

The Graph of a Convex 4-Polytope Is Flat in the Boundary Whenever It Is Linklessly Embeddable: An Answer to a Question of Izmestiev

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Abstract

Problem 7 of the problem list of the Oberwolfach workshop “Discrete Differential Geometry” of 2009 (collected by G. Rote; Oberwolfach Reports 6, p. 108), posed by I. Izmestiev, asks whether there is a convex 4-polytope whose 1-skeleton is linked as a graph in the boundary of the polytope, a 3-sphere, but has a different embedding that is not linked. We show that there is none. Let P be a convex 4-polytope and φ a piecewise linear embedding of its graph in the 3-sphere in which the boundary of every 2-face of P bounds a disc meeting the graph only in this boundary; every flat embedding is of this kind. Then φ is the embedding of the graph in ∂P followed by a piecewise linear homeomorphism of 3-spheres. Hence the following are equivalent: the graph of P has no minor in the Petersen family; it has an embedding without two disjoint cycles of odd linking number; its embedding in ∂P is flat; any two of its disjoint cycles bound disjoint discs in ∂P . This gives the negative answer for each of the usual meanings of “linked”. The proof is a short application of a lemma of Böhme, in the form given by Robertson, Seymour and Thomas, and of Alexander’s theorem on polyhedral spheres; the first statement also follows from that lemma and a uniqueness theorem of Georgakopoulos and Kim. The equivalence relies on the theorem of Robertson, Seymour and Thomas on linkless embeddings, which is deep and is not reproved here. As a consequence, the null-space embedding of the Colin de Verdière matrix that Izmestiev attaches to a 4-polytope is the boundary embedding, hence flat when the graph is linklessly embeddable; the general question of Lovász and Schrijver about null spaces of such matrices remains open. As a consistency test, which is not part of the proof, exact computations for all 275,478 combinatorial types of 4-polytopes with at most nine vertices find two disjoint cycles with non-zero linking number in the boundary exactly when the graph has a minor in the Petersen family. No novelty is claimed beyond the answer to the question. This is an unrefereed note.

1 Introduction

1.1 The question

Let P be a convex 4-polytope, the convex hull of a finite subset of $\mathbb{R}^4$ that is not contained in a hyperplane. Its boundary ∂P is homeomorphic to the 3-sphere, and its vertices and edges form a graph $G(P)$ which lies in ∂P . In the problem session of the Oberwolfach workshop “Discrete Differential Geometry” of January 2009, I. Izmestiev posed the following problem [BKSZ09, Problem 7 of the list collected by G. Rote, p. 108]. The problem list calls a graph embedded in S^3 *linked* if two disjoint cycles of the graph are linked, and it asks whether some convex 4-polytope P has a 1-skeleton that is linked as a graph in $\partial P \approx S^3$ “but has a different embedding that is not linked”. The problem is record OWR-3385-009 of the UnsolvedMath collection [ula26], where it is listed as open.

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The source gives the following background. Let G be a graph with Colin de Verdière number $\mu(G) = 4$ and M a Colin de Verdière matrix of G . The problem list refers to a conjecture of Lovász and Schrijver [LS99] and Lovász [Lov01], according to which the null space of M yields a non-linked embedding of G . A polytope as in the problem would refute it: its graph would have $\mu = 4$, and by a theorem of Izvestiev [Izm10] there is a Colin de Verdière matrix whose null space represents the graph as the skeleton of P , that is, in a linked way. In the two papers this is a hope and a question, respectively; see Section 1.4.

1.2 Definitions

Piecewise linear topology. We work in the piecewise linear (PL) category [RS72]. A polyhedron is a finite union of simplices in some $\mathbb{R}^N$, and a map between polyhedra is PL if it is simplicial with respect to suitable triangulations. A *PL n -ball* (*PL n -sphere*) is a polyhedron that is PL homeomorphic to an n -simplex (to the boundary of an $(n+1)$ -simplex); a *PL disc* is a PL 2-ball. A convex n -polytope is a PL n -ball, and its boundary is a PL $(n-1)$ -sphere. Throughout, M and M' denote PL 3-spheres.

Graphs. Graphs are finite and simple. A *cycle* of a graph is a connected subgraph all of whose vertices have degree two; two cycles are *disjoint* if they have no common vertex. A *minor* of a graph G is a graph obtained from a subgraph of G by contracting edges. The *Petersen family* consists of the seven graphs that can be obtained from the complete graph K_6 by $Y\Delta$ - and ΔY -exchanges [RST93a]. A graph is regarded as a one-dimensional polyhedron; for $G = G(P)$ this is the union $|G|$ of the edges, and $|C| \subseteq |G|$ is the union of the edges of a subgraph C . An *embedding* of G in M is an injective PL map $\varphi: |G| \rightarrow M$. The *boundary embedding* of $G(P)$ is the inclusion $\iota_P: |G(P)| \rightarrow \partial P$.

Panels. Let φ be an embedding of a graph G in M and C a cycle of G . A *panel* of C (with respect to φ) is a PL disc $\Delta \subseteq M$ with $\Delta \cap \varphi(|G|) = \varphi(|C|) = \partial\Delta$. The embedding is *flat* if every cycle of G has a panel. (“Flat” is the term of [RST93a]; in [RST95] such embeddings are called *panelled*.)

Linked. For disjoint oriented simple closed curves C, D in an oriented 3-sphere, $\text{lk}(C, D) \in \mathbb{Z}$ is the linking number: the class of D in $H_1(S^3 \setminus C; \mathbb{Z}) \cong \mathbb{Z}$, a meridian of C corresponding to 1. It is symmetric [Rol03], and we only use whether it is zero, non-zero or odd. The source does not say when two disjoint cycles are “linked”. The usual meanings, for disjoint simple closed curves C, D in a 3-sphere, are: (L3) $\text{lk}(C, D)$ is odd; (L2) $\text{lk}(C, D) \neq 0$; (L1) no embedded 2-sphere that is disjoint from $C \cup D$ separates C from D ; (L0) C and D do not bound disjoint embedded discs. For PL curves each of these properties implies the next (Lemma 5.1). The linking number modulo 2 is used in [RST93a] and discs in [RST95]; three of the notions are compared in Section 16.3 of a draft of the book [Lov19]. An embedded graph is *linked*, in one of these readings, if some pair of disjoint cycles has the respective property.

1.3 Results

Theorem 1.1. *Let P be a convex 4-polytope and let $\varphi: |G(P)| \rightarrow M$ be an embedding of its graph in a PL 3-sphere such that for every 2-face σ of P the boundary cycle C_σ of σ has a panel with respect to φ . Then there is a PL homeomorphism $\Phi: \partial P \rightarrow M$ with $\Phi \circ \iota_P = \varphi$. In particular, if φ is a flat embedding of $G(P)$, then $\varphi = \Phi \circ \iota_P$ for a PL homeomorphism Φ , and the boundary embedding ι_P is flat.*

That two flat embeddings of a 4-connected graph differ by a homeomorphism of the sphere is known [RST93a, (1.4)], [RST95, (1.7)], and graphs of 4-polytopes are 4-connected [Bal61]; the point here is that the boundary embedding is flat as soon as a flat embedding exists.

Corollary 1.2. *For a convex 4-polytope P the following are equivalent.*

- (1) $G(P)$ has no minor in the Petersen family.

- (2) $G(P)$ has an embedding in a PL 3-sphere in which no two disjoint cycles have odd linking number.
- (3) The boundary embedding ι_P is flat.
- (4) Any two disjoint cycles of $G(P)$ have disjoint panels with respect to ι_P ; that is, they bound disjoint PL discs in ∂P each of which meets $|G(P)|$ only in its boundary.

Consequently there is no convex 4-polytope P such that $G(P)$ is linked in ∂P and has an embedding that is not linked, whichever of the readings (L0)–(L3) is used in either place.

Corollary 1.3. *Let $Q \subset \mathbb{R}^4$ be a convex 4-polytope with the origin in its interior, and let M_Q be the Colin de Verdière matrix of corank 4 that [Izm10, Theorem 2.4] attaches to the polar polytope Q^* ; it is a matrix for the graph $G(Q)$. Then every null-space embedding of M_Q in the unit sphere S^3 is the image of the boundary embedding ι_Q under a homeomorphism $\partial Q \rightarrow S^3$. If $G(Q)$ has no minor in the Petersen family, then in this embedding every cycle bounds a disc that meets the embedded graph only in the cycle, and any two disjoint cycles bound disjoint such discs; in particular it is not linked in any of the readings.*

The null-space embedding is recalled in Section 5.3.

1.4 Relation to the literature, and what is not claimed

Links in boundaries of 4-polytopes. Knots and links formed by cycles of the graph of a 4-polytope in its boundary were studied by Eppstein in a blog post [Epp14] and in [Epp20, Section 6.1 of the arXiv version]. He observes that such links are common, that the linked 4-polytope with the fewest vertices is the convex hull of two triangles in orthogonal planes (the polytope $C(6, 4)$ of Section 6.2), that a neighbourly polytope with seven vertices contains a knot because its graph K_7 does in every embedding, and that pyramids and stacked polytopes contain neither knots nor links; and he shows that every knot or link occurs in the boundary of some 4-polytope. In [Epp20] the graphs of the Petersen family are named as forbidden minors for the graphs of pyramids. Beyond this, these texts do not relate links in the boundary to linkless embeddability of the graph.

The ingredients. Theorem 1.1 is a short consequence of Böhme’s lemma in the form of Robertson, Seymour and Thomas (Lemma 2.2); we did not find the theorem stated in the literature that we searched. It is that lemma combined with the uniqueness theorem of Georgakopoulos and Kim [GK23, Theorem 1.3 of the arXiv version] for 2-complexes in the 3-sphere (Remark 4.1); the corresponding part of the proof, Steps 3–5 in Section 4, is given for completeness. The technique, to panel the boundaries of the faces by Böhme’s lemma and to rebuild the complex, is that of [RST93a, (2.2)], where a flat embedding of a 3-connected planar graph is shown to lie on a 2-sphere, and of statement (4.3) of the survey [RST93b]; related uses of the lemma are in [FJP⁺24, Foi22]. The equivalence in Corollary 1.2 rests on the theorem of Robertson, Seymour and Thomas (Theorem 2.1). We claim no novelty beyond the answer to the question.

The null-space question. Lovász and Schrijver [LS99, Remark 2, p. 1019] consider a 4-connected linklessly embeddable graph G and a matrix of corank 4 with the sign pattern of G and one negative eigenvalue; they suggest that its null space may yield a linkless embedding of G in S^3 , and state that they do not know whether this is true. Lovász [Lov01, Section 8, item 4, p. 235] considers a linklessly embeddable graph and a Colin de Verdière matrix of it, and asks whether the embedding in S^3 obtained from the null space is linkless. So the sources state a hope and a question, which the problem list calls a conjecture. Corollary 1.3 gives a positive answer for the matrices that Izmestiev attaches to 4-polytopes, and it closes the route to a counterexample proposed in the problem. It says nothing about other matrices: the general question remains open as far as we know; it is described as open in [SS17] and in [LS17].

Not claimed. If $G(P)$ has a minor in the Petersen family, then every embedding of $G(P)$ is linked, by the theorem of Sachs quoted in Section 2; the question is about the other polytopes. The statements concern the embedding in ∂P , equivalently Schlegel diagrams; other straight-line embeddings of $G(P)$ in $\mathbb{R}^3$ can be linked although $G(P)$ is linklessly embeddable (Section 7). Generalisations to cell decompositions of the 3-sphere are not pursued.

2 The cited theorems

Theorem 2.1 (Robertson, Seymour and Thomas [RST93a, (1.2)], [RST95, (1.2)]). *For a graph G the following are equivalent: (i) G has a flat embedding; (ii) G has an embedding in which any two disjoint cycles have even linking number; (iii) G has no minor in the Petersen family.*

In [RST93a] all embeddings are PL, linking numbers are taken modulo 2, and the embeddings in (ii) are called linkless. The implication (i) $\Rightarrow$ (ii) is clear, since a panel of a cycle is disjoint from every cycle disjoint from it. The implication (ii) $\Rightarrow$ (iii) is due to Sachs [Sac83, Sac84], for K_6 also to Conway and Gordon [CG83]: every embedding of a graph of the Petersen family contains two disjoint cycles with odd linking number, and property (ii) is inherited by minors. The implication (iii) $\Rightarrow$ (i) is the main theorem of [RST95]; it is deep, and we do not reprove it. In [RST95, (1.2)] the place of (ii) is taken by the property that G is *semi-flat*: it has an embedding in which every family of pairwise disjoint cycles is a trivial link, that is, each of these cycles bounds a disc that meets the union of the family only in this cycle. This property follows from (i) and implies (ii), by Lemma 5.1(a); so the theorem holds with every condition between these two in the place of (ii).

Lemma 2.2 (Böhme’s lemma, in the form [RST95, (2.8), p. 195]). *Let φ be an embedding of a graph G in M and let $C_1, \dots, C_k$ be cycles of G such that for $i < j$ the graph $C_i \cap C_j$ is empty or connected. If each C_i has a panel with respect to φ , then there are panels $\Delta_1, \dots, \Delta_k$ of $C_1, \dots, C_k$ such that $\Delta_i \cap \Delta_j = \varphi(|C_i \cap C_j|)$ for all $i < j$.*

The lemma is attributed to Böhme [Böh90] in [RST93a, RST95], with a reference to Saran [Sar89] in [RST93a]; it is proved in [RST95, Section 2]. Only statement (2.8) there has the form above, in which panels are assumed for the cycles $C_1, \dots, C_k$ alone. The versions in [RST93a, (2.1)] and in [RST93b] assume a flat embedding; they suffice for the second sentence of Theorem 1.1 and for the corollaries, but not for its first sentence.

Category. In [RST95] all embeddings and all subspaces considered are assumed to be tame, and the proofs use piecewise linear general position. We use Theorem 2.1 and Lemma 2.2 in the PL category, as stated above: embeddings are PL and panels are PL discs. For the lemma this is the form in which it is proved in [RST95, Section 2]: the proof moves one given panel into general position with respect to another, for which it refers to [RS72], and replaces subdiscs of the first by displaced copies of subdiscs of the second, which turns PL panels into PL panels. In the statements (1.2) of [RST93a, RST95] the discs are not required to be PL; that a graph without a minor in the Petersen family has a PL embedding in which every cycle has a PL panel is the form in which we read them.

We use three classical facts. *Alexander’s theorem* [Ale24]: if $\Sigma \subseteq M$ is a PL 2-sphere, then $M \setminus \Sigma$ has exactly two components, and the closure of each of them is a PL 3-ball with boundary Σ . *Extension*: a PL homeomorphism between the boundaries of two PL n -balls extends to a PL homeomorphism between the balls [RS72]. *Invariance of domain*: an injective continuous map between 3-manifolds without boundary is open [Hat02].

3 Faces of a 4-polytope

Let P be a convex 4-polytope. Its faces of dimension 0, 1, 2, 3 are the vertices, edges, 2-faces and facets. We use the following standard facts; see the lectures on faces and on graphs of polytopes in [Zie95].

- (P1) Every face is the convex hull of the vertices of P that it contains. The intersection of two faces is a face or empty. The faces of a face F are the faces of P contained in F . Every proper face is the intersection of the facets containing it. A face that contains a relative interior point of a segment $[x, y] \subseteq P$ contains x and y .
- (P2) The relative interiors of the proper faces form a partition of ∂P . The relative boundary ∂F of a face F is the union of the proper faces of F ; for a facet it is the union of the 2-faces contained in it.
- (P3) A 2-face σ is a convex polygon; its vertices and edges form a cycle C_σ of $G(P)$, and $|C_\sigma| = \partial\sigma$. Every edge lies in a 2-face.
- (P4) The graph of a polytope is connected. If v is a vertex of P and ℓ is an affine function with $\ell(w) \geq \ell(v)$ for all neighbours w of v , then $\ell \geq \ell(v)$ on P .

Let $X \subseteq \partial P$ be the union of the 2-faces of P ; by (P3) it contains $|G(P)|$.

Lemma 3.1. *Let $\sigma \neq \tau$ be 2-faces of P . Then $C_\sigma \neq C_\tau$, and the graph $C_\sigma \cap C_\tau$ is empty, or a single vertex, or a single edge with its two ends.*

Proof. By (P1), σ is the convex hull of the vertices of C_σ , so $C_\sigma \neq C_\tau$. The set $\sigma \cap \tau$ is a face of σ and of τ different from both, hence empty, a vertex or an edge. The common vertices of C_σ and C_τ are the vertices of P in $\sigma \cap \tau$, and a common edge is an edge of P contained in $\sigma \cap \tau$. So $C_\sigma \cap C_\tau$ is empty in the first case and a single vertex in the second; in the third case it consists of the edge $\sigma \cap \tau$ and its two ends. $\square$

Lemma 3.2. (a) *For every facet F we have $X \cap F = \partial F$, and every point of $\partial P \setminus X$ lies in the relative interior of exactly one facet.* (b) *If $F \neq F'$ are facets, then $F \cap F' \subseteq \partial F \cap \partial F'$ and $\partial F \neq \partial F'$.*

Proof. (a) Let $x \in X \cap F$, say $x \in \sigma$ for a 2-face σ . Then $\sigma \cap F$ is a face of F of dimension at most 2, so it lies in ∂F by (P2). Conversely $\partial F \subseteq X$ by (P2). By (P2) a point $x \in \partial P$ lies in the relative interior of exactly one proper face; if this face is not a facet, it lies in a 2-face by (P1) and (P3), and $x \in X$; if it is a facet F , then $x \notin \partial F = X \cap F$. (b) The set $F \cap F'$ is a proper face of both facets, so it lies in $\partial F \cap \partial F'$. A convex polytope of positive dimension is the convex hull of its relative boundary, so $\partial F = \partial F'$ would imply $F = F'$. $\square$

Lemma 3.3. *For every facet F of P the set $X \setminus \partial F$ is non-empty and path-connected.*

Proof. Let ℓ be an affine function with $\ell \leq 0$ on P and $F = P \cap \{\ell = 0\}$, and let V' be the set of the vertices with $\ell < 0$, that is, of the vertices not in F . It is non-empty.

(i) *The subgraph of $G(P)$ induced on V' is connected.* Let $m = \min_P \ell < 0$. The face $P \cap \{\ell = m\}$ has all its vertices in V' , and its graph is connected by (P4) and consists of edges of P by (P1). If $v \in V'$ and $\ell(v) > m$, then by (P4) some neighbour w of v has $\ell(w) < \ell(v)$; in particular $w \in V'$. Repeating this, we reach a vertex of $P \cap \{\ell = m\}$ from v along a path in V' .

(ii) An edge with both ends in V' lies in X and in $\{\ell < 0\}$, hence in $X \setminus \partial F$.

(iii) Let $x \in X \setminus \partial F$, say $x \in \sigma$ for a 2-face σ . Then $x \notin F$ by Lemma 3.2(a), so $\ell(x) < 0$, and some vertex u of σ has $\ell(u) < 0$. The segment $[x, u]$ lies in σ and in $\{\ell < 0\}$, hence in $X \setminus \partial F$.

By (iii) every point of $X \setminus \partial F$ is joined in this set to a vertex in V' , and by (i) and (ii) any two vertices in V' are joined in it. $\square$

4 Proof of Theorem 1.1

Let $G = G(P)$ and $\iota = \iota_P$, and let $\varphi: |G| \rightarrow M$ be an embedding in which the cycle C_σ has a panel for every 2-face σ of P .

Step 1: panels for all 2-faces at once. By Lemma 3.1 the cycles C_σ , where σ runs through the 2-faces of P , satisfy the hypothesis of Lemma 2.2. Hence there are panels Δ_σ of the cycles C_σ with

$$\Delta_\sigma \cap \varphi(|G|) = \varphi(|C_\sigma|) = \partial\Delta_\sigma \quad \text{and} \quad \Delta_\sigma \cap \Delta_\tau = \varphi(|C_\sigma \cap C_\tau|) \quad (\sigma \neq \tau). \quad (1)$$

Step 2: the 2-skeleton embeds. For every 2-face σ the map φ restricts to a PL homeomorphism $\partial\sigma \rightarrow \partial\Delta_\sigma$; let $h_\sigma: \sigma \rightarrow \Delta_\sigma$ be a PL homeomorphism extending it. Define $\Phi_2: X \rightarrow M$ by $\Phi_2 = h_\sigma$ on σ . This is well defined and PL: two different 2-faces meet in a vertex or an edge or not at all, and on $|G|$ all the maps h_σ agree with φ . It is injective. Indeed, let $x \in \sigma$ and $y \in \tau$ with $\Phi_2(x) = \Phi_2(y) = z$. If $\sigma = \tau$, then $x = y$. Otherwise $z \in \Delta_\sigma \cap \Delta_\tau \subseteq \partial\Delta_\sigma \cap \partial\Delta_\tau$ by (1), so $x \in \partial\sigma$ and $y \in \partial\tau$, hence $\varphi(x) = z = \varphi(y)$ and $x = y$. Put $A = \Phi_2(X)$, the union of the panels Δ_σ . By (P3), $\Phi_2 \circ \iota = \varphi$.

Step 3: every facet has an empty side. Let F be a facet. Its boundary ∂F is a PL 2-sphere, the union of the 2-faces of F , and Φ_2 is PL and injective on it. So $\Sigma_F = \Phi_2(\partial F)$, the union of the panels Δ_σ with $\sigma \subseteq F$, is a PL 2-sphere in M . By Alexander's theorem $M \setminus \Sigma_F$ has two components, and the closure of each is a PL 3-ball with boundary Σ_F . Since Φ_2 is injective, $A \setminus \Sigma_F = \Phi_2(X \setminus \partial F)$; this set is connected and non-empty by Lemma 3.3, so it lies in one of the two components. Let O_F be the other one. Then $O_F \cap A = \emptyset$, and $B_F = O_F \cup \Sigma_F$ is a PL 3-ball with boundary Σ_F .

Step 4: different facets have disjoint empty sides. Let $F \neq F'$ be facets. The connected set O_F is disjoint from $\Sigma_{F'} \subseteq A$, so it lies in one component of $M \setminus \Sigma_{F'}$: either $O_F \subseteq O_{F'}$ or $O_F \cap O_{F'} = \emptyset$. The same holds with F and F' exchanged. If O_F and $O_{F'}$ had a common point, they would therefore be equal, and so would their frontiers Σ_F and $\Sigma_{F'}$. As Φ_2 is injective, this would give $\partial F = \partial F'$, contradicting Lemma 3.2(b). Hence $O_F \cap O_{F'} = \emptyset$.

Step 5: extension over the facets. For every facet F let $g_F: F \rightarrow B_F$ be a PL homeomorphism that extends the PL homeomorphism $\Phi_2|_{\partial F}: \partial F \rightarrow \Sigma_F$. Define $\Phi: \partial P \rightarrow M$ by $\Phi = g_F$ on F . By Lemma 3.2, two different facets meet inside X , where all the maps g_F agree with Φ_2 ; so Φ is well defined and PL, and $\Phi = \Phi_2$ on X . It is injective: g_F maps the relative interior of F onto O_F ; the sets O_F are pairwise disjoint (Step 4) and disjoint from $A = \Phi(X)$ (Step 3); Φ_2 is injective on X (Step 2); and by Lemma 3.2(a) every point of ∂P lies in X or in the relative interior of exactly one facet. By invariance of domain $\Phi(\partial P)$ is open in M ; it is also compact, and M is connected, so Φ is onto. Thus Φ is a PL homeomorphism, and $\Phi \circ \iota = \Phi_2 \circ \iota = \varphi$.

The flat case. Let φ be flat and let Δ be a panel of a cycle C of G with respect to φ . Then $\Phi^{-1}(\Delta)$ is a PL disc in ∂P with boundary $\Phi^{-1}(\varphi(|C|)) = |C|$, and $\Phi^{-1}(\Delta) \cap |G| = \Phi^{-1}(\Delta \cap \varphi(|G|)) = |C|$. So ι is flat. $\square$

Remark 4.1 (How Theorem 1.1 follows from the literature). Steps 1 and 2 extend φ , by Böhme's lemma, to a PL embedding Φ_2 of the 2-skeleton X , as in [RST93a, (2.2)] and [RST93b, (4.3)]. The theorem of Georgakopoulos and Kim [GK23, Theorem 1.3 of the arXiv version] says: if X is a finite, simply connected 2-complex all of whose link graphs are 3-connected, then any two locally flat embeddings of X in S^3 differ by a homeomorphism of S^3 ; as they remark, PL embeddings are locally flat. The 2-skeleton of P is such a complex: it is simply connected because ∂P arises from it by attaching 3-cells, and the link graph of a vertex is the graph of a vertex figure of P , a 3-polytope, which is 3-connected [Bal61]. Applied to Φ_2 and to the inclusion of X in ∂P , their theorem gives a homeomorphism $\Phi: \partial P \rightarrow M$ with $\Phi|_X = \Phi_2$: the first statement of Theorem 1.1 with a homeomorphism in place of a PL homeomorphism. Steps 3–5 prove this special case directly.

5 The corollaries

5.1 The readings of “linked”

Lemma 5.1. *Let C and D be disjoint simple closed curves in a 3-sphere. (a) If C is null-homotopic in the complement of D , in particular if C bounds a disc disjoint from D , then $\text{lk}(C, D) = 0$. (b) If an embedded 2-sphere disjoint from $C \cup D$ separates C from D , then $\text{lk}(C, D) = 0$. (c) If the 3-sphere and the curves are PL and C, D bound disjoint PL discs, then a PL 2-sphere disjoint from $C \cup D$ separates C from D . Hence $(\text{L3}) \Rightarrow (\text{L2}) \Rightarrow (\text{L1})$ and $(\text{L2}) \Rightarrow (\text{L0})$, and for PL curves $(\text{L1}) \Rightarrow (\text{L0})$, the discs in (L0) being PL.*

Proof. (a) Then C is null-homologous in the complement of D , so $\text{lk}(D, C) = 0$, and the linking number is symmetric. (b) Let Σ be the sphere and U the component of $S^3 \setminus \Sigma$ that contains C . The first homology group of $S^3 \setminus \Sigma$ vanishes [Hat02, Chapter 2, classical applications], so C bounds a singular 2-chain in U , which is disjoint from D . (c) A regular neighbourhood of the disc of C that is disjoint from the disc of D is a PL 3-ball [RS72]; its boundary is the required sphere. $\square$

5.2 Proof of Corollary 1.2

Let (F) be the statement that $G(P)$ has a flat embedding. By Theorem 2.1, (1), (2) and (F) are equivalent. (F) implies (3) by Theorem 1.1, and (3) implies (F) because ∂P is a PL 3-sphere. (3) implies (4): two disjoint cycles C, D have panels, and $C \cap D$ is the empty graph, so Lemma 2.2 gives panels Δ_C, Δ_D with $\Delta_C \cap \Delta_D = \emptyset$. (4) implies (2), by Lemma 5.1(a) applied to the embedding ι_P .

For the last sentence, let ψ be an embedding of $G(P)$ in a PL 3-sphere that is not linked in the reading $(\text{L}j)$, where $j \in \{0, 1, 2, 3\}$, the discs in (L0) being PL. As (L3) implies $(\text{L}j)$, no two disjoint cycles have odd linking number with respect to ψ . So (2) holds, hence (4). By (4) no pair of disjoint cycles has property (L0) with respect to ι_P , and as $(\text{L}k)$ implies (L0) for PL curves, ι_P is not linked in any reading $(\text{L}k)$. The argument applies to every notion of a linked pair that is implied by (L3) and implies (L0) , for instance to the notion that one of the two curves is not null-homotopic in the complement of the other. $\square$

The answer does not change if the other embedding is only a topological embedding ψ : parts (a) and (b) of Lemma 5.1 hold for arbitrary curves, and a PL embedding close to ψ [RS72] has the same linking numbers of disjoint cycles, as these do not change under a homotopy during which the two curves remain disjoint [Rol03]; so (2) holds again.

5.3 Izmistiev’s matrices: proof of Corollary 1.3

Let G be a graph with vertices $1, \dots, n$. Following [LS99], let $\mathcal{M}(G)$ be the set of symmetric $n \times n$ matrices M with exactly one negative eigenvalue, of multiplicity one, such that $M_{ij} < 0$ if ij is an edge and $M_{ij} = 0$ if $i \neq j$ are not adjacent. A *Colin de Verdière matrix* of G is a matrix in $\mathcal{M}(G)$ with the Strong Arnold Property; $\mu(G)$ is the largest corank of such a matrix, and $\mu(G) \leq 4$ if and only if G is linklessly embeddable [LS98]. If $M \in \mathcal{M}(G)$ has corank r and the columns of an $n \times r$ matrix form a basis of $\ker M$, its rows $u_1, \dots, u_n \in \mathbb{R}^r$ are a *null-space representation* of G , determined up to a linear automorphism of $\mathbb{R}^r$. The *null-space embedding* maps the vertex i to $u_i / \|u_i\| \in S^{r-1}$ and an edge to the shortest arc of a great circle between the images of its ends [LS99, Lov01], provided that this is defined and injective.

Theorem 5.2 (Izmestiev [Izm10, Theorem 2.4 and its proof]). *Let $R = \{p \in \mathbb{R}^d : \langle v_i, p \rangle \leq x_i^0, i = 1, \dots, n\}$ be a convex d -polytope with outer facet normals $v_1, \dots, v_n$, and let G be the graph on $1, \dots, n$ in which i and j are adjacent if the facets with the normals v_i and v_j meet in a face of dimension $d - 2$. Then minus the Hessian, at $x = x^0$, of the volume of $\{p : \langle v_i, p \rangle \leq x_i\}$ as a function of x is a*

Colin de Verdière matrix of G of corank d , and its kernel consists of the vectors $(\langle v_i, p \rangle)_{i=1, \dots, n}$ with $p \in \mathbb{R}^d$.

Proof of Corollary 1.3. Let $q_1, \dots, q_n$ be the vertices of Q . The polar polytope is $Q^* = \{p : \langle q_i, p \rangle \leq 1, i = 1, \dots, n\}$; its facets correspond to the vertices of Q , with outer normals q_i , and two facets of Q^* meet in a 2-face if and only if the corresponding vertices of Q are adjacent. So Theorem 5.2, applied to $R = Q^*$, gives a Colin de Verdière matrix M_Q of $G(Q)$ of corank 4 whose kernel consists of the vectors $(\langle q_i, p \rangle)_i$. For the basis of the kernel that belongs to the standard basis of $\mathbb{R}^4$ the null-space representation is $u_i = q_i$; for another basis it is $u_i = Aq_i$ with an invertible matrix A . (Another choice of the lengths of the normals multiplies the u_i by positive numbers and changes nothing below.)

Let $\rho(x) = x/\|x\|$. The polytope AQ contains the origin in its interior, so ρ maps $\partial(AQ)$ homeomorphically onto S^3 . If $[q_i, q_j]$ is an edge of Q , then $[Aq_i, Aq_j]$ is an edge of AQ ; it lies in $\partial(AQ)$, so it is not contained in a line through the origin, and ρ maps it onto the shortest arc of a great circle between $\rho(Aq_i)$ and $\rho(Aq_j)$. Hence the null-space embedding is $(\rho \circ A) \circ \iota_Q$, and $\rho \circ A$ is a homeomorphism from ∂Q onto S^3 . If $G(Q)$ has no minor in the Petersen family, then ι_Q is flat and has property (4) of Corollary 1.2; the images of the panels under $\rho \circ A$ are the discs required. So no pair of disjoint cycles is linked in the reading (L0), nor in the readings (L2) and (L3), by Lemma 5.1(a), nor in the reading (L1): the sphere of Lemma 5.1(c) for two disjoint cycles in ∂Q is mapped by $\rho \circ A$ to an embedded sphere that separates their images. $\square$

If $G(Q)$ is linklessly embeddable, the matrix M_Q satisfies the hypotheses of the two texts quoted in Section 1.4: $G(Q)$ is 4-connected [Bal61], and M_Q is a Colin de Verdière matrix in $\mathcal{M}(G(Q))$ of corank $4 = \mu(G(Q))$. For $d = 3$ the construction is that of Lovász [Lov01], as shown in [Izm10, Section 2.3].

6 Examples

The certificates of this section can be checked directly; the exact program of this note, `check_note.py`, verifies them.

6.1 A criterion that needs neither of the cited theorems

Lemma 6.1. *Let v be a vertex of a convex 4-polytope P such that every edge of P that does not contain v lies in a facet that contains v . Then the boundary embedding ι_P is flat.*

Proof. Let st be the union of the facets that contain v , let ast be the union of the other facets, and let $S = \text{st} \cap \text{ast}$. For $x \in \partial P$ and $0 \leq r \leq 1$ write $r \cdot x = v + r(x - v)$.

(i) *st is the cone over S with apex v :* every point of $\text{st} \setminus \{v\}$ has a unique representation $r \cdot x$ with $x \in S$ and $0 < r \leq 1$. Let $y \in \text{st} \setminus \{v\}$, say $y \in F$ for a facet $F \ni v$, and let x be the last point of F on the ray from v through y . The face H of P that contains x in its relative interior lies in F and does not contain v , since otherwise the ray could be continued in H beyond x . By (P1) H lies in a facet not containing v , so $x \in S$ and $y \in [v, x]$. For the uniqueness let $x \neq x'$ be points of S on one ray from v , with $x \in [v, x']$. Then x' lies in a facet $F \ni v$ and x in a facet $F' \not\ni v$. The face $F \cap F'$ contains the relative interior point x of the segment $[v, x'] \subseteq F$, hence v by (P1), which is impossible.

(ii) *S is a PL 2-sphere and a union of 2-faces of P , each of which lies in a facet containing v ; and st , ast are PL 3-balls with boundary S .* Since ast is closed and $v \notin \text{ast}$, the set st is a neighbourhood of v in ∂P . By (i) it is the cone over S with apex v , so S is PL homeomorphic to the link of a point in a PL 3-manifold, a PL 2-sphere [RS72]. It is a 2-manifold and a union of faces of P of dimension at most 2, hence a union of 2-faces; and a 2-face in S lies in one of the facets containing v , because a polygon is not a union of finitely many proper faces. The sets $\text{st} \setminus S$ and $\text{ast} \setminus S$ are open and disjoint,

their union is $\partial P \setminus S$, and the first is connected by (i). By Alexander's theorem they are the two components of $\partial P \setminus S$, and st and ast are PL 3-balls.

(iii) $|G(P)| \subseteq \text{st}$; the graph $G(P) - v$ lies in S ; an edge $[v, w]$ meets S only in w . An edge that does not contain v lies in a facet containing v by hypothesis, and in a facet not containing v by (P1). An edge $[v, w]$ lies in st, and $w \in S$. The rest follows from (i).

(iv) A cycle C that does not contain v has a panel. By (iii) and (ii), $|C|$ is a PL circle in the PL sphere S ; it bounds a PL disc $D \subseteq S$. Let $c: S \times [0, 1] \rightarrow \text{ast}$ be a PL collar of the boundary of the ball ast, with $c(x, 0) = x$ [RS72]. Then $\Delta = c(|C| \times [0, \frac{1}{2}]) \cup c(D \times \{\frac{1}{2}\})$ is a PL disc with boundary $|C|$, and $\Delta \setminus |C|$ lies in $\text{ast} \setminus S$, which contains no point of the graph by (iii).

(v) A cycle C through v has a panel. Let a, b be the neighbours of v on C , and let $\pi \subseteq S$ be the path of C from a to b that avoids v . Let N be a regular neighbourhood of π in S that contains no vertex of P outside π ; it is a PL disc with π in its interior [RS72]. Extend π at both ends to an arc that meets ∂N exactly in its two ends. This arc cuts N into two PL discs; let D' be one of them. Its boundary is the union of π and an arc λ from a to b . Triangulate D' so that every triangle lies in a 2-face of P , the path π is a subcomplex, and every simplex with all its vertices on π lies in π (pass to a barycentric subdivision). Put $r_y = 1$ for the vertices y of the triangulation on π and $r_y = \frac{1}{2}$ for the others. For a triangle $\tau = \text{conv}\{y_1, y_2, y_3\}$ of the triangulation let $\hat{\tau} = \text{conv}\{r_{y_1} \cdot y_1, r_{y_2} \cdot y_2, r_{y_3} \cdot y_3\}$; by (ii) it lies in a facet containing v , hence in st. The plane of τ meets P in a 2-face in S , so it does not contain v . Hence for $x = \sum_i \lambda_i y_i \in \tau$ the segment $[v, x]$ meets $\hat{\tau}$ in exactly one point $\varrho(x) \cdot x$, where $\varrho(x) = (\sum_i \lambda_i / r_{y_i})^{-1}$; and $\varrho(x) = 1$ exactly if x lies in the face of τ spanned by its vertices on π , that is, for $x \in \tau \cap \pi$. These functions agree on common faces and define $\varrho: D' \rightarrow [\frac{1}{2}, 1]$ with $\varrho^{-1}(1) = \pi$. By (i) the triangles $\hat{\tau}$ form a simplicial complex isomorphic to the triangulation of D' , so their union $R = \{\varrho(x) \cdot x : x \in D'\}$ is a PL disc. Likewise $T = \{r \cdot x : x \in \lambda, 0 \leq r \leq \varrho(x)\}$, the union of the triangles $\text{conv}\{v, r_y \cdot y, r_{y'} \cdot y'\}$ for the edges yy' of λ , is a PL disc, bounded by $[v, a]$, $[v, b]$ and the arc $\hat{\lambda} = \{\varrho(x) \cdot x : x \in \lambda\}$. By (i) the discs R and T meet exactly in $\hat{\lambda}$, which lies in the boundary of both. So $\Delta = R \cup T$ is a PL disc with boundary $[v, a] \cup \pi \cup [b, v] = |C|$. By (iii) a point of $\Delta \cap |G(P)|$ lies in S or on an edge $[v, w]$. Now $\Delta \cap S = \pi$, because $\varrho < 1$ outside π . If $w \neq a, b$ is a neighbour of v , then $[v, w]$ meets T only in v , since $w \notin \lambda$, and it meets R at most in the point $\varrho(w) \cdot w$, where $w \in D'$; then $w \in \pi$ and this point is w . Hence $\Delta \cap |G(P)| = |C|$. $\square$

The lemma is a form of the familiar fact that a planar graph in a plane or on a sphere, with one further vertex joined to it by straight segments, is flatly embedded [RST93b, (1.2)]; compare [Epp14, Epp20] for pyramids. It applies to the following polytopes, for which flatness of the boundary embedding is thus proved without Theorem 2.1 and without Lemma 2.2.

- The 4-simplex, and every pyramid with apex v : an edge that does not contain v is an edge of the base and lies in the pyramid over a 2-face of the base.
- The prism over a tetrahedron $abcd$ with top $a'b'c'd'$, and $v = a$. The facets through a are the bottom and the prisms over abc , abd , acd . The edges bc , bd , cd lie in the bottom, and each of the six edges of the top and of the edges bb' , cc' , dd' lies in one of the three prisms.
- The product of two triangles with vertices (i, j) , $i, j \in \{0, 1, 2\}$, and $v = (0, 0)$. The facets are $e \times \Delta$ and $\Delta \times e$ for the edges e of a triangle Δ . An edge $\{(i, j), (i', j)\}$ lies in the facet $\Delta \times \{0, j\}$ if $j \neq 0$ and in $\Delta \times \{0, 1\}$ if $j = 0$; both contain v . The same holds for the edges $\{(i, j), (i, j')\}$.

The hypothesis fails for every vertex of the product of a triangle and a square and of the polar of the hypersimplex $\Delta(2, 5)$. For both, the complete searches of Section 7 find no two disjoint cycles with non-zero linking number in the boundary; so (2) holds for ι_P itself, and the boundary embedding is flat by Corollary 1.2.

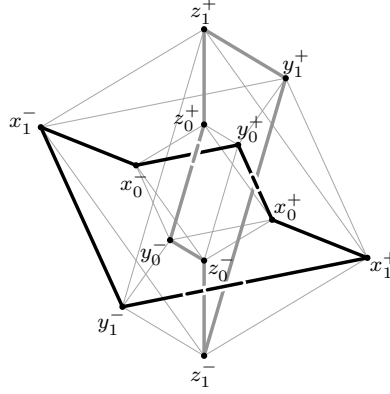


Figure 1: The prism over the octahedron as two concentric octahedra (a Schlegel diagram based at an octahedral facet): level 0 inside, level 1 outside. The cycle C is black, the cycle D grey; at a crossing the curve nearer to the viewer is drawn on top.

6.2 The cyclic polytope $C(6, 4)$

Let $P = C(6, 4)$ be the convex hull of six points $1, \dots, 6$ on the moment curve, in this order. Its graph is K_6 , a member of the Petersen family. By Gale's evenness condition [Zie95] the facets are the complements of the pairs $\{i, j\}$ with $j - i$ odd; there are nine of them. As P is simplicial, a triangle is a 2-face if and only if it lies in a facet, that is, if and only if its complement contains two numbers of different parity. So all triangles are 2-faces except 135 and 246. The only pairs of disjoint cycles of K_6 are the ten pairs of complementary triangles. If one triangle of a pair is the boundary of a 2-face, this 2-face is a disc disjoint from the other triangle by (P1), and the linking number vanishes (Lemma 5.1(a)). By the theorem of Conway and Gordon and of Sachs, the sum of the ten linking numbers is odd in every embedding of K_6 [CG83, Sac83]. Hence $\text{lk}(135, 246)$ is odd: the boundary embedding is linked, in accordance with Corollary 1.2. The computed value is ± 1 .

6.3 The prism over the octahedron

Let P be the product of the octahedron with vertices $\pm x, \pm y, \pm z$ and the segment $[0, 1]$; write $x_t^+, x_t^-, \dots$ for the vertices on level $t \in \{0, 1\}$. It has 12 vertices and 30 edges: on each level two vertices are adjacent unless they are antipodal, and every vertex is joined to its copy on the other level. Its facets are the two octahedra and eight prisms over triangles.

A K_6 -minor. The six sets

$$\{x_0^-, x_1^-, y_0^-\}, \quad \{y_0^+, z_0^+, z_0^-\}, \quad \{y_1^-, z_1^-\}, \quad \{z_1^+\}, \quad \{y_1^+\}, \quad \{x_0^+, x_1^+\}$$

partition the vertex set, each induces a connected subgraph, and any two of them are joined by an edge; the fifteen edges $x_0^- z_0^-, x_1^- y_1^-, x_1^- z_1^+, x_1^- y_1^+, y_0^- x_0^+; z_0^- z_1^-, z_0^+ z_1^+, y_0^+ y_1^+, z_0^+ x_0^+; y_1^- z_1^+, z_1^- y_1^+, y_1^- x_1^+; z_1^+ y_1^+, z_1^+ x_1^+; y_1^+ x_1^+$ are witnesses. So K_6 is a minor of $G(P)$.

A linked pair. The two disjoint 6-cycles

$$C = (x_0^+, y_0^+, x_0^-, x_1^-, y_1^-, x_1^+), \quad D = (z_0^+, y_0^-, z_0^-, z_1^-, y_1^+, z_1^+)$$

have linking number ± 1 in ∂P (Figure 1). To see this, let O be the octahedron. Both cycles lie in $\partial O \times [0, 1]$, the union of the eight prisms. Move the parts of C and D on level 0 to level ε and those on level 1 to level $1 - \varepsilon$, for a small $\varepsilon > 0$, and shorten the vertical edges accordingly; the two curves stay disjoint. For a point $q \in \partial O$ outside the planes $z = 0$ and $x = 0$ the set $U = (\partial O \setminus \{q\}) \times (0, 1)$ is

an open subset of ∂P homeomorphic to $\mathbb{R}^2 \times (0, 1)$, and it contains both curves. The linking number can be computed in U , where it is the number of crossings, counted with sign, at which C lies above D in the projection to $\partial O \setminus \{q\}$ [Rol03]; the vertical edges are tilted a little for this, which creates no crossing. The projection of C is the circle $\partial O \cap \{z = 0\}$ and that of D is the circle $\partial O \cap \{x = 0\}$. They cross at y^+ and at y^- . Over y^+ the cycle C is on the lower and D on the upper level; over y^- it is the other way round. So there is exactly one crossing at which C lies above D , and $\text{lk}(C, D) = \pm 1$. Thus P is linked and has a K_6 -minor, in accordance with Corollary 1.2.

6.4 The truncated 4-simplex and the hypersimplex

Let P be the truncated 4-simplex, the convex hull of the permutations of $(0, 0, 0, 1, 2)$ in the hyperplane $x_1 + \dots + x_5 = 3$ of $\mathbb{R}^5$. Its 20 vertices are the pairs (i, j) with $i \neq j$ in $\{0, \dots, 4\}$ (the positions of the entries 2 and 1), and (i, j) is adjacent to (j, i) and to the three vertices (i, k) with $k \neq i, j$. It is the polar of the stacked polytope obtained by stacking on all five facets of a simplex. In a comment of 14 December 2014 to [Epp14], Eppstein expects the following pair to be linked: the two edge-disjoint 5-cycles 01234 and 02413 of K_5 give the disjoint 10-cycles

$$\begin{aligned} C &= ((0, 1), (1, 0), (1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 0), (0, 4)), \\ D &= ((0, 2), (2, 0), (2, 4), (4, 2), (4, 1), (1, 4), (1, 3), (3, 1), (3, 0), (0, 3)). \end{aligned}$$

Indeed $\text{lk}(C, D) = \pm 1$: this is the result of the original programs, of the second verification run and of the third, each with two methods (Section 7). The graph has a K_6 -minor: the six sets

$$\begin{aligned} \{(0, j), (j, 0)\} \ (j = 1, 2, 3, 4), \quad & \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3)\}, \\ & \{(3, 1), (1, 3), (1, 4), (4, 1), (4, 2), (2, 4)\} \end{aligned}$$

partition the vertex set and induce connected subgraphs (the last two induce paths, in the order written), and any two of them are joined by an edge: $(0, j)(0, k)$; $(j, 0)(j, k)$ with $(j, k) = (1, 2), (2, 1), (3, 2), (4, 3)$; $(j, 0)(j, k)$ with $(j, k) = (1, 3), (2, 4), (3, 1), (4, 1)$; and $(1, 2)(1, 3)$. By the theorem of Sachs every embedding of this graph is linked. So polars of stacked polytopes can be linked, although stacked polytopes are not [Epp14]. A comment to [Epp14] signed by G. Kalai asks whether the hypersimplex $\Delta(2, 5)$ and its polar contain links. According to the computations of Section 7, $\Delta(2, 5)$ is linked and has a K_6 -minor, and its polar is not linked and has no minor in the Petersen family.

7 Computations and reproducibility

The computations of this section are a consistency test. No proof in this note depends on them.

What is tested. For a 4-polytope P with rational vertices the programs decide, in exact arithmetic, whether some pair of disjoint cycles of $G(P)$ has non-zero linking number in ∂P (*linked*: a pair with odd linking number is exhibited; *not linked*: a complete search, which uses that the linking number is bilinear on cycle spaces), and whether $G(P)$ has a minor in the Petersen family (a model is exhibited and verified, or all seven graphs are excluded). By Corollary 1.2 the two answers must agree; a polytope with a linked pair and without a minor would be one as asked for in the problem. That a minor forces a linked pair is the theorem of Sachs and tests the programs. The test concerns linking numbers; it does not establish flatness.

Programs. Four sets of programs were written independently of each other. Those with which the results were first obtained (folder `original/` of the package) compute linking numbers in a Schlegel diagram, as intersection numbers of one cycle with a cone over the other and by signed crossings in a

projection, and search for minors with a SAT solver. The first verification run (`verification_run_A/`) intersects two cones in $\mathbb{R}^4$ with apices in the interior of P and searches for minors exhaustively. The second (`verification_run_B/`) uses a cellular formula on the face lattice, which needs no coordinates, and signed crossings; it searches for minors exhaustively up to ten vertices and with a SAT solver beyond, and it compares combinatorial types by canonical forms. In the first and the third set every exhibited pair is confirmed by the second of the two methods. Negative answers of the SAT solver are not certified. The third verification run (`independent_run_2/`) wrote its programs from the statements of this note before it opened the others. In a Schlegel diagram it counts the signed crossings of a generic projection twice, with either curve on top, and compares with the Gauss integral, computed in floating point; it searches for minors exhaustively, over the partitions of the vertex set into connected sets; and it sorts polytopes into combinatorial types by isomorphism of their vertex-facet incidences.

Lists of polytopes. For nine vertices the list is that of Firsching [Fir20]: the ancillary file of the arXiv version contains rational coordinates for the 274,148 combinatorial types, and the 41,866 non-polytopal 3-spheres; its completeness is his theorem. The numbers of types with 5, ..., 8 vertices, also for each f -vector, are taken from the tables of [Fir20]; the classification is attributed there to Perles for seven vertices (see [Grü03]) and to Altshuler and Steinberg [AS85] for eight. The second run enumerated the types with at most seven vertices by Gale diagrams and found 1,294 pairwise non-isomorphic polytopes with eight vertices, as many for each f -vector as in the table; so all types were processed, given the published numbers.

vertices	types	not linked, no minor	linked, K_6 -minor
5	1	1	0
6	4	3	1
7	31	19	12
8	1,294	302	992
9	274,148	8,755	265,393
total	275,478	9,080	266,398

Table 1: The combinatorial types of 4-polytopes with at most nine vertices (second verification run; the rows for 5 to 8 vertices also third run).

Results. Table 1 gives the result of the second run: the two answers agree without exception. The original programs treated all 30,266 types with nine vertices and at most 26 edges, with the same verdict for every polytope (8,755 not linked; 21,511 linked, each with a K_6 -minor), a sample of 6,000 of the others, all 495 types with eight vertices and at most 22 edges (302 and 193), and the types with at most seven vertices. On 498,389 pairs of disjoint cycles in 680 polytopes the linking numbers of the original programs and of the second run agree up to one sign for each polytope. Both, and the third run, treated the 347 classes of 4-dimensional 0/1-polytopes under the symmetries of the cube (160 not linked, 187 linked). The number 347 results from the enumeration of the orbits in each of these three sets of programs. It is the value in entry A105231 of [OEI26], which refers to Aichholzer, and the sum of the numbers of classes with 5, ..., 16 vertices in Tables 1 and 4 of the arXiv version of [CG14], with which the third run agrees for each number of vertices; the arXiv version of [Zie00] has 349. The second run also treated 194 products, prisms, pyramids, bipyramids, cyclic and stacked polytopes and their polars with up to 36 vertices, all types with at most seven facets, 1,279 of the types with eight facets, all types with nine facets and at most ten vertices, and 2,400 random polytopes with 10 to 14 vertices. Counted with repetitions of types, it processed 306,927 polytopes; each of the 292,737 linked ones has a verified minor in the Petersen family, and for 980 of those without a linked pair it

did not decide the absence of a minor, which follows from the theorem of Sachs. Outside the scope of Theorem 1.1, it treated the 41,866 non-polytopal 3-spheres with nine vertices of [Fir20]: 41,861 are linked and have a K_6 -minor, and five are not linked and have no minor. The first run found the same pattern in three samples of 1,500 polytopes with at most ten vertices; among polytopes without a minor it found a vertex as in Lemma 6.1 in 198 of 207 and no knotted cycle in 564, as far as three values of the Alexander polynomial detect; and it confirmed the kernel in Theorem 5.2 numerically for seven polytopes. As a control, the original programs and those of the second run report a linked straight-line embedding of the graph of the prism over the tetrahedron, not a Schlegel diagram, as linked and without a minor.

The third run sorted about 3.1 million point sets (random subsets of small grids and of similar sets, random Gale diagrams with small integer vectors, vertex deletions) into combinatorial types. It found all types with at most eight vertices again, as many for each f -vector as in Table 6 of the arXiv version of [Fir20], and it took 300 types with nine vertices from the same sets. For these 1,630 polytopes it computed the linking numbers of all 1,086,276 pairs of disjoint cycles, of which 42,057 are non-zero, and decided the existence of a minor exhaustively. The rows of Table 1 for 5, $\dots$, 8 vertices are reproduced; of the 300 types with nine vertices 117 are not linked and 183 are linked, each with a K_6 -minor; there is no exception. A vertex as in Lemma 6.1 exists in 298 of the 302 types with eight vertices that are not linked, and in none of the linked ones.

The package. The reproducibility package contains the four sets of programs with their recorded outputs, the program `check_note.py`, which verifies the certificates of Section 6 and compares the numbers of this section, except those of the third run, with the recorded outputs, and the script `run_quick.sh`, which repeats, in one process, every check of the first three sets that finishes within a few minutes; a complete run reproduced the recorded outputs in all 41 comparisons, up to the recorded running times. The folder `independent_run_2/` contains the programs of the third run, the 1,630 point sets and the recorded outputs, with the commands that repeat them. The lists of polytopes with nine vertices are not included; `README.md` says how to obtain them.

Verification. The results were first obtained, with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs.

The first run compared the statement with the source, read the cited statements in the sources ([RST93a] completely; pp. 185–197 of [RST95]; [Izm10]; [LS99]), and derived every step of the proofs of Theorem 1.1 and of the corollaries again. It found no gap. It required that Lemma 2.2 be cited in the form (2.8) of [RST95] only, that the statements be formulated in the piecewise linear category, with the reason why Σ_F is a PL sphere in Step 3, that a remark on cell decompositions be proved or omitted (it is omitted here), and that references quoted from memory be verified. It contributed Lemma 5.1, Lemma 6.1 and the certificates of Sections 6.2 and 6.3. The second run repeated the computations with programs of its own, as described above, ran the original programs again, with identical outputs, and read them. It searched the literature and required the credits to Eppstein, to Georgakopoulos and Kim, to the survey [RST93b], and to Foisy, Saran, Balinski and Grünbaum, the statement of how Theorem 1.1 follows from the literature, the example of Section 6.4, and a source for the number of classes of 0/1-polytopes. All corrections required by the two runs were made.

When this note was written, several things were added or changed after these two runs. The paper [Lov01], which had not been accessible to them, was read, and the description of the null-space question in Section 1.4 follows it. Part (v) of the proof of Lemma 6.1 constructs the panel in a different way than the first run did. The K_6 -minor of Section 6.4, the program `check_note.py` and Figure 1 were new.

The third run examined the final text, and these points first. It read Problem 7 again in the publisher’s file and in the copy of the collector and compared the statement of the question with it.

It read again the passages of [RST93a, RST95, RST93b, GK23, Epp14, Epp20, LS99, Lov01, Izm10] and of the draft of [Lov19] that are used here, and compared what this note says about each of them. It derived all proofs again line by line, with the category in which each step is stated, in particular part (v) of Lemma 6.1 and Remark 4.1. It found no wrong statement and no gap in a proof. It required corrections of the text, which were made: the exact form of [RST95, (1.2)] after Theorem 2.1; what Section 1.4 says about the texts of Eppstein; the reference to [RST93b, (1.2)] for Lemma 6.1; the last step of the proof of Corollary 1.3; the description of the consistency test in the abstract; and the sources for the number 347, where it resolved a discrepancy between them as far as Section 7 says. With its programs it confirmed all certificates of Section 6: the hypothesis of Lemma 6.1 for the examples, the two K_6 models with their witness edges, the linking numbers ± 1 of Sections 6.2 to 6.4, each in two or three Schlegel diagrams, and Figure 1, in which the projections of C and D cross exactly four times and all crossings are shown as one viewer sees them; and it obtained the results reported in Section 7. It read `check_note.py`, ran it, and ran the short mode of `run_quick.sh` from the archive of the package: 25 comparisons, none different. It verified the bibliography and repeated the literature search. No proof depends on a computation.

Scope and priority. The note answers Problem 7 of [BKSZ09] in the negative, for the readings (L0)–(L3) of “linked” and under the conventions of Sections 1.2 and 2; in particular it uses the two results of Robertson, Seymour and Thomas in the piecewise linear category. It does not answer the general question of Lovász and Schrijver on null spaces, which it settles only for the matrices of [Izm10]. The problem is due to Izmestiev; the notions and theorems on flat and linkless embeddings are due to Sachs, Conway and Gordon, Böhme, Saran, and Robertson, Seymour and Thomas; the uniqueness theorem for 2-complexes is due to Georgakopoulos and Kim; links in the boundaries of 4-polytopes, with the cases of pyramids and stacked polytopes and the examples of Sections 6.2 and 6.4, were studied by Eppstein. Theorem 1.1 and Corollary 1.2 are short consequences of these results. We did not find them, or an answer to Problem 7, in the literature searched; no further novelty is claimed.

What was read: Problem 7 in [BKSZ09]; [RST93a] and [Lov01] completely; pp. 185–197 of [RST95] and its list of references (the remaining pages, with the proof of the main theorem, by nobody); pp. 125–130 and 133–134 of [RST93b]; pp. 1017–1020 of [LS99]; Sections 1 and 2 of [Izm10], without the appendix on the signature; the introduction and Section 2 of the arXiv version of [GK23]; [Epp14] with its comments and Section 6.1 of the arXiv version of [Epp20]; the passages of [SS17, LS17] on the open question; the introduction and the tables of the arXiv version of [Fir20]; the tables of the arXiv version of [CG14] and the passage on the number of classes in that of [Zie00]; the abstracts of [FJP⁺24, Foi22]; and Section 16.3 of a draft of the book [Lov19], dated February 2019, on the web page of its author. That section treats flat embeddings and Böhme’s lemma, and it remarks that skeletons of 3-polytopes are flat and that skeletons of 4-polytopes may or may not be linklessly embeddable; it does not consider the embedding in the boundary. Not read: the survey [Ram05], to which the publisher gave no automated access; the originals [Böh90, Sar89, Sac83, Sac84, CG83]; the printed version of [Lov19]; the journal versions of the papers that were read in their arXiv versions; [AS85]; the paper of Aichholzer named in [OEI26]. The papers [Ale24, Bal61, LS98] and the books [Zie95, Grü03, RS72, Hat02, Rol03] are cited for standard facts and were not compared. The proofs of the cited theorems were not checked, except that the first and the third run read the proof of Lemma 2.2 in [RST95, Section 2].

The searches were made on 10 October 2026, and repeated by the third run on the same day: queries to arXiv, zbMATH, Crossref, OpenAlex, Semantic Scholar, DataCite and Zenodo, the lists of works citing [Izm10] (18 in OpenAlex, 25 in Semantic Scholar), [LS99] (40), [Lov01] (31), [LS98] (76), [GK23] (4 and 6) and [Epp20] (4 and 3), and eight web searches. Apart from the works discussed in Section 1.4, none of the titles and abstracts seen addresses Problem 7 or the boundary embedding of

polytope graphs. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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