

The Harrell–Stubbe Gap Inequality for Dirichlet Eigenvalues Is Never Saturated

Alper Ferudun*

September 30, 2026

Abstract

Let $E_1 \leq E_2 \leq \dots$ be the eigenvalues of the Dirichlet Laplacian on a bounded open set $\Omega \subset \mathbb{R}^n$, and let $M_p(J) = \left(\frac{n+2p}{n} \frac{1}{J} \sum_{j \leq J} E_j^p \right)^{1/p}$. Harrell and Stubbe proved that $M_1(J)^2 - M_2(J)^2 \geq \frac{1}{4}(E_{J+1} - E_J)^2$, and in a problem list of the 2009 Oberwolfach workshop on low eigenvalues of Laplace and Schrödinger operators they asked whether some Ω and J saturate this inequality. We show that the answer is no: the inequality is strict for every bounded open set and every J . Written in terms of the mean and the variance of $E_1, \dots, E_J$, the inequality is a gap estimate of Cheng and Yang, so that estimate is strict as well. The proof analyses the case of equality in the Harrell–Stubbe trace identity behind H. C. Yang’s inequality. Equality would put each function $x_k u_1$, where $u_1 \geq 0$ is a first eigenfunction, into a finite sum of eigenspaces. Then every partial derivative of u_1 would lie in $H_0^1(\Omega)$, so $\int_{\Omega} \Delta u_1 = 0$, which is impossible because $\Delta u_1 = -E_1 u_1$. The same argument shows that Yang’s first inequality is strict whenever $E_{J+1} > E_1$, and hence for every J when Ω is connected; in 2002 Ashbaugh left the strictness of this inequality undecided. By contrast, for the harmonic oscillator and on spheres the analogous bounds are equalities at every spectral gap, that is, for every J with $E_J < E_{J+1}$. This is an unrefereed note.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set, and let $0 < E_1 \leq E_2 \leq \dots$ be the eigenvalues of the Dirichlet Laplacian on Ω , repeated according to multiplicity. Following Harrell and Stubbe [HS09, (2)], put

$$M_p(J) = \left(\frac{n+2p}{n} \frac{1}{J} \sum_{j=1}^J E_j^p \right)^{1/p} \quad (J \geq 1, p > 0). \quad (1)$$

They proved [HS97] the gap inequality

$$M_1(J)^2 - M_2(J)^2 \geq \frac{1}{4} (E_{J+1} - E_J)^2, \quad (2)$$

and in the problem list of the 2009 Oberwolfach workshop on low eigenvalues of Laplace and Schrödinger operators they asked [HS09, p. 415]: “Can one find Ω and J such that the inequality [(2)] is saturated?” That section of the list concerns the Dirichlet Laplacian on bounded domains in $\mathbb{R}^n$, $n \geq 2$. The question is record OWR-3389-016 of the UnsolvedMath dataset [ula26]. The report prints the left-hand side of (2) as $M_1^2(J) - M_2(J)$, here and in (4) below. With M_2 as in (1) this is not homogeneous in the eigenvalues, while the commutator method gives $M_2(J)^2$ (Section 3; compare

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

[Ash99, (3.32)–(3.33)] and [HS11, (3.12)]). So we read $M_2(J)$ as a misprint for $M_2(J)^2$. The dataset record writes $M_{1/2}(J) - M_2(J)$, which we regard as an extraction error.

The inequality (2) comes from H. C. Yang's inequality [Yan91] in the form given by Harrell and Stubbe [HS97]; see also [Ash99, Ash02, LP02, HS11]. For $z \in \mathbb{R}$ put

$$Q_J(z) = \sum_{j=1}^J (z - E_j) \left(z - \left(1 + \frac{4}{n}\right) E_j \right) = J(z^2 - 2M_1(J)z + M_2(J)^2). \quad (3)$$

Then $Q_J(z) \leq 0$ for $E_J \leq z \leq E_{J+1}$ (see Section 3), and for $z = E_{J+1}$ this is Yang's first inequality [Ash02, (1.7)]. Hence the quadratic (3) has real roots $r_{\pm} = M_1(J) \pm (M_1(J)^2 - M_2(J)^2)^{1/2}$, and

$$r_- \leq E_J \leq E_{J+1} \leq r_+, \quad (4)$$

which gives (2). Both (2) and (4) are recorded in [HS09, p. 415]. For $J = 1$, (2) is the Payne–Pólya–Weinberger bound $E_2 \leq (1 + \frac{4}{n})E_1$ ([PPW56]; see [Ash02, (1.5)] for general n), which is known to be strict [Ash02, Section 3].

Let $\bar{E}_J = \frac{1}{J} \sum_{j \leq J} E_j$ and $V_J = \frac{1}{J} \sum_{j \leq J} (E_j - \bar{E}_J)^2$. Then

$$M_1(J)^2 - M_2(J)^2 = \left(1 + \frac{2}{n}\right)^2 \bar{E}_J^2 - \left(1 + \frac{4}{n}\right) (V_J + \bar{E}_J^2) = \left(\frac{2\bar{E}_J}{n}\right)^2 - \left(1 + \frac{4}{n}\right) V_J,$$

so (2) is the same inequality as the gap estimate

$$E_{J+1} - E_J \leq 2 \left[\left(\frac{2\bar{E}_J}{n}\right)^2 - \left(1 + \frac{4}{n}\right) V_J \right]^{1/2}$$

that Cheng and Yang derived from Yang's first inequality [CY05] (we quote it from [CZY16, (1-8)]).

Theorem 1.1. *For every bounded open set $\Omega \subset \mathbb{R}^n$ and every $J \geq 1$,*

$$M_1(J)^2 - M_2(J)^2 > \frac{1}{4} (E_{J+1} - E_J)^2.$$

So no Ω and J saturate (2), and the answer to the question of Harrell and Stubbe is negative.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $J \geq 1$ and $E_J \leq z \leq E_{J+1}$. If $z > E_1$, then $Q_J(z) < 0$.*

Corollary 1.3. *Yang's first inequality $Q_J(E_{J+1}) \leq 0$ is strict whenever $E_{J+1} > E_1$, so for every J if Ω is connected. For every $z > E_1$, $\sum_j (z - E_j)_+^2 < \frac{4}{n} \sum_j (z - E_j)_+ E_j$.*

The first statement is Theorem 1.2 with $z = E_{J+1}$, together with $E_1 < E_2$ for connected Ω (Section 2). The second is a strict form of the Riesz-mean version of Yang's inequality [HS11, (3.11)]; it follows from Theorem 1.2 with J the number of j with $E_j < z$. Ashbaugh [Ash02, Section 3] proved that the inequalities of Payne–Pólya–Weinberger, of Hile and Protter [HP80] and Yang's second inequality are strict, and he left the strictness of Yang's first inequality undecided. Corollary 1.3 settles this. The hypothesis $E_{J+1} > E_1$ cannot be dropped (Remark 5.1). For Schrödinger operators $-\Delta + V$ with $V \geq M > 0$, Ashbaugh and Hermi [AH04, Theorem 4.1 and the Remark after it] showed that the classical inequalities of Payne–Pólya–Weinberger, Hile–Protter and Yang, in which the potential does not appear, hold with strict inequality. Their argument uses the lower bound $M > 0$ of the potential, so it does not cover $V = 0$, the case treated here. By the computation above, Theorem 1.1 also shows that the gap estimate of Cheng and Yang is strict for every bounded open set and every J .

The proofs use the trace identity of Harrell and Stubbe [HS97] (Proposition 3.2); the new ingredient is the analysis of its case of equality (Theorem 1.2). Harrell and Stubbe observe [HS10, Section 1] that for the Schrödinger operators of [HS97] the gap bounds become equalities for the harmonic oscillator, and that for Laplacians on embedded manifolds they become equalities for spheres (see [ESHI09, Corollary 2.4]); Provenzano and Stubbe [PS25] proved the corresponding equalities for all compact rank-one symmetric spaces. So the question is a real one; Remark 5.2 explains the difference.

2 Preliminaries

Throughout, $\Omega \subset \mathbb{R}^n$ is a bounded open set, not necessarily connected, and no regularity of $\partial\Omega$ is assumed. Functions are real, and $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ refer to $L^2(\Omega)$. $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, and the Dirichlet Laplacian H is the self-adjoint operator associated with the form $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$ on $H_0^1(\Omega)$. Thus $D(H) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$, with the distributional Laplacian, and $Hu = -\Delta u$. As $H_0^1(\Omega)$ embeds compactly into $L^2(\Omega)$ (extend by zero to a ball and use Rellich's theorem), $L^2(\Omega)$ has an orthonormal basis of real eigenfunctions $u_1, u_2, \dots$, $Hu_l = E_l u_l$. By the Poincaré inequality $E_1 > 0$, and

$$a(v, v) = \sum_l E_l \langle v, u_l \rangle^2 \quad (v \in H_0^1(\Omega)). \quad (5)$$

Lemma 2.1. (a) If $u \in D(H)$, then $x_k u \in D(H)$ and $H(x_k u) = x_k H u - 2\partial_k u$ ($1 \leq k \leq n$).

(b) If $v \in H_0^1(\Omega)$, then $\int_\Omega \partial_i v \, dx = 0$ and $2 \int_\Omega x_k v \partial_k v \, dx = - \int_\Omega v^2 \, dx$ ($1 \leq i, k \leq n$).

(c) The eigenfunctions can be chosen so that, in addition, $u_1 \geq 0$ almost everywhere.

Proof. (a) If $\varphi_m \in C_c^\infty(\Omega)$ tend to u in $H^1(\Omega)$, so do $x_k \varphi_m \in C_c^\infty(\Omega)$ to $x_k u$, as x_k is bounded on Ω ; so $x_k u \in H_0^1(\Omega)$. In the sense of distributions $\Delta(x_k u) = x_k \Delta u + 2\partial_k u \in L^2(\Omega)$.

(b) Both identities hold for $v \in C_c^\infty(\Omega)$ (integrate by parts), and both sides are continuous on $H^1(\Omega)$ because Ω has finite measure and x_k is bounded on Ω ; by density they hold on $H_0^1(\Omega)$.

(c) Let u be a real eigenfunction for E_1 with $\|u\| = 1$. By [GT01, Lemma 7.6], $|u| \in H^1(\Omega)$ and $|\nabla|u|| \leq |\nabla u|$ almost everywhere. Also $|u| \in H_0^1(\Omega)$: if $\varphi_m \in C_c^\infty(\Omega)$ tend to u in $H^1(\Omega)$, the functions $|\varphi_m|$ are Lipschitz with compact support in Ω , hence in $H_0^1(\Omega)$, bounded in $H^1(\Omega)$, and convergent to $|u|$ in $L^2(\Omega)$; so a subsequence converges weakly in $H_0^1(\Omega)$, to $|u|$. Now $v = |u|$ has $\|v\| = 1$ and $a(v, v) \leq a(u, u) = E_1$. By (5), $a(v, v) - E_1 = \sum_l (E_l - E_1) \langle v, u_l \rangle^2$ is a sum of nonnegative terms; so every term vanishes, and v lies in the eigenspace of E_1 . Take an orthonormal basis of this eigenspace with first element v , and orthonormal bases of the other eigenspaces. $\square$

If Ω is connected, E_1 is simple with a positive eigenfunction [GT01, Theorem 8.38], and (c) is automatic. If not, the eigenspace of E_1 can contain functions of both signs (for two disjoint congruent balls, a first eigenfunction on one ball minus its translate on the other), and (c) is a choice of basis. The choice is allowed: the identities of Section 3 hold for every orthonormal basis of eigenfunctions, while (2) and Q_J involve only the eigenvalues. From now on we use a basis as in (c).

3 The commutator identity

For $1 \leq k \leq n$ and $j, l \geq 1$ let $g_{jl}^k = \langle x_k u_j, u_l \rangle = g_{lj}^k$.

Lemma 3.1. For all j and k , with absolutely convergent series,

$$\sum_l (E_l - E_j) (g_{jl}^k)^2 = 1, \quad \sum_l (E_l - E_j)^2 (g_{jl}^k)^2 = 4 \|\partial_k u_j\|^2. \quad (6)$$

Proof. By Lemma 2.1(a) and the self-adjointness of H , $(E_l - E_j)g_{jl}^k = \langle H(x_k u_j) - x_k H u_j, u_l \rangle = -2\langle \partial_k u_j, u_l \rangle$. So $(E_l - E_j)g_{jl}^k$ and g_{jl}^k are the coefficients of $-2\partial_k u_j$ and of $x_k u_j$ in the basis (u_l) . By Parseval's identity the first sum is $\langle -2\partial_k u_j, x_k u_j \rangle = \|u_j\|^2 = 1$ (Lemma 2.1(b)), and the second is $\|2\partial_k u_j\|^2$. Absolute convergence follows from the Cauchy-Schwarz inequality. $\square$

Proposition 3.2. *For $J \geq 1$ and $z \in \mathbb{R}$,*

$$-n Q_J(z) = \sum_{k=1}^n \sum_{j=1}^J \sum_{l>J} (z - E_j)(E_l - E_j)(E_l - z) (g_{jl}^k)^2. \quad (7)$$

Proof. Let $c_{jl}(z) = (z - E_j)(E_l - E_j)(E_l - z)$. Multiply the second identity in (6) by $z - E_j$ and the first by $(z - E_j)^2$, subtract, and sum over $j \leq J$:

$$4 \sum_{j \leq J} (z - E_j) \|\partial_k u_j\|^2 - \sum_{j \leq J} (z - E_j)^2 = \sum_{j \leq J} \sum_{l \geq 1} c_{jl}(z) (g_{jl}^k)^2.$$

As $c_{lj}(z) = -c_{jl}(z)$ and $g_{lj}^k = g_{jl}^k$, the terms with $l \leq J$ cancel in pairs. Summing over k and using $\sum_k \|\partial_k u_j\|^2 = a(u_j, u_j) = E_j$, the left-hand side becomes $4 \sum_{j \leq J} (z - E_j) E_j - n \sum_{j \leq J} (z - E_j)^2 = -n Q_J(z)$. $\square$

For $E_J \leq z \leq E_{J+1}$ every term on the right of (7) is nonnegative, since $E_j \leq E_J \leq z$ for $j \leq J$ and $z \leq E_{J+1} \leq E_l$ for $l > J$. This gives $Q_J \leq 0$ on $[E_J, E_{J+1}]$, hence (4) and (2).

The identity (7) is not new: it is the case of the Dirichlet Laplacian of the trace identity of Harrell and Stubbe [HS97]. Stubbe displays that identity in his abstract in the same report [Stu09, (4)]. Take $\alpha = 1$ and $d = n$ there, write J for the number of eigenvalues and l for the second summation index (called n and k there), and let the sum over l run over all $l > J$ (its upper limit is misprinted as n). Then $T_j = \|\nabla u_j\|^2 = E_j$ and, by the proof of Lemma 3.1, $T_{jl} = \sum_k \langle u_j, \partial_k u_l \rangle^2 = \frac{1}{4} \sum_k (E_l - E_j)^2 (g_{jl}^k)^2$, so [Stu09, (4)] becomes (7). An analogous identity for the functions $e^{-iq \cdot x}$ in place of x_k , with the remainder written out, is [HS11, (3.9)]; see also [Ash02, LP02]. We have included the short proof to keep the note self-contained. What is new here is the analysis of the case of equality in (7) (Theorem 1.2).

4 Proofs

Proof of Theorem 1.2. Suppose that $E_J \leq z \leq E_{J+1}$, $z > E_1$ and $Q_J(z) = 0$. Then every term on the right of (7), being nonnegative, vanishes. Take $j = 1$ and any l with $E_l > z$. Then $l > J$, and $z - E_1$, $E_l - E_1$ and $E_l - z$ are positive. Hence $g_{1l}^k = 0$ for all k and all l with $E_l > z$, that is,

$$x_k u_1 \in W := \text{span}\{u_l : E_l \leq z\} \quad (1 \leq k \leq n).$$

The space W is finite dimensional, $W \subset D(H) \subset H_0^1(\Omega)$ and $HW \subset W$. By Lemma 2.1(a), $2\partial_k u_1 = E_1 x_k u_1 - H(x_k u_1) \in W \subset H_0^1(\Omega)$. By Lemma 2.1(b) with $v = \partial_k u_1$, $\int_\Omega \partial_k (\partial_k u_1) dx = 0$ for each k . The weak derivatives of $\partial_k u_1 \in H_0^1(\Omega)$ are its distributional derivatives, so $\sum_k \partial_k (\partial_k u_1) = \Delta u_1 = -E_1 u_1$; summing over k therefore gives $E_1 \int_\Omega u_1 dx = 0$. This is impossible, since $E_1 > 0$ and $u_1 \geq 0$ with $\|u_1\| = 1$. $\square$

Proof of Theorem 1.1. If $E_{J+1} = E_1$, then $E_1 = \dots = E_{J+1} =: E$; as $(1 + \frac{2}{n})^2 - (1 + \frac{4}{n}) = 4/n^2$, we get $M_1(J)^2 - M_2(J)^2 = 4E^2/n^2 > 0 = \frac{1}{4}(E_{J+1} - E_J)^2$. If $E_{J+1} > E_1$, then $Q_J(E_{J+1}) < 0$ by Theorem 1.2, so $E_{J+1} < r_+$, while $E_J \geq r_-$ by (4). Hence $E_{J+1} - E_J < r_+ - r_- = 2(M_1(J)^2 - M_2(J)^2)^{1/2}$. $\square$

5 Remarks

Remark 5.1. (i) If Ω is the union of two disjoint congruent balls, then $E_1 = E_2$ and $Q_1(E_2) = 0$: Yang's first inequality is an equality for $J = 1$, while (2) is strict. (ii) In (4) the upper estimate $E_{J+1} \leq r_+$ is always strict (by Theorem 1.2, or since $r_+ = (1 + \frac{4}{n})E_1$ if $E_{J+1} = E_1$). The lower estimate $r_- \leq E_J$ is strict if $E_J > E_1$ (Theorem 1.2 with $z = E_J$), but it is an equality if $E_J = E_1$, for instance for $J = 1$, because then $Q_J(z) = J(z - E_1)(z - (1 + \frac{4}{n})E_1)$.

Remark 5.2. For $-\Delta + |x|^2$ on $\mathbb{R}^n$ the argument of Section 3 applies to the Hermite functions (they are Schwartz functions, so the identities of Lemma 2.1(a),(b) hold for them), except that $\|\nabla u_j\|^2 = E_j/2$ (virial theorem) instead of E_j . This gives (7) with $\tilde{Q}_J(z) = \sum_{j \leq J} j(z - E_j)(z - (1 + \frac{2}{n})E_j)$ in place of Q_J , and (2) with $(n + p)/n$ in place of $(n + 2p)/n$ in (1); we write $\tilde{M}_p(J)$ for these means. The eigenvalues are $2m + n$ ($m \geq 0$), and x_k maps the eigenspace of $2m + n$ into those of $2m + n \pm 2$. So if $E_J < E_{J+1}$, then $g_{jl}^k \neq 0$ with $j \leq J < l$ only if $E_j = E_J$ and $E_l = E_{J+1}$; every term of the analogue of (7) vanishes at $z = E_J$ and at $z = E_{J+1}$, and the analogue of (2) is an equality. For $n = 2$ it is also an equality $0 = 0$ for some J inside an eigenspace: if $J = 2k(k + 1)$ with $k \geq 1$, then $E_J = E_{J+1} = 4k + 2$, $\sum_{j \leq J} E_j = \frac{8}{3}k(k + 1)(2k + 1)$ and $\sum_{j \leq J} E_j^2 = 4k(k + 1)(2k + 1)^2$, and hence $\tilde{M}_1(J) = \tilde{M}_2(J)$. So the case $E_J = E_{J+1}$ of Theorem 1.1 has content; for the Dirichlet Laplacian, strictness in this case comes from Theorem 1.2 when $E_{J+1} > E_1$. On $S^n \subset \mathbb{R}^{n+1}$ the coordinate functions likewise map spherical harmonics of degree ℓ to degrees $\ell \pm 1$, and the analogue of Yang's first inequality is an equality at every spectral gap [ESHI09, Corollary 2.4]. Provenzano and Stubbe prove, for all compact rank-one symmetric spaces, that the corresponding quadratic vanishes at both ends of every spectral gap [PS25, Theorem 1.4], so that the analogue of (2) is an equality there. They deduce from this a selection rule of this kind [PS25, (2.10)], and they show that the equality fails for the square and the equilateral flat tori [PS25, Appendix B]. For the Dirichlet Laplacian on a bounded Euclidean domain no such selection rule holds: by the proof of Theorem 1.2, the functions $x_1 u_1, \dots, x_n u_1$ never all lie in a finite sum of eigenspaces.

Remark 5.3 (numerical illustration). Let $\rho_J = (E_{J+1} - E_J)^2 / (4(M_1(J)^2 - M_2(J)^2)) < 1$. For $J \leq 2000$, the maximum of ρ_J is $9/16$ for $(0, \pi)$, $(0, \pi)^2$ and $(0, \pi)^3$, $1/4$ for $(0, \pi) \times (0, \pi/\sqrt{2})$ (exact arithmetic), 0.5919 for the unit disk and 0.6151 for the unit ball in $\mathbb{R}^3$ (double precision, Bessel zeros), in each case attained at $J = 1$. For the interval, $\rho_J = 3(2J + 1)/((J + 1)(3J + 5)) \rightarrow 0$. For $J = 1$, $\rho_1 = \frac{n^2}{16}(E_2/E_1 - 1)^2$, which among bounded domains in $\mathbb{R}^n$ is largest for the ball [AB92], where $E_2/E_1 = (j_{n/2,1}/j_{n/2-1,1})^2$. For the unit ball, $\rho_1 = 0.5625, 0.5919, 0.6151, 0.7062, 0.8828, 0.9673$ for $n = 1, 2, 3, 10, 100, 1000$, and $\rho_1 \rightarrow 1$ as $n \rightarrow \infty$ by the asymptotics of $j_{\nu,1}$ [NIS26, (10.21.40)]. So Theorem 1.1 cannot be improved to $\rho_J \leq c < 1$ with c independent of the dimension. For the harmonic oscillator on $\mathbb{R}$ ($E_j = 2j - 1$) the analogue of ρ_J equals 1 for every J .

Verification. The proofs do not use computation; the computations described here only illustrate the results. The programs and their recorded outputs are included in the source files of this note (folder `reproducibility`).

1. A program by the author, `saturation_ratios.py` (Python with numpy and scipy, a few seconds), computes ρ_J and $Q_J(E_{J+1})$ for $J \leq 2000$ for the domains of Remark 5.3 and for two disjoint disks, and confirms $\rho_J < 1$, $Q_J(E_{J+1}) < 0$ whenever $E_{J+1} > E_1$, and $Q_1(E_2) = 0$ for the two disks. It also checks the formula for the interval, the equalities of Remark 5.2 exactly on $\mathbb{R}$ and $\mathbb{R}^2$ for $J \leq 2000$, and, on $(0, \pi)$, where g_{jl}^1 is explicit, (6) for $j \leq 5$ and (7) for $J \leq 4$ at $z = E_{J+1}$ and at the midpoint of $[E_J, E_{J+1}]$ (10^6 terms and a tail estimate).
2. Two independent verification runs, both AI-assisted, re-read the question on the rendered page of the report and checked the proofs line by line. Neither found a mathematical error.

- The first run asked for three changes, which were made: the corrected statement of (2) with a note on the misprint; the scope, and a justification of the nonnegative first eigenfunction on disconnected sets (Lemma 2.1(c) and the paragraph after it); and a search for explicit strictness statements (see below).
- The second run asked for changes of credit and wording, which were made in this version: the credit for (7) to [HS97], the equivalence with the gap estimate of [CY05], and references to [AH04, PS25]. Before it read the program above, it wrote its own code and recomputed the numbers of Remarks 5.2 and 5.3. In exact arithmetic it confirmed $\rho_J < 1$, and $Q_J(E_{J+1}) < 0$ whenever $E_{J+1} > E_1$, on boxes of dimension 1 to 7, on unions of two or three disjoint congruent boxes and on 60 random disjoint unions of boxes ($J \leq 2000$ for the domains of Remark 5.3, and $J \leq 300$ to $J \leq 1200$ for the others). It treated the disk and the ball in $\mathbb{R}^3$ with Bessel zeros, checked (6) and (7) on $(0, \pi)$ and (7) on the square, and checked the equalities of Remark 5.2 on $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ and on the spheres $S^1, \dots, S^6$. It also pointed out the equalities $0 = 0$ inside eigenspaces on $\mathbb{R}^2$ (Remark 5.2).

Scope and priority. Theorem 1.1 answers the question of [HS09, p. 415] for the Dirichlet Laplacian on every bounded open subset of $\mathbb{R}^n$, for every $n \geq 1$ (the report assumes $n \geq 2$) and every J , including J with $E_J = E_{J+1}$. We do not treat Neumann conditions, Schrödinger operators, domains in manifolds or unbounded domains; for some operators of these kinds the analogous bounds are equalities (Remark 5.2). The identity (7) is due to Harrell and Stubbe [HS97], and (2) is equivalent to the gap estimate of Cheng and Yang [CY05]; what is new here is the analysis of the case of equality (Theorem 1.2). The strictness result of Ashbaugh and Hermi [AH04, Theorem 4.1 and the Remark after it] requires a potential $V \geq M > 0$ and does not cover the Dirichlet Laplacian ($V = 0$). We read the question on the rendered page of the report, and checked [Ash99, Ash02, AH04, LP02, HS10, HS11, ESHI09, CC08, CZY16, PS25] and several later papers on universal inequalities for statements on equality or strictness. We did not have access to [HS97, Yan91] or to the papers [CY05, CY07] of Cheng and Yang. We took the form of Yang’s inequality from [Ash99, Ash02, HS11], the trace identity of [HS97] from [Stu09], and the gap estimate of [CY05] from [CZY16]. We searched arXiv, Crossref, zbMATH, OpenCitations (the works citing the report), Semantic Scholar (the works it lists as citing [HS97] and [Ash02], and the texts of those that looked relevant) and the web, and found no explicit statement of Theorem 1.1, Theorem 1.2 or Corollary 1.3. The argument is elementary, and the result may be known to experts. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the proof, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AB92] Mark S. Ashbaugh and Rafael D. Benguria. A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions. *Annals of Mathematics*, 135(3):601–628, 1992. doi:10.2307/2946578.
- [AH04] Mark S. Ashbaugh and Lotfi Hermi. A unified approach to universal inequalities for eigenvalues of elliptic operators. *Pacific Journal of Mathematics*, 217(2):201–219, 2004. doi:10.2140/pjm.2004.217.201.
- [Ash99] Mark S. Ashbaugh. Isoperimetric and universal inequalities for eigenvalues. In *Spectral Theory and Geometry (Edinburgh, 1998)*, volume 273 of *London Mathematical Society Lecture Note*

- Series*, pages 95–139. Cambridge University Press, Cambridge, 1999. Cited from the version arXiv:math/0008087. doi:10.1017/CB09780511566165.007.
- [Ash02] Mark S. Ashbaugh. The universal eigenvalue bounds of Payne–Pólya–Weinberger, Hile–Protter, and H C Yang. *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)*, 112(1):3–30, 2002. doi:10.1007/BF02829638.
 - [CC08] Daguang Chen and Qing-Ming Cheng. Extrinsic estimates for eigenvalues of the Laplace operator. *Journal of the Mathematical Society of Japan*, 60(2):325–339, 2008. doi:10.2969/jmsj/06020325.
 - [CY05] Qing-Ming Cheng and Hongcang Yang. Estimates on eigenvalues of Laplacian. *Mathematische Annalen*, 331(2):445–460, 2005. doi:10.1007/s00208-004-0589-z.
 - [CY07] Qing-Ming Cheng and Hongcang Yang. Bounds on eigenvalues of Dirichlet Laplacian. *Mathematische Annalen*, 337(1):159–175, 2007. doi:10.1007/s00208-006-0030-x.
 - [CZY16] Daguang Chen, Tao Zheng, and Hongcang Yang. Estimates of the gaps between consecutive eigenvalues of Laplacian. *Pacific Journal of Mathematics*, 282(2):293–311, 2016. doi:10.2140/pjm.2016.282.293.
 - [ESHI09] Ahmad El Soufi, Evans M. Harrell, and Saïd Ilias. Universal inequalities for the eigenvalues of Laplace and Schrödinger operators on submanifolds. *Transactions of the American Mathematical Society*, 361(5):2337–2350, 2009. Cited from the version arXiv:0706.0910. doi:10.1090/S0002-9947-08-04780-6.
 - [GT01] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer, Berlin, 2001. Reprint of the 1998 edition. doi:10.1007/978-3-642-61798-0.
 - [HP80] G. N. Hile and M. H. Protter. Inequalities for eigenvalues of the Laplacian. *Indiana University Mathematics Journal*, 29(4):523–538, 1980. doi:10.1512/iumj.1980.29.29040.
 - [HS97] Evans M. Harrell and Joachim Stubbe. On trace identities and universal eigenvalue estimates for some partial differential operators. *Transactions of the American Mathematical Society*, 349(5):1797–1809, 1997. doi:10.1090/S0002-9947-97-01846-1.
 - [HS09] Evans M. Harrell and Joachim Stubbe. Problems on eigenvalues of Schrödinger operators related to commutator methods. In *Low Eigenvalues of Laplace and Schrödinger Operators (organised by M. Ashbaugh, R. Benguria, R. Laugesen and T. Weidl)*, volume 6 of *Oberwolfach Reports*, pages 414–416. European Mathematical Society, 2009. Issue 1, Report No. 6/2009, pp. 355–428; item (5) of Section 8 of the problem list on pp. 399–425. doi:10.4171/OWR/2009/06.
 - [HS10] Evans M. Harrell and Joachim Stubbe. Universal bounds and semiclassical estimates for eigenvalues of abstract Schrödinger operators. *SIAM Journal on Mathematical Analysis*, 42(5):2261–2274, 2010. Cited from the version arXiv:0808.1133. doi:10.1137/090763743.
 - [HS11] Evans M. Harrell and Joachim Stubbe. Trace identities for commutators, with applications to the distribution of eigenvalues. *Transactions of the American Mathematical Society*, 363(12):6385–6405, 2011. Cited from the version arXiv:0903.0563. doi:10.1090/S0002-9947-2011-05252-9.

- [LP02] Michael Levitin and Leonid Parnovski. Commutators, spectral trace identities, and universal estimates for eigenvalues. *Journal of Functional Analysis*, 192(2):425–445, 2002. Cited from the version arXiv:math/0102144. doi:10.1006/jfan.2001.3913.
- [NIS26] NIST. NIST Digital Library of Mathematical Functions. <https://dlmf.nist.gov/>, 2026. Release 1.2.8 of 2026-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl and M. A. McClain, eds.
- [PPW56] L. E. Payne, G. Pólya, and H. F. Weinberger. On the ratio of consecutive eigenvalues. *Journal of Mathematics and Physics*, 35:289–298, 1956. doi:10.1002/sapm1956351289.
- [PS25] Luigi Provenzano and Joachim Stubbe. Sum rules and sharp eigenvalue bounds for compact homogeneous irreducible Riemannian manifolds. *Annals of Global Analysis and Geometry*, 68(3), 2025. Article no. 14. Cited from the version arXiv:2503.02716 (v1, titled “Semiclassical eigenvalue bounds for compact homogeneous irreducible Riemannian manifolds”). doi:10.1007/s10455-025-10018-z.
- [Stu09] Joachim Stubbe. Universal monotonicity for eigenvalue means. In *Low Eigenvalues of Laplace and Schrödinger Operators (organised by M. Ashbaugh, R. Benguria, R. Laugesen and T. Weidl)*, volume 6 of *Oberwolfach Reports*, pages 364–366. European Mathematical Society, 2009. Issue 1, Report No. 6/2009, pp. 355–428; talk abstract. doi:10.4171/OWR/2009/06.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record OWR-3389-016.
- [Yan91] Hongcang Yang. An estimate of the difference between consecutive eigenvalues. Preprint IC/91/60, International Centre for Theoretical Physics, Trieste, 1991. Revised version (1995): *Estimates of the difference between consecutive eigenvalues*.