_{ii} = 0$. Define an alternating bilinear bracket by

$$\begin{aligned} [X_i, X_j] &= C_{ij}, \\ [X_i, C_{ij}] &= Y_i, \quad [X_j, C_{ij}] = -Y_j & (i < j), \\ [X_h, C_{ij}] &= 0 & (h \notin \{i, j\}), \end{aligned} \tag{3}$$

with all Y_i central and $[C_{ij}, C_{ab}] = 0$.

Lemma 2. *These formulas define a Lie algebra of nilpotency class three. Every permutation σ of $\mathbb{N}_0$ acts by the Lie automorphism*

$$X_i \mapsto X_{\sigma(i)}, \quad Y_i \mapsto Y_{\sigma(i)}, \quad C_{ij} \mapsto C_{\sigma(i)\sigma(j)}.$$

Proof. For three distinct X 's the Jacobi terms vanish individually. With repeated X 's they cancel by alternation. A triple containing a C has zero Jacobi terms because the relevant brackets are central or between two C 's. A triple containing a Y also has zero Jacobi terms. This proves Jacobi on the basis and hence everywhere. The second lower central term lies in $\text{span}_{\mathbb{F}_p}\{C_{ij}, Y_i\}$, the third in $\text{span}_{\mathbb{F}_p}\{Y_i\}$, and the fourth is zero; the third is nonzero. The bracket can equivalently be written $[X_i, [X_i, X_j]] = Y_i$ for $i \neq j$, with $[X_h, [X_i, X_j]] = 0$ for pairwise distinct indices. These identities and the centrality conditions are permutation invariant, including the antisymmetric convention on the C 's. $\square$

On $\mathfrak{g}$ define

$$u * v = u + v + \frac{1}{2}[u, v] + \frac{1}{12}([u, [u, v]] + [v, [v, u]]). \tag{4}$$

No exponential map over a finite field is needed here. The usual class-three multiplication identity follows directly by expansion:

$$(u * v) * w - u * (v * w) = -\frac{1}{6}([u, [v, w]] + [v, [w, u]] + [w, [u, v]]) = 0. \tag{5}$$

The subtraction in (5) is vector subtraction. All terms with four brackets vanish, and the denominators are invertible in $\mathbb{F}_p$. Thus (4) defines a group G , with identity 0 and inverse $-u$. The m th group power of u is mu , so G has exponent p . It is countable and locally finite, since every finite set of elements is contained in a finite-dimensional bracket-closed coordinate subspace.

Set $g_i = X_i \in G$, and use the convention $[a, b]_{\text{grp}} = a * b * (-a) * (-b)$ for group commutators. For $i \neq j$, direct substitution gives

$$[g_i, g_j]_{\text{grp}} = C_{ij} + \frac{1}{2}(Y_i - Y_j), \quad [g_i, [g_i, g_j]_{\text{grp}}]_{\text{grp}} = Y_i. \quad (6)$$

In the second formula the Lie bracket is already central, so all higher corrections vanish.

For $I \subseteq \mathbb{N}_0$, let $H_I = \langle g_i : i \in I \rangle$ and put

$$K_I = \text{span}_{\mathbb{F}_p} \{X_i, Y_i : i \in I\} + \text{span}_{\mathbb{F}_p} \{C_{ij} : i < j, i, j \in I\}. \quad (7)$$

The space K_I is closed under (4) and inversion.

Lemma 3. *One has*

$$H_\emptyset = \{0\}, \quad H_{\{i\}} = \mathbb{F}_p X_i, \quad H_I = K_I \quad \text{if } |I| \geq 2.$$

In particular $G = H_{\mathbb{N}_0}$ and

$$H_{\{0,1\}} \cap H_{\{1,2\}} = \mathbb{F}_p X_1 \oplus \mathbb{F}_p Y_1. \quad (8)$$

Proof. The empty and singleton cases follow from the power formula. For $|I| \geq 2$, the inclusion $H_I \subseteq K_I$ follows from closure. For each $i \in I$, choose $j \in I \setminus \{i\}$. Equation (6) supplies Y_i and Y_j . Since these elements are central, correcting the first commutator supplies C_{ij} . Powers give all their scalar multiples. To obtain an arbitrary vector in K_I , first multiply powers of the X_i to prescribe its X coefficients. Next multiply powers of the C_{ij} to correct the C coefficients; this may change only the Y coefficients. Finally correct the latter with central Y_i powers. Only finitely many operations are required, also for infinite I . This proves equality. Formula (8) is then the intersection of two explicit coordinate subspaces. $\square$

The group in (8) is additive and has order p^2 . It properly contains $H_{\{1\}}$, of order p . Coordinate support also gives

$$H_I \cap H_J = \{0\} \quad (I \cap J = \emptyset), \quad \bigcap_{n \geq 0} H_{\{n, n+1, \dots\}} = \{0\}. \quad (9)$$

3 The canonical filtration and its tail

Write λ for the left regular representation of G on $\ell^2(G)$, and put $M = L(G)$ with its faithful normal trace $\tau(T) = \langle T\delta_0, \delta_0 \rangle$. Countability gives separable predual. The increasing finite subgroups on $\{0, \dots, n\}$ generate G , so their finite-dimensional von Neumann algebras have weakly dense union in M ; in particular M is hyperfinite. We do not assert that M is a factor.

We recall the Fourier-support facts needed to avoid confusing group intersections with von Neumann algebra intersections. For a subgroup $H \leq G$, its trace-preserving conditional expectation is

$$E_H(\lambda(g)) = \begin{cases} \lambda(g), & g \in H, \\ 0, & g \notin H. \end{cases} \quad (10)$$

One way to obtain it is compression to $\ell^2(H)$ followed by the standard inclusion of $L(H)$ into $L(G)$. On $L^2(M, \tau) = \ell^2(G)$ it is the orthogonal projection onto $\ell^2(H)$. If T has Fourier support in H ,

then $(T - E_H(T))\delta_0 = 0$, whence $T = E_H(T)$. Indeed δ_0 is separating: right translations commute with $L(G)$ and move δ_0 to every regular basis vector. Consequently, for any family of subgroups,

$$\bigcap_a L(H_a) = L\left(\bigcap_a H_a\right), \quad L(H) \vee L(K) = L(\langle H, K \rangle). \quad (11)$$

If $H \cap K = \{0\}$, group-algebra multiplication gives $\tau(ab) = \tau(a)\tau(b)$ for $a \in \mathbb{C}[H]$, $b \in \mathbb{C}[K]$. The identity extends to $L(H), L(K)$ by L^2 approximation and Cauchy–Schwarz.

Let $B = L(C_p)$ with its normalized trace, and define $\iota_i : B \rightarrow M$ by mapping the cyclic generator to $\lambda(g_i)$. These are injective and trace preserving since g_i has order exactly p . Their ranges are expected by (10). Their canonical filtration is $A_I = L(H_I)$ by (11). Lemma 3 proves minimality.

Every index permutation in Lemma 2 induces a trace-preserving automorphism of $L(G)$ carrying ι_i to $\iota_{\sigma(i)}$. Thus all mixed moments are invariant under finite permutations, proving exchangeability. Equations (9) and (11) give the scalar tail

$$\bigcap_{n \geq 0} A_{\{n, n+1, \dots\}} = L\left(\bigcap_{n \geq 0} H_{\{n, n+1, \dots\}}\right) = \mathbb{C}1.$$

The same equations give scalar independence for all disjoint index sets.

Finally, (8) and (11) imply

$$A_{\{0,1\}} \cap A_{\{1,2\}} = L(\mathbb{F}_p X_1 \oplus \mathbb{F}_p Y_1) \supsetneq L(\mathbb{F}_p X_1) = A_{\{1\}}. \quad (12)$$

The unitary $u = \lambda(Y_1)$ witnesses strictness: it lies in the larger algebra and $E_{H_{\{1\}}}(u) = 0$. The dimensions are p^2 and p because the indicated groups have those orders.

Joins hold by construction. If $I_n \uparrow \mathbb{N}_0$, every generator belongs to some H_{I_n} , giving $\bigvee_n A_{I_n} = M$. This proves all assertions of Theorem 1, including the impossibility of retaining singleton ranges: the join axiom would force the two pair algebras in (12), contradicting the meet axiom.

Corollary 4. *The coordinate expectations commute, but their product need not be the expectation onto the coordinate intersection:*

$$E_{H_{\{0,1\}}} E_{H_{\{1,2\}}} = E_{\mathbb{F}_p X_1 \oplus \mathbb{F}_p Y_1} \neq E_{H_{\{1\}}}.$$

The difference between the last two expectations has L^2 -operator norm one.

Proof. The expectations are diagonal Fourier projections. Their product projects onto the group intersection. The difference is a nonzero orthogonal projection, with norm-one witness $\lambda(Y_1)$. $\square$

4 The least enlarged factorization

The scope of the counterexample can be described constructively.

Proposition 5. *The family $D_I = L(K_I)$, with K_I from (7), is a noncommutative factorization of (M, τ) . It equals A_I except on singleton sets, where*

$$D_{\{i\}} = L(\mathbb{F}_p X_i \oplus \mathbb{F}_p Y_i).$$

It is the least factorization containing all original singleton ranges: if (F_I) is a factorization and $A_{\{i\}} \subseteq F_{\{i\}}$ for all i , then $D_I \subseteq F_I$ for all I .

Proof. Coordinate supports give $K_I \cap K_J = K_{I \cap J}$. The group generated by K_I, K_J is $K_{I \cup J}$: the X_i, Y_i are already present, and commutators followed by central corrections supply the cross C_{ij} . Equations (10)–(11) now prove the meet, join, expectedness and disjoint independence axioms for D . Generation along increasing sets follows from finite support. The comparison with A is Lemma 3.

For minimality of this enlargement, fix i and choose distinct j, h different from i . The join axiom implies $A_{\{i,j\}} \subseteq F_{\{i,j\}}$ and $A_{\{i,h\}} \subseteq F_{\{i,h\}}$. Since $\lambda(Y_i)$ lies in both smaller pair algebras, the meet axiom puts it in $F_{\{i\}}$. Therefore $D_{\{i\}} \subseteq F_{\{i\}}$. The generation argument above gives $D_I = \bigvee_{i \in I} D_{\{i\}}$. Monotonicity and weak closure of F_I finish the proof. $\square$

Thus the construction settles the coordinate-preserving question completely but is not a counterexample to the existence of some enlarged factorization. This distinction should be retained when comparing the theorem with Conjecture 5 of [1]. Likewise, the central Y_i mean that the example does not address that report's separate question concerning type-II or type-III factors.

5 Exact reproducibility

The accompanying standard-library Python script uses rational sparse polynomials and finite-field arithmetic. For three generic nine-coordinate vectors it expands all nine associator coordinates to zero. It also checks all 2744 basis Jacobi triples with four indices over $\mathbb{F}_5$, and enumerates each of the three two-generator subgroups on three indices, each of order 3125. The relevant intersection has order 25 and the singleton subgroup order 5. Additional tests cover 486 permuted basis brackets, 36 ordered commutator formulas at $p = 5, 7, 11$, and 900 seeded exact checks of associativity, inverses and exponent. Normal and optimized Python runs give identical output. The entire group on three indices, of order 5^9 , is not exhaustively enumerated. These checks support, but do not replace, the general proofs above.

The author used AI assistance for exploration, drafting, literature searches and exact computational checks. This manuscript is unrefereed, has not been independently reviewed or verified by a proof assistant, and makes no absolute priority claim.

References

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