

Intrinsic Negative Sobolev Norms at a Fixed First Eigenvalue

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Abstract

We construct smooth unit-area metrics on the two-sphere with first nonzero Laplace eigenvalue exactly 6π , for which the intrinsic H^{-1} norm of the conformal factor tends to infinity in every round conformal parametrization. More precisely, the infimum of these norms over all conformal parametrizations is at least $n/(4\pi)$ along an explicit polynomial family with one scalar parameter chosen by continuity. The proof combines continuity of the weighted Poincaré constant under $L^{3/2}$ perturbations, concentration in antipodal polynomial caps, and the Jacobian formula for Möbius maps. The dual pairing uses the volume measure of the varying metric. This distinction is essential: the conformal area measure has a bounded negative Sobolev norm relative to a suitably normalized round metric, consistently with known stability results. We explain the resulting qualification for the norm printed in a 2009 Oberwolfach conjecture.

1 The norm and the result

Let g_0 be the curvature-one metric on $\mathbb{S}^2$, with area measure dA_0 and total area 4π . For a smooth positive function s , write $g_s = sg_0$. Thus $dA_{g_s} = s dA_0$. Our convention for scalar functions is

$$\|s\|_{H^{-1}(g_s)} = \sup_{0 \neq f \in C^\infty(\mathbb{S}^2)} \frac{|\int_{\mathbb{S}^2} f s dA_{g_s}|}{(\int_{\mathbb{S}^2} (|\nabla_{g_s} f|^2 + f^2) dA_{g_s})^{1/2}}. \quad (1)$$

In particular, the numerator is $|\int f s^2 dA_0|$, not $|\int f s dA_0|$. We use the inhomogeneous Sobolev space, so constants are admissible tests.

If Φ is a conformal or anticonformal diffeomorphism of the round sphere, define s^Φ by

$$\Phi_*(s dA_0) = s^\Phi dA_0.$$

The map Φ is an isometry from g_s to g_{s^Φ} . Conversely, every round conformal presentation of g_s has this form, allowing either orientation.

Theorem 1. *For every sufficiently large integer n , there is a smooth positive antipodally symmetric function s_n such that*

$$\int_{\mathbb{S}^2} s_n dA_0 = 1, \quad \lambda_1(g_{s_n}) = 6\pi, \quad \inf_{\Phi} \|s_n^\Phi\|_{H^{-1}(g_{s_n^\Phi})} \geq \frac{n}{4\pi}.$$

Here the infimum is over all conformal and anticonformal diffeomorphisms of the round sphere. In particular, no finite bound depending only on the first eigenvalue can hold for (1), even after choosing the conformal parametrization.

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The construction specifies

$$\begin{aligned} z(x) &= x_3, & q_t(x) &= \frac{2-t+5tz(x)^4}{8\pi} \quad (0 \leq t \leq 1), \\ N_n &= n^5, & b_n(x) &= \frac{2N_n+1}{4\pi} z(x)^{2N_n}, \\ s_{t,n} &= (1-n^{-2})q_t + n^{-2}b_n, & s_n &= s_{t_n,n}. \end{aligned} \tag{2}$$

The parameter t_n is chosen by the intermediate value theorem. We do not assert an explicit formula or uniqueness for t_n .

2 An elementary spectral continuity estimate

For a smooth positive probability density w on $\mathbb{S}^2$, put

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} |\nabla_{g_0} f|^2 dA_0, \quad \text{Var}_w(f) = \int_{\mathbb{S}^2} f^2 w dA_0 - \left(\int_{\mathbb{S}^2} f w dA_0 \right)^2.$$

Conformal invariance of the Dirichlet energy in dimension two and the variational characterization of the first eigenvalue give

$$\lambda_1(g_w)^{-1} = \sup_{\substack{\int f dA_0 = 0 \\ \mathcal{E}(f) = 1}} \text{Var}_w(f). \tag{3}$$

The choice of round mean zero is harmless because both variance and energy are invariant under addition of constants.

Lemma 2. *There is a constant C , depending only on g_0 , such that for any smooth positive probability densities v, w ,*

$$\left| \lambda_1(g_w)^{-1} - \lambda_1(g_v)^{-1} \right| \leq C \|w - v\|_{3/2} (1 + \|w\|_{3/2} + \|v\|_{3/2}).$$

All norms on the right are relative to dA_0 .

Proof. For a round-mean-zero function with $\mathcal{E}(f) = 1$, the round Poincaré inequality bounds its $H^1(g_0)$ norm. The two-dimensional Sobolev embeddings therefore bound both $\|f\|_3$ and $\|f\|_6$ by a fixed constant. Hölder's inequality gives

$$\left| \int f^2 (w - v) dA_0 \right| \leq \|w - v\|_{3/2} \|f\|_6^2.$$

Writing $m_w = \int f w dA_0$, we also have

$$|m_w^2 - m_v^2| \leq \|w - v\|_{3/2} (\|w\|_{3/2} + \|v\|_{3/2}) \|f\|_3^2.$$

The difference of the two variances satisfies the required uniform bound. Taking suprema in (3) proves the assertion. $\square$

Every q_t in (2) is a probability density and $q_t \geq 1/(8\pi)$. At $t = 0$ the metric is the round metric scaled to area one, and hence

$$\lambda_1(g_{q_0}) = 8\pi, \quad \lambda_1(g_{q_1}) \leq \frac{56\pi}{11} < 6\pi. \tag{4}$$

Indeed, z has zero q_1 -mean and

$$\mathcal{E}(z) = \frac{8\pi}{3}, \quad \int_{\mathbb{S}^2} z^2 q_1 dA_0 = \frac{11}{21}.$$

The second inequality in (4) is its Rayleigh quotient, not an assertion that z is an eigenfunction.

3 Polynomial concentration and exact eigenvalue tuning

For any nonnegative integer m ,

$$\int_{\mathbb{S}^2} z^{2m} dA_0 = \frac{4\pi}{2m+1}. \quad (5)$$

Thus b_n is an even probability density. Its relevant norms are

$$\|b_n\|_{3/2} = \frac{2N_n + 1}{(4\pi)^{1/3}(3N_n + 1)^{2/3}}, \quad \|b_n\|_2^2 = \frac{(2N_n + 1)^2}{4\pi(4N_n + 1)}. \quad (6)$$

Since $N_n = n^5$, these identities imply

$$\sup_{0 \leq t \leq 1} \|s_{t,n} - q_t\|_{3/2} \leq n^{-2}(\|b_n\|_{3/2} + \sup_t \|q_t\|_{3/2}) = O(n^{-1/3}). \quad (7)$$

The $L^{3/2}$ norms of $s_{t,n}$ are consequently bounded uniformly in $t, n \geq 2$.

Lemma 2 shows that the reciprocal eigenvalues for $s_{t,n}$ converge uniformly in t to those for q_t . In particular, for all sufficiently large n ,

$$\lambda_1(g_{s_{0,n}}) > 6\pi, \quad \lambda_1(g_{s_{1,n}}) < 6\pi.$$

For each such n , the same lemma gives continuity in t . There is therefore a $t_n \in (0, 1)$ with $\lambda_1(g_{s_{t_n,n}}) = 6\pi$. The density $s_n = s_{t_n,n}$ is smooth, even, of mass one, and strictly positive because $n \geq 2$ and $q_t \geq 1/(8\pi)$.

On the other hand, all terms in the mixture are nonnegative, so

$$\int_{\mathbb{S}^2} s_n^2 dA_0 \geq n^{-4} \|b_n\|_2^2 = \frac{n^{-4}(2n^5 + 1)^2}{4\pi(4n^5 + 1)} \geq \frac{n}{4\pi}. \quad (8)$$

The last inequality follows from $(2N+1)^2 - N(4N+1) = 3N+1 > 0$. This separates the topology controlling the eigenvalue from the larger moment detected by the intrinsic dual pairing.

4 Conformal reparametrizations

Lemma 3. *If $w \geq 0$ is an antipodally symmetric smooth function on $\mathbb{S}^2$, then every conformal or anticonformal diffeomorphism Φ satisfies*

$$\int_{\mathbb{S}^2} (w^\Phi)^2 dA_0 \geq \int_{\mathbb{S}^2} w^2 dA_0.$$

Proof. Write $\Phi^* dA_0 = J_\Phi dA_0$. A round-sphere Möbius transformation has Jacobian

$$J_\Phi(x)^{-1} = (a + b \cdot x)^2, \quad a \geq 1, \quad a^2 - |b|^2 = 1. \quad (9)$$

For completeness, a determinant-one complex matrix representing the map has a singular value decomposition. Its unitary factors act as round rotations, while its positive diagonal factor acts in a stereographic coordinate as $\zeta \mapsto r\zeta$. For this dilation,

$$J(\zeta) = \frac{r^2(1 + |\zeta|^2)^2}{(1 + r^2|\zeta|^2)^2}, \quad J^{-1/2} = \frac{r + r^{-1}}{2} + \frac{r - r^{-1}}{2} x_3.$$

Rotations give (9) with an arbitrary direction for b . An orientation-reversing map is obtained by composing with a round reflection, which has Jacobian one.

The change-of-variables formula gives

$$\int_{\mathbb{S}^2} (w^\Phi)^2 dA_0 = \int_{\mathbb{S}^2} \frac{w(x)^2}{J_\Phi(x)} dA_0.$$

Because w^2 is even, $\int xw(x)^2 dA_0 = 0$. Hence the right-hand side equals

$$a^2 \int_{\mathbb{S}^2} w^2 dA_0 + \int_{\mathbb{S}^2} (b \cdot x)^2 w^2 dA_0 \geq \int_{\mathbb{S}^2} w^2 dA_0.$$

□

Proof of Theorem 1. The construction and the exact eigenvalue condition were proved above. Put $\sigma = s_n^\Phi$. Since g_σ has area one, the test function $f = 1$ has $H^1(g_\sigma)$ norm one. Consequently (1), Lemma 3, and (8) give

$$\|\sigma\|_{H^{-1}(g_\sigma)} \geq \int_{\mathbb{S}^2} \sigma^2 dA_0 \geq \int_{\mathbb{S}^2} s_n^2 dA_0 \geq \frac{n}{4\pi}.$$

The estimate is simultaneous for all Φ , as required. □

5 The different estimate with a round background

The preceding theorem does not contradict coercivity in a fixed background negative Sobolev norm. Karpukhin, Nahon, Polterovich and Stern prove a stronger quantitative stability estimate for Hersch's inequality in that setting [2, Theorem 1.2 and Remark 2.3]. The following elementary estimate illustrates the distinction.

Proposition 4. *Let $w > 0$ be smooth with $\int w dA_0 = 1$ and $\int xw(x) dA_0 = 0$. Write $\lambda = \lambda_1(g_w)$. Then*

$$\int_{\mathbb{S}^2} f^2 w dA_0 \leq \frac{2}{\lambda} \int_{\mathbb{S}^2} (|\nabla_{g_0} f|^2 + f^2) dA_0, \quad \|w dA_0\|_{H^{-1}(g_0)} \leq \sqrt{\frac{2}{\lambda}}.$$

Proof. Let $A = \int f^2 w dA_0$, $m = \int f w dA_0$, and $B = \int f x w dA_0 \in \mathbb{R}^3$. The weighted Poincaré inequality yields $A - m^2 \leq \mathcal{E}(f)/\lambda$. Balance and Cauchy–Schwarz give

$$|B|^2 = \left| \int (f - m) x w dA_0 \right|^2 \leq A - m^2 \leq \mathcal{E}(f)/\lambda.$$

Apply the weighted Poincaré inequality to each $f x_i$ and sum over $i = 1, 2, 3$. The identities $\sum x_i^2 = 1$, $\sum x_i \nabla x_i = 0$, and $\sum |\nabla x_i|^2 = 2$ give

$$\lambda(A - |B|^2) \leq \mathcal{E}(f) + 2 \int f^2 dA_0.$$

It follows that $\lambda A \leq 2\mathcal{E}(f) + 2 \int f^2 dA_0$. Finally, $|\int f w dA_0|^2 \leq A$ proves the dual norm bound. □

Hersch's balancing lemma supplies this normalization for any smooth positive probability measure; see, for example, its use in [2, Section 2.2]. Our densities are already balanced by antipodal symmetry. Their round-background dual norms stay bounded while the norms in Theorem 1 diverge.

There is a third convention worth distinguishing. The area measure dA_{g_s} , viewed as a functional on $H^1(g_s)$, has norm exactly one when g_s has area one. Cauchy–Schwarz gives the upper bound, and the constant test gives equality. This functional also differs from the scalar s in (1).

6 Relation to the source question

Nadirashvili’s Conjecture 3 in the 2009 Oberwolfach problem list [1, p. 408] proposes a bound on conformal factors at fixed unit area and fixed first eigenvalue. The displayed norm uses $H^{-1}(\mathbb{S}^2, g)$, and the bound is written $C_{\lambda_1(g)}$. The subscript means dependence on the eigenvalue; it is not the product of a universal constant and the eigenvalue. The frozen UnsolvedMath record OWR-3389-011 loses this distinction.

Theorem 1 disproves the scalar-function estimate when that printed norm is given the intrinsic convention (1). If a fixed round background and the measure $s dA_0$ were intended instead, Proposition 4 and the prior stability theorem apply. We do not infer the author’s intended pairing from the typography, nor present the fixed-background estimate as a new resolution. The mathematical assertion established here is the explicitly defined intrinsic counterexample.

The verification script checks exact polynomial moments, the endpoint Rayleigh bound, concentration estimates and the Möbius Jacobian identity. These diagnostics are not a computation of t_n or a substitute for the continuity proof. AI tools assisted the derivation, literature search, checking and exposition; the author takes responsibility for the manuscript. This is an unrefereed preprint, and no absolute priority claim is made.

References

- [1] M. Ashbaugh, R. Benguria, R. Laugesen and T. Weidl (organizers), *Low Eigenvalues of Laplace and Schrödinger Operators*, Oberwolfach Reports **6** (2009), no. 1, 355–428. Problem section 5 by N. Nadirashvili, Conjecture 3, p. 408. [doi:10.4171/OWR/2009/06](https://doi.org/10.4171/OWR/2009/06).
- [2] M. Karpukhin, M. Nahon, I. Polterovich and D. Stern, *Stability of isoperimetric inequalities for Laplace eigenvalues on surfaces*, Journal of Differential Geometry **129** (2025), no. 2, 415–490. <https://doi.org/10.4310/jdg/1738163208>. Author manuscript: [arXiv:2106.15043](https://arxiv.org/abs/2106.15043).