

A Star-Shaped Spherical Heptagon Whose Development Self-Intersects

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Abstract

Problem 13 in G. Rote’s list of open problems from the 2009 Oberwolfach workshop *Discrete Differential Geometry*, posed by J. O’Rourke, asks when a simple piecewise-geodesic curve on the sphere, developed onto the plane with the same edge lengths and turning angles, does not intersect itself. The source records that closed convex polygons develop simply and asks: what about star-shaped polygons? A closed polygon has to be cut open at a point to be developed. The theorems of O’Rourke and Schevon (1989) and O’Rourke (2003) on closed curves name no cut point, and O’Rourke and Vilcu (2011) single out the closed curves whose development is simple for every cut point; we read the question in this way, as the source states no convention. We give a simple spherical heptagon with rational vertices, contained in the open hemisphere about the north pole N and star-shaped with respect to N , whose development cut at one of its vertices is not simple: its first and last edges cross transversally. So star-shaped polygons need not develop without self-intersection, even if only cuts at vertices are considered, whereas on a cone closed curves that are star-shaped from the apex do (O’Rourke and Vilcu). The proof is a certificate in exact rational and interval arithmetic, checked by a short program. The developments cut at the other six vertices are simple, and a second certified heptagon shows that cutting at the vertex farthest from N can fail as well. Whether every star-shaped spherical polygon has some cut point with a simple development remains open; numerical searches found no polygon all of whose cut points are bad. General conditions for a simple development are not treated. This is an unrefereed note.

1 Introduction

1.1 The question

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere and $N = (0, 0, 1)$. A curve on $\mathbb{S}^2$ made of great-circle arcs is *developed* onto the plane by drawing the planar chain with the same edge lengths and turning angles, or, equivalently, by rolling the sphere on the plane along the curve [O’R03, §4]. Problem 13 in the list of open problems collected by G. Rote at the 2009 Oberwolfach workshop *Discrete Differential Geometry* [Rot09, p. 110], due to J. O’Rourke, asks for conditions which guarantee that the development of a simple piecewise-geodesic curve on $\mathbb{S}^2$ does not intersect itself, records that this holds for closed convex polygons, and asks “What about star-shaped polygons?”. The question is part of record OWR-3385-011 of the UnsolvedMath collection [ula26], listed there as open; the record also contains Problem 14 of [Rot09] (A. Bobenko), which is not treated here.

A closed polygon has to be cut open at a point before it can be developed, and the development depends on that point; the source does not say which cut points are meant. O’Rourke’s theorems

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on closed curves name no cut point. O’Rourke and Schevon [OS89] proved that closed convex curves on convex polyhedra develop without self-intersection (as quoted in [O’R03, §4] and [OV11, §7.1]), and O’Rourke [O’R03, Theorem 3] proved it for *slice curves*, the intersections of a polytope with a plane, developed from a corner c_0 of the curve; no corner is singled out [O’R03, §4]. O’Rourke and Vilcu write C_x for the development of a closed curve C cut at x [OV11, §7.1], and they are especially interested in the curves for which C_x is simple “for every choice of x ” [OV11, §7.2]. This every-cut convention is O’Rourke’s usage, not a definition stated in the source. We read the question as

(Q) *Is the development of every star-shaped simple closed spherical polygon simple, for every choice of the cut point?*

and discuss the weaker reading, in which one good cut point suffices, in Section 5.

On a cone the corresponding statement holds; a cone in the sense of [OV11, §1] has cone angle at most 2π . If a simple closed curve C on such a cone encloses the apex and meets every generator once, so that C “could be said to be star-shaped from” the apex [OV11, footnote 2], then C_x is simple for every x , because the cone cut open along the generator through x unrolls without overlap [OV11, §7.1]. This argument has no counterpart on $\mathbb{S}^2$: a curve “lives on a cone” in their sense if a neighbourhood on one side embeds isometrically in a cone [OV11, §1], which no open set of $\mathbb{S}^2$ does. In a MathOverflow question that has no answer [O’R12], O’Rourke asked for conditions under which a curve on a cone develops to a simple curve, and expected the analogous problem on the sphere to be difficult.

1.2 The result

Theorem 1.1. *Let $Q = (v_0, \dots, v_6)$ be the spherical heptagon whose rational vertices are listed in Table 1, with edges $e_i = [v_i, v_{i+1}]$ (indices modulo 7; the definitions are recalled in Section 2).*

- (a) *Q is simple and lies in the open hemisphere $\{z > 0\}$ centred at N . The closed region R bounded by Q that contains N is star-shaped with respect to N , and N is an interior point of R .*
- (b) *The development of Q cut at v_0 is not simple: its first edge, the image of e_0 , and its last edge, the image of e_6 , cross transversally.*
- (c) *More generally, for every point p of the open arc of Q that runs from the point of e_6 at arc-length fraction 0.90076... through v_0 to the point of e_0 at fraction 0.06699..., the development of Q cut at p is not simple; this arc has length greater than 0.118.*
- (d) *The developments of Q cut at $v_1, \dots, v_6$ are simple.*

Corollary 1.2. *In the reading (Q), and also if only cuts at vertices are considered, the answer to O’Rourke’s question is no: a simple closed spherical polygon that lies in an open hemisphere, and is star-shaped with respect to the centre of that hemisphere, which is an interior point of its region, need not develop without self-intersection.*

The proof (Section 3) is a certificate: part (a) is decided in exact rational arithmetic, and parts (b)–(d) by interval arithmetic with outward rounding. The crossing is far from degenerate: the four orientation determinants that decide it are at least 0.034 in absolute value. Figure 1 shows Q and its development cut at v_0 . A tempting rule for choosing a good cut point is to cut at the vertex farthest from the star centre; in a random sample this rule never failed (Section 5). It does fail in general.

Proposition 1.3. *The heptagon $Q' = (v'_0, \dots, v'_6)$ of Table 3 lies in the open hemisphere about N and is star-shaped with respect to N , an interior point of its region. Its vertex farthest from N is v'_5 , and the development of Q' cut at v'_5 is not simple. More precisely, the developments of Q' cut at v'_k are not simple for $k \neq 3$, and the development cut at v'_3 is simple.*

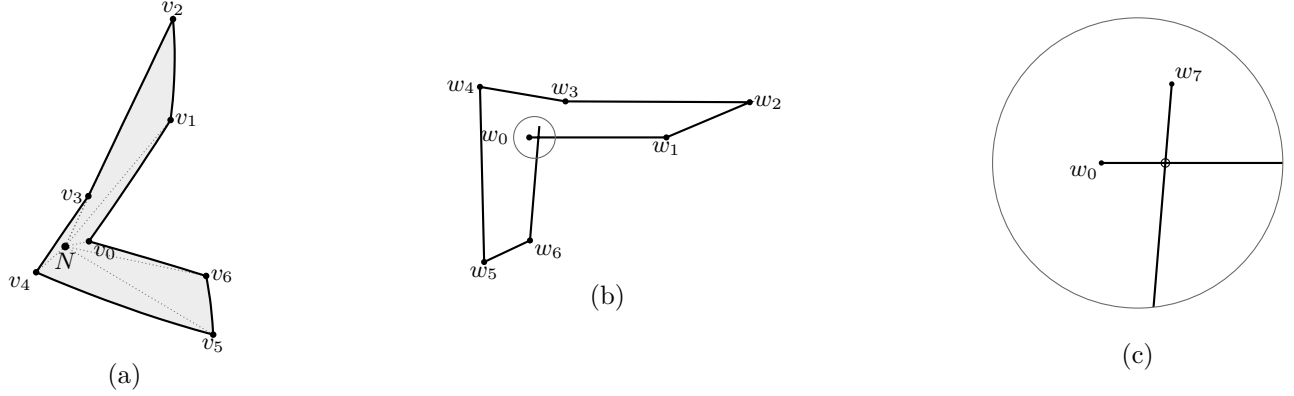


Figure 1: (a) The heptagon Q of Table 1 in the azimuthal equidistant projection about N : the point with polar distance r and longitude φ is drawn at $(r \cos \varphi, r \sin \varphi)$, so that meridians become rays from N (each edge is drawn through ten of its points). The dotted rays from N to the vertices show that Q is a polar graph about N (Lemma 2.2). (b) The development D_0 of Q cut at v_0 , with vertices $w_0, \dots, w_7$; the grey circle marks the region enlarged in (c). (c) Near w_0 the last edge w_6w_7 (the image of e_6) crosses the first edge w_0w_1 (the image of e_0) at the point marked by a small circle, at distance 0.0529 from w_0 .

1.3 What is new and what is not

The question is O'Rourke's [Rot09]; the polyhedral theorems behind it are due to O'Rourke and Schevon [OS89] and O'Rourke [O'R03], the cone analogue to O'Rourke and Vilcu [OV11], and MathOverflow question 100496 [O'R12] asks a related question for cones. This note contributes a certified star-shaped counterexample on the sphere for the every-cut reading, the certified failure of the farthest-vertex rule, and numerical evidence on the some-cut reading. No printed answer to the star-shaped question was found in the sources listed in Section 6; a search that finds nothing is not a proof of novelty. The record OWR-3385-011 is answered only in part: the some-cut variant, the request for general conditions and Problem 14 of [Rot09] remain open.

2 Spherical polygons and their developments

2.1 Polygons and turning angles

For $p, q \in \mathbb{S}^2$ with $q \neq \pm p$ let $[p, q]$ be the minor great-circle arc from p to q , of length $\arccos(p \cdot q) \in (0, \pi)$. A *(spherical) polygon* $Q = (v_0, \dots, v_{n-1})$, $n \geq 3$, with $v_{i+1} \neq \pm v_i$, is the closed curve formed by the edges $e_i = [v_i, v_{i+1}]$ of lengths l_i (indices modulo n); it is *simple* if non-consecutive edges are disjoint and consecutive edges meet only at their common vertex. At v_i the vectors

$$A_i = (v_i \cdot v_{i-1}) v_i - v_{i-1}, \quad B_i = v_{i+1} - (v_i \cdot v_{i+1}) v_i$$

are orthogonal to v_i and point in the directions in which e_{i-1} arrives and e_i leaves. Put $k_i = A_i \cdot B_i$ and $s_i = v_i \cdot (A_i \times B_i)$; then $k_i^2 + s_i^2 = |A_i|^2 |B_i|^2$, with $|A_i|^2 = 1 - (v_i \cdot v_{i-1})^2$ and $|B_i|^2 = 1 - (v_i \cdot v_{i+1})^2$. The *turning angle* $\tau_i \in (-\pi, \pi]$ is defined by

$$\cos \tau_i = \frac{k_i}{|A_i| |B_i|}, \quad \sin \tau_i = \frac{s_i}{|A_i| |B_i|}; \quad (1)$$

it is the angle from A_i to B_i in the tangent plane oriented by the outward normal v_i , positive for a left turn. If Q is simple and R is the closed region on its left, the interior angle of R at v_i is $\pi - \tau_i$,

and $\sum_i \tau_i = 2\pi - \text{area}(R)$ by the Gauss–Bonnet theorem. For rational vertices the numbers $v_i \cdot v_j$, k_i , s_i , $|A_i|^2$ and $|B_i|^2$ are rational, while l_i and τ_i are in general transcendental.

2.2 Star-shaped polygons

Definition 2.1. Let Q be a simple polygon and R one of the two closed regions of $\mathbb{S}^2$ bounded by Q . The region R is *star-shaped with respect to* a point $c \in R$ if $-c \notin R$ and the arc $[c, x]$ lies in R for every $x \in R \setminus \{c\}$.

The source does not define star-shaped polygons on $\mathbb{S}^2$. The polygons of this note satisfy the most restrictive version we know: c is an interior point of R , and R lies in the open hemisphere centred at c . We use polar coordinates about N : a point of $\mathbb{S}^2 \setminus \{\pm N\}$ has a polar distance $r \in (0, \pi)$ and a longitude $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$; its *horizontal projection* is $h = (x, y) = \sin r (\cos \varphi, \sin \varphi)$.

Lemma 2.2 (polar graphs). *Let $v_0, \dots, v_{n-1} \in \mathbb{S}^2 \setminus \{\pm N\}$ have horizontal projections $h_i = (x_i, y_i)$ such that*

(i) $h_i \times h_{i+1} = x_i y_{i+1} - y_i x_{i+1} > 0$ for every i , so that the longitude increases from v_i to v_{i+1} by an angle $g_i \in (0, \pi)$, and

(ii) $g_0 + \dots + g_{n-1} = 2\pi$.

Then $Q = (v_0, \dots, v_{n-1})$ is a simple polygon, N lies on its left, and the closed region R bounded by Q that contains N is star-shaped with respect to N , with N in its interior. If moreover all v_i lie in the open hemisphere $\{z > 0\}$, then so do Q and R .

Proof. By (i), $v_{i+1} \neq \pm v_i$, and the points of e_i are $p(s) = ((1-s)v_i + sv_{i+1})/|(1-s)v_i + sv_{i+1}|$, $s \in [0, 1]$. The horizontal projection of $p(s)$ is a positive multiple of $(1-s)h_i + sh_{i+1} \neq 0$, so e_i avoids $\pm N$, and the longitude of $p(s)$, the direction of a point moving along a segment that avoids the origin, increases strictly from φ_i to $\varphi_i + g_i$. As a point of $\mathbb{S}^2 \setminus \{\pm N\}$ is determined by its longitude and polar distance, e_i is the graph of a continuous function of the longitude on $[\varphi_i, \varphi_i + g_i]$. By (ii) these intervals, taken consecutively, cover the circle of longitudes once, and consecutive ones share an endpoint, where both functions take the polar distance of the common vertex. So Q is the graph of a continuous function $r: \mathbb{R}/2\pi\mathbb{Z} \rightarrow (0, \pi)$, hence simple, and the region $R = \{N\} \cup \{(r, \varphi) : 0 < r \leq r(\varphi)\}$ contains, with each of its points $x = (r, \varphi)$, the meridian segment $[N, x] = \{(\rho, \varphi) : 0 \leq \rho \leq r\}$; also $-N \notin R$. Thus R is star-shaped with respect to N , which is an interior point because $r(\varphi)$ has a positive minimum, and R lies on the left of Q , since the longitude increases counterclockwise when seen from outside above N . If all $z_i > 0$, each edge, a minor arc between two points of the open hemisphere $\{z > 0\}$, lies in it, and so does each meridian segment from N to a point of Q ; hence $R \subset \{z > 0\}$. $\square$

Only this direction is used. We do not claim the converse: a polygon that is star-shaped with respect to N need not satisfy (i), for instance if an edge lies on a meridian through N .

2.3 Developments

We identify the plane with $\mathbb{C}$. A *planar chain* $(w_0, \dots, w_m)$, with the segments $[w_j, w_{j+1}]$, is *simple* if non-consecutive segments are disjoint and consecutive segments meet only at their common point.

Definition 2.3. Let $Q = (v_0, \dots, v_{n-1})$ be a polygon with edge lengths l_i and turning angles τ_i .

(a) The *development of Q cut at the vertex v_k* is the chain $D_k = (w_0, \dots, w_n)$ with

$$w_0 = 0, \quad w_{j+1} = w_j + l_{k+j} u_j \quad (0 \leq j \leq n-1), \quad w_n = 1, \quad u_j = u_{j-1} e^{i\tau_{k+j}} \quad (1 \leq j \leq n-1).$$

Its n segments have the lengths $l_k, l_{k+1}, \dots, l_{k-1}$ of the edges $e_k, e_{k+1}, \dots, e_{k-1}$, and its turning angles at $w_1, \dots, w_{n-1}$ are $\tau_{k+1}, \dots, \tau_{k-1}$; the turning angle τ_k at the cut vertex is not used.

- (b) For $0 < t < 1$ let p be the point of e_k at distance tl_k from v_k . The *development of Q cut at p* is the chain with $n + 1$ segments of lengths $(1 - t)l_k, l_{k+1}, \dots, l_{k-1}, tl_k$ and the n turning angles $\tau_{k+1}, \dots, \tau_{k-1}, \tau_k$, built by the same rule.

A point p of Q is a *bad cut point* if the development of Q cut at p is not simple, and a *good cut point* otherwise. We say that Q *develops without self-intersection* if every point of Q is a good cut point.

This is the development of the source, with the same lengths and angles, cut open as in [O'R03, §4] and [OV11, §7.1]; the first and third verification runs (Section 6) also obtained the developments of Theorem 1.1 by simulating the rolling sphere, in agreement with Definition 2.3 up to a reflection. Reversing the orientation of Q or reflecting the sphere replaces each development by a congruent chain, possibly reflected and reversed, so whether a cut point is good does not depend on these choices. Since two chains with the same lengths and turning angles differ by a rotation followed by a translation, we have:

Lemma 2.4. *Let Γ be an arc of Q , traversed in the direction of Q , with endpoints at vertices or at interior points of edges, and let p be a point of Q that is not an interior point of Γ . Then the development of Q cut at p contains, as a connected part, the image under a rotation followed by a translation of the planar chain with the lengths and turning angles of Γ .*

Proof. The development cut at p traverses Q once, from p back to p ; since p is not interior to Γ , the arc Γ is traversed in one piece, with its own lengths and its turning angles at its interior vertices. $\square$

Disjointness and crossings of segments are certified by the orientation determinant $\text{or}(a, b, c) = \text{Im}(\overline{(b - a)}(c - a))$, which is positive if c lies to the left of the directed line from a to b :

- (D) If $\text{or}(a, b, c)$ and $\text{or}(a, b, d)$ are nonzero and of the same sign, or $\text{or}(c, d, a)$ and $\text{or}(c, d, b)$ are, then the segments $[a, b]$ and $[c, d]$ are disjoint.
- (X) If $\text{or}(a, b, c)$ or $\text{or}(a, b, d) < 0$ and $\text{or}(c, d, a)$ or $\text{or}(c, d, b) < 0$, then $[a, b]$ and $[c, d]$ are not parallel and meet in exactly one point, which is interior to both; it is $a + \lambda(b - a) = c + \mu(d - c)$ with $\lambda = \text{or}(c, d, a) / (\text{or}(c, d, a) - \text{or}(c, d, b)) \in (0, 1)$ and $\mu = \text{or}(a, b, c) / (\text{or}(a, b, c) - \text{or}(a, b, d)) \in (0, 1)$. We then say that the segments *cross transversally*.

Consecutive segments of a development meet only at their common point unless the turning angle there equals π ; this does not happen when $s_i \neq 0$ for all i .

3 The certificate

The computations of this section are made by the program `check_note.py` of the reproducibility package (Section 6), which uses exact rational arithmetic and an interval arithmetic of its own and runs in a few seconds.

3.1 The vertices

The vertices of Q are the points

$$v_i = \frac{(2\xi_i, 2\eta_i, 1 - \xi_i^2 - \eta_i^2)}{1 + \xi_i^2 + \eta_i^2} \quad (2)$$

for the rational numbers ξ_i, η_i of Table 1; this is the inverse of the stereographic projection from the south pole, so the v_i are rational points of $\mathbb{S}^2$. Table 1 lists them as integer vectors (X_i, Y_i, Z_i) divided by D_i , and $X_i^2 + Y_i^2 + Z_i^2 = D_i^2$ holds exactly.

i	$1000\xi_i$	$1000\eta_i$	X_i	Y_i	Z_i	D_i	$r_i \approx$	$\varphi_i \approx$
0	64	14	32000	7000	248927	251073	0.1308	0.2154
1	305	366	610000	732000	773019	1226981	0.8892	0.8761
2	346	730	86500	182500	43423	206577	1.3590	1.1282
3	63	137	63000	137000	488631	511369	0.2993	1.1398
4	-79	-70	-158000	-140000	988859	1011141	0.2103	3.8667
5	431	-257	86200	-51400	74819	125181	0.9302	5.7455
6	401	-84	802000	-168000	832143	1167857	0.7777	6.0767

Table 1: The heptagon Q : $v_i = (X_i, Y_i, Z_i)/D_i$, obtained from (2). The polar distances r_i and the longitudes φ_i about N are rounded and only describe the shape.

3.2 Proof of Theorem 1.1(a)

All Z_i are positive, so the vertices lie in $\{z > 0\}$; the smallest z -coordinate is $z_2 = 43423/206577 > 0.21$, so all polar distances are less than $1.36 < \pi/2$. The seven rational numbers $x_i y_{i+1} - y_i x_{i+1}$ are positive, which is (i) of Lemma 2.2. When all steps g_i lie in $(0, \pi)$, their sum is 2π times the number of steps that pass through the longitude 0, that is, that go from $\{y < 0\} \cup \{y = 0, x < 0\}$ to its complement. The signs of $y_0, \dots, y_6$ are $+, +, +, +, -, -, -$, so only the step from v_6 to v_0 does, and (ii) holds. Part (a) follows from Lemma 2.2. Independently, the program checks with exact orientation tests that the gnomonic image $(x/z, y/z)$ of Q , a heptagon with rational vertices, is simple and counterclockwise, with the origin strictly to the left of each directed edge; since the gnomonic projection maps $\{z > 0\}$ bijectively onto the plane, minor arcs onto segments and meridian arcs from N onto rays from the origin, this gives (a) again.

3.3 Interval arithmetic

The numbers $c_i = v_i \cdot v_{i+1}$, k_i and s_i are rational and are computed exactly. Every other quantity is enclosed in an interval whose endpoints are dyadic rationals; the result of every arithmetic operation is rounded outward to multiples of 2^{-320} . Square roots are bounded by integer square roots. The lengths are $l_i = \arccos c_i = 2 \arctan \sqrt{(1 - c_i)/(1 + c_i)}$; the arctangent of $x \geq 0$ is computed from $\arctan x = \frac{\pi}{2} - \arctan \frac{1}{x}$ if $x > 1$, from $\arctan x = 2 \arctan(x/(1 + \sqrt{1 + x^2}))$, applied until the argument is at most $1/8$, and from the alternating Taylor series, whose partial sums lie alternately above and below the value; π is computed by Machin's formula $\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$. No angles are added in the developments: by (1), $e^{i\tau_j} = (k_j + is_j)/\sqrt{k_j^2 + s_j^2}$, and the directions u_j and the points w_j of Definition 2.3 are obtained by multiplying and adding complex intervals. A sign is accepted only if the whole interval has it. All enclosures used below have width less than 10^{-80} .

i	l_i	τ_i	$\pi - \tau_i$ (degrees)	sign of s_i
0	0.788896063149	-1.917573753423	289.869	-
1	0.520324652474	0.399668806274	157.101	+
2	1.059709072630	2.737397626102	23.159	+
3	0.498849449922	-0.163758055996	189.383	-
4	1.007747884058	1.762853468190	78.996	+
5	0.291463381859	1.986779351882	66.166	+
6	0.660047704558	1.049457433073	119.871	+

Table 2: Edge lengths l_i of $e_i = [v_i, v_{i+1}]$, turning angles τ_i and interior angles of R for the heptagon Q (rounded from the enclosures). The signs of s_i are exact.

Table 2 gives the lengths and angles. The signs of the s_i are exact; since all $s_i \neq 0$, no turning angle is 0 or π , and consecutive segments of every development of Q meet only at their common point. The vertices v_0 and v_3 are reflex, with interior angles 289.869° and 189.383° ; in particular Q is not convex.

3.4 Proof of Theorem 1.1(b)

Let $D_0 = (w_0, \dots, w_7)$ be the development of Q cut at v_0 (Figure 1(b); the program prints enclosures of the w_j , for instance $w_7 \approx (0.058189, 0.065285)$). The enclosures of the four orientation determinants of the first segment $[w_0, w_1]$ (the image of e_0) and the last segment $[w_6, w_7]$ (the image of e_6) are

$$\begin{aligned} \text{or}(w_0, w_1, w_6) &\in [-0.46747394338566341173, -0.46747394338566341172], \\ \text{or}(w_0, w_1, w_7) &\in [0.05150318903423074704, 0.05150318903423074705], \\ \text{or}(w_6, w_7, w_0) &\in [0.03476828703784041768, 0.03476828703784041769], \\ \text{or}(w_6, w_7, w_1) &\in [-0.48420884538205374109, -0.48420884538205374108]. \end{aligned}$$

The signs are $-, +, +, -$. By (X) the two segments cross transversally, at the point $w_0 + \lambda(w_1 - w_0) = w_6 + \mu(w_7 - w_6)$ with $\lambda \in [0.066993, 0.066994]$ and $\mu \in [0.900760, 0.900761]$. Dividing the determinants by the segment lengths l_0 and l_6 , the distances of w_6 and w_7 from the line through w_0, w_1 are 0.5926 and 0.0653, and those of w_0 and w_1 from the line through w_6, w_7 are 0.0527 and 0.7336. The first and the last of the seven segments are not consecutive, so D_0 is not simple. The other 14 pairs of non-consecutive segments of D_0 satisfy (D), so this is the only crossing.

Proof of Theorem 1.1(d). For $k = 1, \dots, 6$, each of the 15 pairs of non-consecutive segments of D_k satisfies (D) with certified signs, and consecutive segments meet only at their common point because all $s_i \neq 0$. So $D_1, \dots, D_6$ are simple.

3.5 Proof of Theorem 1.1(c)

The crossing point $X = w_0 + \lambda(w_1 - w_0) = w_6 + \mu(w_7 - w_6)$ is the image of the point x of e_0 at arc-length fraction λ and of the point y of e_6 at fraction μ . Let p lie on the open arc of Q from y through v_0 to x ; the case $p = v_0$ is part (b). If p is the point of e_0 at fraction $t \in (0, \lambda)$, the arc Γ of Q from p through $v_1, \dots, v_6$ to v_0 has the lengths and turning angles of the part of D_0 that starts at $w_0 + t(w_1 - w_0)$. In that part the first segment $[w_0 + t(w_1 - w_0), w_1]$ and the last segment $[w_6, w_7]$ cross transversally at X , since $t < \lambda$; by Lemma 2.4, the first and the seventh of the eight segments of the development cut at p cross transversally. If p is the point of e_6 at fraction $t \in (\mu, 1)$, the arc from v_0 through $v_1, \dots, v_6$ to p corresponds to the part of D_0 that ends at $w_6 + t(w_7 - w_6)$, in which $[w_0, w_1]$ and $[w_6, w_6 + t(w_7 - w_6)]$ cross transversally at X , since $\mu < t$; so the second and the eighth segments of the development cut at p cross. In both cases the development cut at p is not simple. The arc has length $\lambda l_0 + (1 - \mu)l_6 \in [0.11835, 0.11836]$.

A consistency check. As a check of the conventions (the region on the left of Q , the signs of the τ_i), the program verifies $\sum_i \tau_i + \text{area}(R) = 2\pi$, with $\tau_i = \text{atan2}(s_i, k_i)$ and $\text{area}(R)$ the sum of the areas E_i of the triangles $Nv_i v_{i+1}$, which tile R by the proof of Lemma 2.2; here $\tan(E_i/2) = \det(N, v_i, v_{i+1}) / (1 + N \cdot v_i + v_i \cdot v_{i+1} + v_{i+1} \cdot N)$ [VS83] is rational, with a positive denominator. It finds $\sum_i \tau_i \approx 5.854824876101$ and $\text{area}(R) \approx 0.428360431078$; the enclosure of $\sum_i \tau_i + \text{area}(R) - 2\pi$ has width less than 10^{-90} and contains 0.

4 Further observations

Remark 4.1 (numerical). Dense floating-point sampling of the cut points of Q (2000 points per edge in the first verification run, 30,001 points of e_0 and of e_6 in the second, 4000 points per edge with the ends refined by bisection in the third) finds the following bad cut points and no others, where t is the arc-length fraction from v_i on e_i (end points rounded):

$$v_0; \quad e_0 : t \in (0, 0.0670) \cup (0.1992, 0.2550); \quad e_6 : t \in (0.7195, 0.8079) \cup (0.9008, 1).$$

The intervals next to v_0 form the arc of Theorem 1.1(c); in the two others the two pieces of the cut edge, the first and the last segment of the development, cross each other. The bad cut points make up about 4.6% of the perimeter 4.827. These statements are numerical, except that `check_note.py` certifies the cut points at $t = \frac{1}{20}, \frac{1}{5}, \frac{201}{1000}$ on e_0 and $t = \frac{18}{25}, \frac{403}{500}, \frac{19}{20}$ on e_6 as bad, and those at $t = \frac{1}{10}, \frac{1}{2}$ on e_0 , $t = \frac{1}{2}$ on e_3 and $t = \frac{17}{20}$ on e_6 as good.

i	$10^7 \xi'_i$	$10^7 \eta'_i$	$r'_i \approx$	$\varphi'_i \approx$	$\pi - \tau'_i$ (deg)	development cut at v'_i
0	9645622	84179	1.5348	0.0087	36.8	not simple: $(e'_2, e'_5), (e'_2, e'_6)$
1	4975153	2486604	1.0152	0.4635	143.5	not simple: $(e'_2, e'_5), (e'_2, e'_6)$
2	2713466	1700638	0.6198	0.5598	169.2	not simple: $(e'_2, e'_5), (e'_2, e'_6)$
3	-158044	-20350	0.0319	3.2696	48.3	simple
4	7579252	-2376842	1.3425	5.9793	142.9	not simple: $(e'_5, e'_3), (e'_6, e'_3)$
5	9676263	-1423511	1.5486	6.1371	37.0	not simple: $(e'_5, e'_3), (e'_6, e'_3)$
6	158279	-3080	0.0317	6.2637	351.1	not simple: $(e'_6, e'_3), (e'_6, e'_5), (e'_2, e'_5)$

Table 3: The heptagon Q' : v'_i is given by (2) with ξ'_i, η'_i in place of ξ_i, η_i . Polar distances r'_i , longitudes φ'_i and interior angles are rounded. The last column lists the pairs of edges whose images cross transversally (certified).

Proof of Proposition 1.3. The program checks, exactly as for Q , the conditions of Section 3 for the vertices v'_i , rational points with denominators up to about 10^{14} that it prints; so Q' is simple, lies in the open hemisphere about N , and is star-shaped with respect to N , an interior point of its region. The smallest z -coordinate is $z'_5 = 434355078771/19565644921229$, so v'_5 is the vertex farthest from N , at polar distance 1.5486. All s'_i are nonzero. With the arithmetic of Section 3.3, the development cut at v'_5 has two transversal crossings, of the images of e'_5 and e'_3 and of the images of e'_6 and e'_3 , with all eight determinants at least 0.24 in absolute value. The other vertex cuts behave as listed in Table 3; in the cut at v'_3 all 15 pairs of non-consecutive segments satisfy (D). The second and third verification runs certified the same statements with interval arithmetics of their own. $\square$

Remark 4.2. Theorem 1.1 does not need a large polygon. For $s \in \{\frac{1}{2}, \frac{1}{5}, \frac{1}{20}\}$ let Q_s be given by (2) with $s\xi_i, s\eta_i$ in place of ξ_i, η_i . The program certifies that Q_s satisfies the hypotheses of Lemma 2.2 with all $z > 0$, that its development cut at v_0 is not simple (the first and the last segment cross transversally), and that its other vertex cuts are simple. The smallest of the four determinants is about $3.7 \cdot 10^{-3}$, $1.1 \cdot 10^{-4}$ and $4.6 \cdot 10^{-7}$; numerically it decreases roughly like s^4 , and a floating-point computation of the first verification run followed a similar family down to $s = 0.02$ with the same outcome.

5 The weaker reading: some cut point

If one good cut point suffices, Theorem 1.1 does not answer the question: Q has good cut points by Theorem 1.1(d), and Q' has the good cut point v'_3 .

Question 5.1. *Let Q be a simple spherical polygon that lies in an open hemisphere and is star-shaped with respect to the centre of that hemisphere, which is an interior point of its region. Does Q have a cut point at which its development is simple?*

For curves on cones, O’Rourke and Vilcu ask for the class of curves with “at least one cut-point x that leads to simple development” [OV11, §8.3]. We report numerical evidence on Question 5.1; all of it was computed in floating point, and none of it is certified. The fraction of bad cut points of a polygon (by arc length) was computed from the crossings in the developments of its sub-arcs, as in the proof of Theorem 1.1(c); for the best polygons of the last stage the verification runs also developed the polygon at many sampled cut points, with agreement within 10^{-4} .

Among 55,123 random polar graphs about N with 6, 8 or 10 vertices and polar distances below 1.55, 1.1% had a bad vertex cut. Among 12,000 random heptagons that are polar graphs about N with polar distances below 1.55, 145 had a bad cut point, the largest fraction of bad cut points was 0.43, and in none of the 145 was the cut at the vertex farthest from N bad; a search aimed at this rule then found Q' . Hill climbing over polar graphs with 6 to 9 vertices aimed at a large fraction of bad cut points, with all polar distances at least $r_{\min}$ and the interior angles away from 0 and 2π (about two hours on three processor cores; runs that allowed polar distances up to 3 ended in the open hemisphere). The best fractions were 0.965, 0.993 and 0.998 for $r_{\min} = 0.03$, 0.005 and 0.001; in each optimum the good cut points form one arc, of length about 6 to 11 times $r_{\min}$, next to a vertex at polar distance of the order of $r_{\min}$, and for the two smaller values of $r_{\min}$ the polygon consists of two lobes joined by a narrow neck at N . The second verification run searched with another method (8 to 16 vertices, 240 sampled cut points per polygon, two runs of 20 minutes) and reached 0.985 in the open hemisphere and 0.995 with one vertex at polar distance 1.94, again with one short arc of good cut points next to a vertex very close to N .

No polygon in which every cut point is bad was found. The near-misses degenerate in the same way: as $r_{\min} \rightarrow 0$ the two lobes meet at N , and the limit is neither simple nor star-shaped with respect to an interior point of its region. This is consistent with a positive answer to Question 5.1; it is not a proof. In these data the arc of good cut points shrank roughly in proportion to $r_{\min}$, which suggests, numerically, that the good cut points can form an arbitrarily small part of the polygon; we have not proved this.

6 Computations and reproducibility

The reproducibility package contains the program of this note, `check_note.py` (Python 3, standard library only, a few seconds, no files read or written), with its recorded output; it checks every certified statement of Sections 3 and 4, tests its own arithmetic, and checks, as a control, that every vertex cut of a convex hexagon about N is simple. The package also contains the programs with which the results were first obtained (floating-point developments; certificates of Theorem 1.1(a), (b) and (d) in the ball arithmetic of Arb, through `python-flint`, and in the interval arithmetic of `mpmath`; a law-of-cosines check of the angles; the search programs of Section 5 with their results) and programs of the three verification runs. Its `README.md` gives command lines, running times and what each program checks.

Verification. The results were first obtained with proofs and with the programs just listed. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first re-derived Lemma 2.2, the definition of the development and the description of the bad cut points behind Theorem 1.1(c) and Section 5; it certified the example with an exact rational interval arithmetic of its own, without angles, simulated the rolling sphere, checked the gnomonic image exactly and the area of R by l’Huillier’s formula, and compared the statements about O’Rourke’s work with the texts of

[Rot09, O’R03, OV11, O’R12]. The second wrote its own interval arithmetic on rationals and certified Theorem 1.1(a), (b) and (d), the Gauss–Bonnet relation, the cut points of Remark 4.1, the heptagon Q' and the copies of Remark 4.2; it tested deliberately wrong inputs and convex controls, re-ran the original programs (byte-identical outputs), re-evaluated the saved search results, ran the search of Section 5, and repeated the literature search. Both runs confirmed Theorem 1.1(a), (b) and (d) and found no error; part (c) is a special case of the description of bad cut points re-derived by the first run. All corrections they required were made; they concern wording and labelling: the partial status of the record, the every-cut convention as O’Rourke’s usage, the intervals of Remark 4.1 (the first computations had used a nearby floating-point polygon), the hypotheses of the cone result of [OV11] and its scope, the title of the arXiv version of [OV14], the direction of Lemma 2.2, and the labelling of numerical statements. The program `check_note.py` was written after the two runs, together with this text.

A third verification run, also AI-assisted, examined this text and the package. It read the sources again (both versions of the problem list, [O’R03, §§1 and 4], [OV11] and [O’R12]), re-derived the proofs of Lemmas 2.2 and 2.4 and of Theorem 1.1(c), and checked the interval arithmetic of `check_note.py` line by line. With programs of its own (Python standard library) it certified Theorem 1.1(a)–(d), Tables 1 to 3, the numbers of Section 3, Proposition 1.3 and the certified statements of Remarks 4.1 and 4.2 by other methods: lengths and angles are enclosed between rationals at which the cosine is bounded by a Taylor polynomial with remainder, the developed directions are exact Gaussian rationals divided by one square root each, and the Gauss–Bonnet relation is checked as an exact identity between Gaussian rationals. It developed Q by rolling the sphere (the mirror image of Definition 2.3, to 50 digits), recomputed the intervals of Remark 4.1 by bisection, re-evaluated the saved polygons of Section 5 by direct developments, re-ran the whole package from its archive (all recorded outputs were reproduced), and repeated the literature search. It found no mathematical error. The corrections it required were made: the quantifier in Theorem 1.1(c); three values that `check_note.py` now prints and checks (the length of the arc in Theorem 1.1(c), the vertices of Q' with z'_5 , and the bound 0.24 in the proof of Proposition 1.3); the notation for the stereographic data; the description of the cut convention of [OS89, O’R03] and of the cones of [OV11]; and the description of the arithmetic in Section 3.3.

Scope and priority. The note answers the star-shaped question of Problem 13 of [Rot09] negatively in the reading in which a closed polygon develops without self-intersection only if its development is simple for every cut point; this is O’Rourke’s usage [O’R03, OV11], not a definition stated in the source. The record OWR-3385-011 [ula26] is answered only in part: the some-cut variant (Question 5.1), the request for general conditions and Problem 14 of [Rot09] (A. Bobenko) remain open. The question is due to J. O’Rourke. The convex case recorded in the source is the spherical counterpart of the theorem of O’Rourke and Schevon [OS89] on closed convex curves on convex polyhedra, which O’Rourke extended to slice curves [O’R03]; an arm lemma for convex chains redrawn on a larger sphere is in [ACC⁺08]; we neither use nor reprove the convex case. The cone analogue is due to O’Rourke and Vilcu [OV11, §7.1 and footnote 2], and MathOverflow question 100496 [O’R12] asks a related question for cones. The contribution of this note is a certified star-shaped counterexample on the sphere for the every-cut version, the certified failure of the farthest-vertex rule, and numerical evidence on the some-cut version. No printed answer to the star-shaped question was found in the sources searched on 16 October 2026: the report containing [Rot09], with [O’R09]; [O’R03]; [OV11]; the abstract of [ACC⁺08]; MathOverflow, through the StackExchange API; the works citing [O’R03] (21), [OV14] (7) and arXiv:1102.0823 (4) listed by Semantic Scholar (titles and abstracts; some later papers of O’Rourke, Vilcu and Itoh in full); the list of open problems on the web site of the book *Geometric Folding Algorithms*; queries to the arXiv API, Crossref, OpenAlex and zbMATH Open; and eight web

searches. The arXiv API and OpenAlex refused part of the requests, and the works citing [OS89] (seven in OpenAlex) were not listed. A search that finds nothing is not a proof of novelty, and no priority is claimed. What was read: Problem 13 in both versions of [Rot09] (identical wording) and [O’R09]; Sections 1 and 4 of [O’R03] (arXiv version 4); [OV11] (arXiv version 2), whose section and footnote numbers we use; [O’R12]; and the abstract of [ACC⁺08]. The journal version [OV14] was not consulted. We did not read [OS89], whose theorem is quoted as stated in [O’R03, §4] and [OV11, §7.1], nor [VS83], whose formula is used only in the consistency check at the end of Section 3.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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