

Nonmonotone Potential Means for a Convex Dirichlet Schrödinger Operator

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Abstract

Lavine asked whether the potential expectations of normalized Dirichlet eigenfunctions increase with the eigenvalue index on a convex domain with a convex potential. On the interval $(0, 1)$ we prove that the affine potential $V(x) = x$ has its second potential expectation larger than its third by more than $1/20000$. The inequality persists throughout the L^∞ ball of radius 10^{-5} about this potential, with gap greater than $1/40000$. In particular, $V(x) = x + 10^{-5}x^2$ is a strictly convex counterexample. The proof combines the classical second-order Stark coefficient with a complex-analytic remainder bound and an elementary eigenvector perturbation estimate.

1 The question and the counterexample

For a real bounded potential V on $(0, 1)$, let

$$H_V = -\frac{d^2}{dx^2} + V, \quad \mathcal{D}(H_V) = H^2(0, 1) \cap H_0^1(0, 1).$$

Write $\mu_1(V) < \mu_2(V) < \dots$ for its eigenvalues and choose real L^2 -normalized eigenfunctions $u_j(V)$. All these eigenvalues are simple. Define

$$M_j(V) = \int_0^1 V(x) u_j(V, x)^2 dx.$$

These expectations do not depend on the choice of sign of u_j .

Lavine's question in [1, Section 3, question (2)], also recorded in [2, Section 10, question (2), p. 424], asks whether $M_j(V)$ is increasing in j for a convex potential on a bounded convex domain. The source explicitly includes dimension one and imposes no reflection symmetry. The following result gives a negative answer even for smooth, positive, strictly convex potentials.

Theorem 1. *For every $0 < t \leq 1$, the potential $V_t(x) = tx$ satisfies*

$$M_2(V_t) - M_3(V_t) > \frac{t^2}{20000}. \quad (1)$$

Furthermore, every real $V \in L^\infty(0, 1)$ satisfying $\|V - x\|_\infty \leq 10^{-5}$ obeys

$$M_2(V) - M_3(V) > \frac{1}{40000}. \quad (2)$$

Consequently the polynomial $V(x) = x + 10^{-5}x^2$ is a strictly convex counterexample to the proposed monotonicity.

The second-order Stark coefficient used below is classical. Belloni and Robinett [3, Section III, equations (40)–(42)] derive it using sum rules and credit earlier calculations by Mavromatis. We give a direct ODE derivation in our normalization, then bound the remainder to obtain an explicit fixed potential and a quantitative neighborhood of counterexamples. No novelty is asserted for the Stark coefficient or the perturbation tools. The conclusion concerns the potential-mean question, not the other gap questions grouped with it in the workshop report. No exhaustive priority claim is made.

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2 The classical coefficient with a uniform remainder

Put $W(x) = x - \frac{1}{2}$ and consider the centered analytic family

$$A(z) = -\frac{d^2}{dx^2} + zW, \quad z \in \mathbb{C},$$

on the same Dirichlet domain. For $z = 0$ its eigenvalues are $E_n = n^2\pi^2$, with normalized eigenfunctions $\phi_n(x) = \sqrt{2}\sin(n\pi x)$. Let $\lambda_n(z)$ denote the branch through E_n .

Lemma 2. *For $n = 2, 3$, the function λ_n is single-valued and analytic on a neighborhood of $|z| \leq 27$. It has the expansion*

$$\lambda_n(z) = E_n + \sum_{m=1}^{\infty} a_{2m,n} z^{2m}, \quad |a_{2m,n}| \leq \frac{27}{2} 27^{-2m}. \quad (3)$$

Its quadratic coefficient is

$$a_{2,n} = \frac{(n\pi)^2 - 15}{48(n\pi)^4}. \quad (4)$$

Proof. The distance from E_n to the other eigenvalues of $A(0)$ is at least $3\pi^2$ for $n = 2, 3$. On the circle $|\zeta - E_n| = 3\pi^2/2$, the resolvent of the self-adjoint operator $A(0)$ has norm at most $2/(3\pi^2)$. Since $\|W\| = 1/2$, the Neumann resolvent series converges there for $|z| < 3\pi^2$. The corresponding Riesz projection is analytic and has rank one, as it does at zero. Thus the eigenvalue inside that circle is analytic throughout $|z| < 3\pi^2$.

The same resolvent argument gives the spectral inclusion

$$\text{spec } A(z) \subseteq \bigcup_{k \geq 1} \{\zeta : |\zeta - E_k| \leq |z|/2\}.$$

For $|z| \leq 27 < 3\pi^2$, the branch inside the chosen contour can belong only to the disk centered at E_n . Therefore

$$|\lambda_n(z) - E_n| \leq |z|/2 \leq 27/2.$$

Reflection $f(x) \mapsto f(1-x)$ conjugates $A(z)$ to $A(-z)$. Uniqueness of the eigenvalue inside the contour gives $\lambda_n(z) = \lambda_n(-z)$. Cauchy's coefficient estimate on $|z| = 27$ proves (3).

For completeness, the quadratic coefficient follows directly from the differentiated eigenvalue equation. Set $a = n\pi$ and $\phi = \sqrt{2}\sin(ax)$. Since $\langle \phi, W\phi \rangle = 0$, the derivative v of a locally normalized eigenfunction at zero is characterized by

$$\left(-\frac{d^2}{dx^2} - a^2\right)v = -W\phi, \quad v(0) = v(1) = 0, \quad \langle \phi, v \rangle = 0.$$

An explicit solution is

$$v(x) = \sqrt{2} \left[\frac{x(1-x)}{4a} \cos(ax) + \frac{x - \frac{1}{2}}{4a^2} \sin(ax) \right]. \quad (5)$$

Substitution checks the equation and endpoint values. The product ϕv is odd about $1/2$, which checks orthogonality. Differentiating the eigenvalue equation a second time gives $a_{2,n} = \langle \phi, Wv \rangle$. Elementary integration yields

$$\langle \phi, Wv \rangle = -\frac{3}{16a^4} + \left(\frac{1}{48a^2} - \frac{1}{8a^4} \right) = \frac{a^2 - 15}{48a^4}.$$

This proves (4). □

For real t in the interval under consideration, $H_{V_t} = A(t) + t/2$ has the same eigenfunctions as $A(t)$. These analytic branches retain their index: the eigenvalues of a real Dirichlet problem are simple, and at $t = 0$ their indices are n . Differentiating the normalized eigenvalue equation, or using the Hellmann–Feynman identity, gives

$$M_n(V_t) = \frac{t}{2} + t\lambda'_n(t). \quad (6)$$

In particular, the common first-order term cancels in the difference.

Proof of (1). Equations (4) and (6) give

$$M_2(V_t) - M_3(V_t) = Ct^2 + \mathcal{R}(t), \quad C = 2(a_{2,2} - a_{2,3}) = \frac{5(12\pi^2 - 65)}{10368\pi^4}.$$

For $0 < t \leq 1$, the coefficient bounds in Lemma 2 imply

$$\begin{aligned} |\mathcal{R}(t)| &\leq 54 \sum_{m=2}^{\infty} m(t/27)^{2m} \\ &\leq 54t^4 \sum_{m=2}^{\infty} m(1/729)^m = \frac{1457}{7154784} t^4. \end{aligned}$$

Here $\sum_{m \geq 2} mq^m = q^2(2 - q)/(1 - q)^2$. The elementary bounds $49/5 < \pi^2 < 10$ give

$$C > \frac{263}{1036800}.$$

Since $t^4 \leq t^2$, it follows that

$$M_2(V_t) - M_3(V_t) > \left(\frac{263}{1036800} - \frac{1457}{7154784} \right) t^2 = \frac{429503}{8585740800} t^2 > \frac{t^2}{20000}.$$

All inequalities use rational bounds; a truncated numerical eigenvalue computation is not needed. $\square$

3 Stability and strict convexity

We include a quantitative stability estimate so that the counterexample does not rely on an affine potential lying on the boundary of the convexity condition.

Lemma 3. *Let $V_0(x) = x$ and let $V = V_0 + B$ with B real and $\delta = \|B\|_{\infty} \leq 10^{-5}$. For $n = 2, 3$,*

$$|M_n(V) - M_n(V_0)| \leq \delta + \frac{4\delta}{25 - \delta}.$$

Proof. Write $A = H_{V_0}$ and $\mu_n = \mu_n(V_0)$. The min-max principle and $\|V_0\|_{\infty} = 1$ show that $|\mu_j - E_j| \leq 1$. Hence, for $n = 2, 3$,

$$\text{dist}(\mu_n, \text{spec } A \setminus \{\mu_n\}) \geq 3\pi^2 - 2 > 25.$$

The same principle gives $|\mu_n(V) - \mu_n| \leq \delta$.

Let $u = u_n(V_0)$ and $w = u_n(V)$ be normalized, with signs chosen so that $\langle u, w \rangle \geq 0$. Let Q be orthogonal projection onto $u^{\perp}$. Projecting $(A + B)w = \mu_n(V)w$ gives

$$(A|_{u^{\perp}} - \mu_n(V))Qw = -QBw.$$

The inverse on $u^{\perp}$ has norm at most $(25 - \delta)^{-1}$. Consequently

$$\|Qw\|_2 \leq \frac{\delta}{25 - \delta}, \quad \|w - u\|_2 \leq 2\|Qw\|_2.$$

For the second inequality, write $\langle u, w \rangle = \sqrt{1 - \|Qw\|_2^2}$ and use $2(1 - \sqrt{1 - r^2}) \leq 2r^2$ for $0 \leq r \leq 1$. Finally,

$$\begin{aligned} |M_n(V) - M_n(V_0)| &\leq |\langle w, Bw \rangle| + |\langle w, V_0 w \rangle - \langle u, V_0 u \rangle| \\ &\leq \delta + 2\|V_0\|_{\infty} \|w - u\|_2 \leq \delta + \frac{4\delta}{25 - \delta}. \end{aligned}$$

$\square$

Proof of (2) and strict convexity. Lemma 3 and (1) at $t = 1$ give

$$M_2(V) - M_3(V) > \frac{1}{20000} - \left(2\delta + \frac{8\delta}{25 - \delta}\right).$$

The expression in parentheses increases with δ and, at $\delta = 10^{-5}$, equals

$$\frac{2899999}{124999950000} < \frac{1}{40000}.$$

Thus (2) follows. Taking $B(x) = 10^{-5}x^2$ gives $\|B\|_\infty = 10^{-5}$ and $V''(x) = 2 \cdot 10^{-5} > 0$. This proves the claimed strictly convex counterexample. $\square$

The source's proposed monotonicity fails between indices two and three. The argument makes no assertion against a separate comparison of the first two potential means. It also does not contradict the fundamental-gap theorem: the gap of the first two total energies and the ordering of all potential expectations are different statements.

The accompanying scripts check the rational bounds, the all-index symbolic ODE identities, and optional sine-basis numerical corroboration. They supplement the written proof and are not formal proof verification. AI assistance was used in developing the argument, checking the algebra, searching the literature, and preparing the manuscript; the author takes responsibility for this self-audited preprint.

References

- [1] American Institute of Mathematics, *Low eigenvalues of Laplace and Schrödinger operators: problem list*, Section 3, question (2), p. 12. <https://aimath.org/WWN/loweigenvalues/loweigenvalues.pdf>.
- [2] *Low Eigenvalues of Laplace and Schrödinger Operators*, Oberwolfach Reports **6** (2009), 355–428, Section 10, question (2), p. 424. [doi:10.4171/OWR/2009/06](https://doi.org/10.4171/OWR/2009/06).
- [3] M. Belloni and R. W. Robinett, *Quantum mechanical sum rules for two model systems*, American Journal of Physics **76** (2008). [doi:10.1119/1.2908194](https://doi.org/10.1119/1.2908194); author preprint [arXiv:0802.2217](https://arxiv.org/abs/0802.2217), Section III.