

A Proof of the Below–Krummeck–Richter–Gebert Conjecture on Cyclic Determinant Quotients

Alper Ferudun*

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Abstract

For points $A_1, \dots, A_n, A'_1, \dots, A'_n$ in K^2 , where K is a field, write $[XY]$ for the 2×2 determinant and let $\alpha(\pi)$ be the cyclic quotient $\prod_k [A_{\pi_k} A_{\pi_{k+1}}] / \prod_k [A'_{\pi_k} A'_{\pi_{k+1}}]$, with indices read cyclically. Below, Krummeck and Richter–Gebert conjectured in 2003 that $\sum \text{sgn}(\pi) \alpha(\pi) = 0$, where the sum runs over the permutations π of $\{1, \dots, n\}$ with $\pi_1 = 1$. Richter–Gebert posed the problem again at Oberwolfach in 2009. It had been proved for $n = 5$ and checked by computer algebra for $n = 6$, and it follows from a symmetry argument when $n \equiv 0, 3 \pmod{4}$. We prove the identity for every $n \geq 3$ and every field, assuming only that $[A'_i A'_j] \neq 0$ for $i \neq j$. Equivalently, the associated bracket polynomial vanishes identically. The proof writes the numerator as the trace of a product of traceless 2×2 matrices. The reciprocals of the denominators are Parke–Taylor factors. They satisfy a shuffle identity, known from the Kleiss–Kuijff relations of gauge theory, and hence are the coefficients of a Lie polynomial. Mapping this Lie polynomial into 2×2 matrices over an exterior algebra, where the commutators of the generators are central, gives zero. The same argument shows that $\sum_{\sigma} \text{sgn}(\sigma) M_{\sigma_2} \cdots M_{\sigma_n} / ([A'_1 A'_{\sigma_2}] \cdots [A'_{\sigma_n} A'_1]) = 0$ for all traceless 2×2 matrices $M_2, \dots, M_n$ when $n \geq 4$, and it recovers the vanishing of the same sum without the matrices. This is an unrefereed note.

1 Introduction

Let K be a field. For $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ in K^2 write $[XY] = x_1 y_2 - x_2 y_1$. For points $X_1, \dots, X_n, Y_1, \dots, Y_n \in K^2$ put

$$\alpha(X_1, \dots, X_n \mid Y_1, \dots, Y_n) = \frac{[X_1 X_2][X_2 X_3] \cdots [X_{n-1} X_n][X_n X_1]}{[Y_1 Y_2][Y_2 Y_3] \cdots [Y_{n-1} Y_n][Y_n Y_1]}.$$

Permutations are written in one-line notation, $\pi = (\pi_1, \dots, \pi_n)$ with $\pi_k = \pi(k)$, and $\text{sgn}(\pi)$ denotes the sign (written $\sigma(\pi)$ in [BKRG03, BKSZ09]). For points $A_1, \dots, A_n, A'_1, \dots, A'_n \in K^2$ let

$$S_n = \sum_{\pi \in S_n, \pi_1=1} \text{sgn}(\pi) \alpha(A_{\pi_1}, \dots, A_{\pi_n} \mid A'_{\pi_1}, \dots, A'_{\pi_n}). \quad (1)$$

Conjecture 1.1 (Below, Krummeck and Richter–Gebert [BKRG03, Conjecture 6.1]). *For all $n \geq 3$ and all points $A_1, \dots, A_n, A'_1, \dots, A'_n \in K^2$ one has $S_n = 0$ (whenever the terms are defined, see (2); for $n \leq 2$ see Remark 1.5).*

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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Conventions. As in [BKRG03, Section 6] and [BKSZ09, Problem 6], the sum runs over the permutations that fix 1. Following [BKRG03, Section 6], a permutation acts on a tuple by $\pi(A_1, \dots, A_n) = (A_{\pi(1)}, \dots, A_{\pi(n)})$. The other possible reading, $(A_{\pi^{-1}(1)}, \dots, A_{\pi^{-1}(n)})$, gives the same sum, because $\pi \mapsto \pi^{-1}$ is a bijection of the set of permutations fixing 1 and $\text{sgn}(\pi^{-1}) = \text{sgn}(\pi)$. For $n \geq 3$ every pair $\{i, j\}$ occurs as a pair of cyclically consecutive entries of some π with $\pi_1 = 1$. So all terms of (1) are defined if and only if

$$[A'_i A'_j] \neq 0 \quad \text{for all } i \neq j. \quad (2)$$

We assume (2) whenever S_n appears. The polynomial form in Corollary 1.3 needs no assumption.

Background. The conjecture comes from the study of phirotopes, the phase analogues of chirotopes for complex point configurations [BKRG03]; see also [AD12]. For $K = \mathbb{C}$ and $A'_i = \overline{A_i}$, the quotient α is a product of squared phases (Remark 1.6). For $n = 5$ the identity then becomes, up to relabelling and a factor 2, the five-point formula (8) of [BKRG03], which characterizes realizability for uniform non-chirotopal rank-2 phirotopes on five points [BKRG03, Lemma 3.4]. Below, Krummeck and Richter-Gebert proved the case $n = 5$ for arbitrary $A_i, A'_i \in \mathbb{C}^2$ [BKRG03, Theorem 5.2]. They noted that the cases $n \equiv 0, 3 \pmod{4}$ are trivial (Remark 1.5), that the case $n = 6$ had been checked with the computer algebra system Mathematica, and that the cases $n \in \{9, 10, 13, 14\}$ had passed numerical tests. In their words, “a general proof still seems to be out of reach” [BKRG03, Section 6]. Richter-Gebert posed the problem again at the 2009 Oberwolfach workshop on discrete differential geometry [BKSZ09, Problem 6], where the cases $n = 5, 6$ are described as proved. The problem is record OWR-3385-008 of the UnsolvedMath dataset [ula26].

Results. We prove the conjecture for every $n \geq 3$ (for $n \leq 2$ it is void or false, see Remark 1.5).

Theorem 1.2. *Let K be a field, let $n \geq 3$, and let $A_1, \dots, A_n, A'_1, \dots, A'_n \in K^2$ satisfy (2). Then $S_n = 0$.*

Following [BKRG03, Section 6], clear denominators. Let $D_n = \prod_{1 \leq i < j \leq n} [A'_i A'_j]$. For π with $\pi_1 = 1$ let $E(\pi)$ be the set of the n pairs $\{\pi_k, \pi_{k+1}\}$ (with $\pi_{n+1} = \pi_1$), and let $d(\pi)$ be the number of $k \in \{1, \dots, n\}$ with $\pi_k > \pi_{k+1}$. Put

$$P_n = \sum_{\pi \in S_n, \pi_1=1} \text{sgn}(\pi) (-1)^{d(\pi)} \prod_{k=1}^n [A_{\pi_k} A_{\pi_{k+1}}] \prod_{\substack{i < j \\ \{i, j\} \notin E(\pi)}} [A'_i A'_j]. \quad (3)$$

This is a polynomial with integer coefficients in the $4n$ coordinates of the points, and $P_n = D_n S_n$ whenever (2) holds. Indeed, a factor $[A'_{\pi_k} A'_{\pi_{k+1}}]$ of the denominator cancels against a factor of D_n , up to the sign -1 when $\pi_k > \pi_{k+1}$.

Corollary 1.3. *For every $n \geq 3$ the polynomial P_n is identically zero. Hence $P_n = 0$ for all points in R^2 , for every commutative ring R .*

Only two properties of the terms are used: the numerators are traces of products of traceless 2×2 matrices, and the denominators satisfy a shuffle identity. This gives a more general statement. For points $B_1, \dots, B_n \in K^2$ with $[B_i B_j] \neq 0$ for $i \neq j$ and a permutation p of $\{1, \dots, n\}$ define the *Parke–Taylor factor*

$$\text{PT}(p) = \frac{1}{[B_{p_1} B_{p_2}][B_{p_2} B_{p_3}] \cdots [B_{p_{n-1}} B_{p_n}][B_{p_n} B_{p_1}]}. \quad (4)$$

We write $\sigma = (\sigma_2, \dots, \sigma_n)$ for an ordering of $\{2, \dots, n\}$, 1σ for the permutation $(1, \sigma_2, \dots, \sigma_n)$, and $\text{sgn}(\sigma) = \text{sgn}(1\sigma)$.

Theorem 1.4. Let $n \geq 3$ and let $B_1, \dots, B_n \in K^2$ with $[B_i B_j] \neq 0$ for $i \neq j$. Let $M_2, \dots, M_n$ be 2×2 matrices over K with trace 0.

(a) If $n \geq 4$, then

$$\sum_{\sigma} \text{sgn}(\sigma) \text{PT}(1\sigma) M_{\sigma_2} M_{\sigma_3} \cdots M_{\sigma_n} = 0.$$

For $n = 3$ the sum equals $\text{PT}(123) \text{tr}(M_2 M_3) I$.

(b) Consequently,

$$\sum_{\sigma} \text{sgn}(\sigma) \text{PT}(1\sigma) \text{tr}(M_1 M_{\sigma_2} \cdots M_{\sigma_n}) = 0$$

for every 2×2 matrix M_1 if $n \geq 4$, and for every traceless M_1 if $n = 3$.

Theorem 1.2 is the case $B_i = A'_i$, $M_i = J A_i A_i^T$ of Theorem 1.4(b), where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (Lemma 4.2). The same method recovers $\sum_{\sigma} \text{sgn}(\sigma) \text{PT}(1\sigma) = 0$ for $n \geq 4$ and related sign identities (Proposition 5.2); these are not new in substance (Remark 5.3).

Remark 1.5. (a) For $n = 1$ the quotient α is $0/0$. For $n = 2$ the sum has the single term $[A_1 A_2][A_2 A_1]/([A'_1 A'_2][A'_2 A'_1]) = ([A_1 A_2]/[A'_1 A'_2])^2$, which is not zero in general. So the conjecture concerns $n \geq 3$, and the list of cases in [BKSZ09, Problem 6] starts at $n = 3$.

(b) Reversing a cycle, $(1, \pi_2, \dots, \pi_n) \mapsto (1, \pi_n, \dots, \pi_2)$, changes every bracket in the numerator and in the denominator of α by the sign -1 , so α is unchanged. The sign of the permutation changes by $(-1)^{(n-1)(n-2)/2}$, which is -1 exactly when $n \equiv 0, 3 \pmod{4}$. In these cases the terms cancel in pairs; this is the symmetry argument of [BKR03, Section 6]. It applies to every weight function $w(i, j) = w(j, i)$ in place of $[A_i A_j]/[A'_i A'_j]$. For $n \equiv 1, 2 \pmod{4}$ the signed cycle sum of a random symmetric weight function is not zero in characteristic 0 (see the Verification paragraph), so the conjecture depends on the special form of the weights. In characteristic 2, however, the pairing proves the identity for every $n \geq 3$, because paired terms are then equal and $2\alpha = 0$; so the proof below matters only in characteristic different from 2.

Remark 1.6. Let $K = \mathbb{C}$, let $A_1, \dots, A_n \in \mathbb{C}^2$ be pairwise linearly independent, and let $\varphi(i, j) = [A_i A_j]/|[A_i A_j]|$ be the phase. Since $[\overline{A_i} \overline{A_j}] = \overline{[A_i A_j]}$, we have $[A_i A_j]/[\overline{A_i} \overline{A_j}] = \varphi(i, j)^2$. So Theorem 1.2 with $A'_i = \overline{A_i}$ gives $\sum_{\pi_1=1} \text{sgn}(\pi) \prod_{k=1}^n \varphi(\pi_k, \pi_{k+1})^2 = 0$. This is the n -point analogue of the five-point formula for realizable uniform rank-2 phirotopes that was asked for in [BKR03, Section 6]. We say nothing about non-realizable phirotopes.

Method and relation to the literature. The proof has three ingredients. (i) With the traceless matrices $M_i = J A_i A_i^T$, the numerator of α is $\text{tr}(M_{\pi_1} \cdots M_{\pi_n})$ (Lemma 4.2). (ii) The Parke–Taylor factors satisfy a shuffle identity (Lemma 3.1). By the multilinear case of Ree’s theorem, which we prove in Lemma 2.2, the element $\sum_{\sigma} \text{PT}(1\sigma) \sigma$ of the free associative algebra is then a Lie polynomial, with an explicit expansion in left-normed brackets (Corollary 3.2). (iii) The homomorphism $c \mapsto M_c \otimes e_c$ into $M_2(K) \otimes \Lambda(e_2, \dots, e_n)$ produces the sign $\text{sgn}(\sigma)$ and the matrix products, and in its image all commutators of generators are central (Lemma 4.3). So it kills every left-normed bracket of length at least 3 (Proposition 4.1).

The factor (4) is the Parke–Taylor factor of gauge theory [PT86], with $[B_i B_j]$ in the role of the spinor bracket $\langle ij \rangle$. Linear relations among Parke–Taylor factors with signs are standard in the theory of scattering amplitudes: the Kleiss–Kuijf relations [KK89], the Bern–Carrasco–Johansson relations [BCJ08], and the orthogonality of Parke–Taylor vectors with respect to the Kawai–Lewellen–Tye form [CHY14]. The last two hold only under momentum conservation or on the support of the scattering equations; this note uses only the Kleiss–Kuijf relations. Lemma 3.1 is known: it is equivalent to the Kleiss–Kuijf relations for Parke–Taylor factors [KK89], see [DDM00], [KS14, Section 4] for proofs and

[FMM23, Section 2.1], [MS23, Section 5.1] for the connection with shuffles, Lie polynomials and Ree's theorem [Ree58]. Corollary 3.2 is the Del Duca–Dixon–Maltoni decomposition [DDM00] for these factors. We include short proofs, so that the note is self-contained and valid in every characteristic. We did not find Theorem 1.2 or Theorem 1.4 in this literature. In physical language, Proposition 4.1 says that a colour-dressed sum of Parke–Taylor factors vanishes for $n \geq 4$ when the colour factors have central commutators. For commuting colour factors this reduces to $\sum_{\sigma} \text{PT}(1\sigma) = 0$, a consequence of the $U(1)$ -decoupling identity. The new ingredient for Conjecture 1.1 is the choice of colour factors $M_i \otimes e_i$. Tensoring with the generators of an exterior algebra, so that products pick up the sign of the permutation, is also the device of Rosset's proof of the Amitsur–Levitzki theorem [Ros76].

2 Words, shuffles and Lie polynomials

Let C be a finite set of letters, and let $K\langle C \rangle$ be the free associative K -algebra on C . Its basis is the set of words, including the empty word 1. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $K\langle C \rangle$ for which the words are orthonormal. A word is *multilinear* if every letter of C occurs in it exactly once, and $W_C \subseteq K\langle C \rangle$ denotes the span of the $|C|!$ multilinear words.

For words $u = u_1 \cdots u_p$ and $v = v_1 \cdots v_q$ the shuffle product $u \sqcup v$ is the sum, over all p -element subsets $I \subseteq \{1, \dots, p+q\}$, of the word that has the letters of u in the positions I and the letters of v in the other positions, both in their original order. If no letter occurs twice in uv , these $\binom{p+q}{p}$ words are distinct. Let $\text{Sh}_C \subseteq W_C$ be the span of all $u \sqcup v$ where u, v are nonempty and uv is multilinear.

The *Lie polynomials* are the elements of the smallest subspace of $K\langle C \rangle$ that contains C and is closed under $[P, Q] = PQ - QP$. For a word $w = w_1 \cdots w_k$ with $k \geq 1$ the left-normed bracket is $\ell[w] = [\cdots [[w_1, w_2], w_3], \dots, w_k]$, with $\ell[w_1] = w_1$.

Lemma 2.1. (a) *If P is a Lie polynomial, then $\langle P, u \sqcup v \rangle = 0$ for all nonempty words u and v .*

(b) *Let $x \in C$ and let Q and Q' be words in which each letter of $C \setminus \{x\}$ occurs exactly once. Then $\ell[xQ] \in W_C$, and $\langle \ell[xQ], xQ' \rangle$ is 1 if $Q = Q'$ and 0 otherwise.*

Proof. (a) Let $\delta: K\langle C \rangle \rightarrow K\langle C \rangle \otimes K\langle C \rangle$ be the algebra homomorphism with $\delta(c) = c \otimes 1 + 1 \otimes c$ for $c \in C$; the product on $K\langle C \rangle \otimes K\langle C \rangle$ is $(P \otimes Q)(P' \otimes Q') = PP' \otimes QQ'$. For a word $w = w_1 \cdots w_k$, expanding $\delta(w) = \prod_i (w_i \otimes 1 + 1 \otimes w_i)$ gives $\delta(w) = \sum_I w_I \otimes w_{I^c}$, where I runs over the subsets of $\{1, \dots, k\}$ and w_I is the subword of w on the positions in I . Extend the form by $\langle P \otimes Q, u \otimes v \rangle = \langle P, u \rangle \langle Q, v \rangle$. Then $\langle \delta(w), u \otimes v \rangle$ is the number of sets I with $w_I = u$ and $w_{I^c} = v$, which is the coefficient of w in $u \sqcup v$. By linearity, $\langle \delta(P), u \otimes v \rangle = \langle P, u \sqcup v \rangle$ for all $P \in K\langle C \rangle$. The elements P with $\delta(P) = P \otimes 1 + 1 \otimes P$ include C , and they are closed under commutators: δ preserves commutators, and $[P \otimes 1 + 1 \otimes P, Q \otimes 1 + 1 \otimes Q] = [P, Q] \otimes 1 + 1 \otimes [P, Q]$ because $P \otimes 1$ commutes with $1 \otimes Q$. Hence every Lie polynomial P satisfies $\delta(P) = P \otimes 1 + 1 \otimes P$, and then $\langle P, u \sqcup v \rangle = \langle P, u \rangle \langle 1, v \rangle + \langle 1, u \rangle \langle P, v \rangle = 0$ for nonempty u and v .

(b) We have $\ell[w_1 \cdots w_j] = \ell[w_1 \cdots w_{j-1}] w_j - w_j \ell[w_1 \cdots w_{j-1}]$. By induction, $\ell[w]$ is a signed sum of the words obtained from w_1 by adding $w_2, \dots, w_k$ one at a time, each at the right end (sign $+$) or at the left end (sign $-$). These words are permutations of w , so $\ell[xQ] \in W_C$. The only one that begins with $w_1 = x$ is the word obtained by always adding on the right, namely xQ , with coefficient 1. $\square$

Lemma 2.2. *Let $\varphi: W_C \rightarrow K$ be a linear form that vanishes on Sh_C , and let $x \in C$. Then*

$$\sum_w \varphi(w) w = \sum_Q \varphi(xQ) \ell[xQ], \quad (5)$$

where w runs over the multilinear words and Q over the orderings of $C \setminus \{x\}$. In particular $\sum_w \varphi(w) w$ is a Lie polynomial.

Proof. First we show that W_C is spanned by Sh_C and the words that begin with x . Let $w = XxY$ be a multilinear word with X nonempty. The element $X \sqcup (xY)$ lies in Sh_C . Grouping its words according to the letters in front of x gives

$$X \sqcup (xY) = \sum_{X=X_1X_2} X_1 x (X_2 \sqcup Y),$$

where the sum is over all ways of writing X as a prefix X_1 followed by a suffix X_2 . The term with X_2 empty is w . All other words on the right have fewer letters in front of x than w . So the claim follows by induction on the number of letters in front of x .

Let Λ and Γ be the left and the right side of (5); both lie in W_C by Lemma 2.1(b). For $s \in W_C$ we have $\langle \Lambda, s \rangle = \varphi(s)$, so Λ is orthogonal to Sh_C . So is Γ , by Lemma 2.1(a). By Lemma 2.1(b), $\langle \Gamma, xQ' \rangle = \varphi(xQ') = \langle \Lambda, xQ' \rangle$ for every ordering Q' of $C \setminus \{x\}$. Hence $\Lambda - \Gamma \in W_C$ is orthogonal to Sh_C and to all words beginning with x , so to all of W_C , and therefore $\Lambda = \Gamma$. $\square$

Over fields of characteristic 0, the statement that an element with zero constant term that is orthogonal to all $u \sqcup v$ with u, v nonempty is a Lie polynomial is Ree's theorem [Ree58]; see also [Reu93]. (The constant term matters: the empty word 1 is orthogonal to all these shuffles.) In the multilinear case the argument above works over every field and gives the explicit expansion (5); compare the basis expansion [FMM23, eq. (2.15)].

3 Parke–Taylor factors

Fix $n \geq 3$ and points $B_1, \dots, B_n \in K^2$ with $[B_i B_j] \neq 0$ for $i \neq j$, and define PT by (4). Let $C = \{2, \dots, n\}$. We identify an ordering $\sigma = (\sigma_2, \dots, \sigma_n)$ of C with the multilinear word $\sigma_2 \sigma_3 \cdots \sigma_n$ in $K\langle C \rangle$.

Lemma 3.1 (shuffle identity). *Let u and v be nonempty words such that uv is a multilinear word in the letters C . Then $\sum_{\sigma \in u \sqcup v} \text{PT}(1\sigma) = 0$.*

Proof. If $g \in GL_2(K)$ and $\lambda_1, \dots, \lambda_n \in K^\times$, replacing each B_i by $\lambda_i g B_i$ multiplies $[B_i B_j]$ by $\lambda_i \lambda_j \det g$. Since every index occurs in exactly two brackets of the cyclic product in (4), every $\text{PT}(1\sigma)$ is multiplied by $(\det g)^{-n} \prod_i \lambda_i^{-2}$, a factor that does not depend on σ . Choose g with $gB_1 = (0, 1)$. For $i \geq 2$ write $gB_i = (y_i, z_i)$; then $[gB_1 gB_i] = -y_i \neq 0$. With $\lambda_1 = 1$ and $\lambda_i = 1/y_i$ we may therefore assume that $B_1 = (0, 1)$ and $B_i = (1, t_i)$ for $i \geq 2$. Then $[B_1 B_i] = -1$, $[B_i B_1] = 1$ and $[B_i B_j] = t_j - t_i$, and the t_i are pairwise distinct. For a word $w = w_1 \cdots w_k$ in C without repeated letters put $O(w) = \prod_{i=1}^{k-1} (t_{w_{i+1}} - t_{w_i})^{-1}$, so that $O(w) = 1$ if $k = 1$. Then $\text{PT}(1\sigma) = -O(\sigma)$, and we have to show $\sum_{\sigma \in u \sqcup v} O(\sigma) = 0$.

Let $F = K(z)$ be the field of rational functions in a variable z . For $\zeta \in F$ and a nonempty word w as above with last letter c and $\zeta \neq t_c$, put $R_\zeta(w) = O(w)/(\zeta - t_c)$, and put $R_\zeta(1) = 1$ for the empty word. We claim that

$$R_\zeta(u) R_\zeta(v) = \sum_{\sigma \in u \sqcup v} R_\zeta(\sigma) \tag{6}$$

for all words u, v such that uv has no repeated letter and all $\zeta \in F$ different from the t_c with c a letter of uv . We use induction on the length of uv . If u or v is empty, there is nothing to prove. Otherwise write $u = u'a$ and $v = v'b$ with letters $a \neq b$. If τ is a word not containing a , then $R_\zeta(\tau a) = R_{t_a}(\tau)/(\zeta - t_a)$; for nonempty τ this is the definition of O , and for empty τ both sides are

$1/(\zeta - t_a)$. The words of $u \sqcup v$ that end with a are the words τa with $\tau \in u' \sqcup v$, and those that end with b are the words τb with $\tau \in u \sqcup v'$. Since the t_c are pairwise distinct, the induction hypothesis applies to (u', v) at $\zeta = t_a$ and to (u, v') at $\zeta = t_b$, and gives

$$\sum_{\sigma \in u \sqcup v} R_\zeta(\sigma) = \frac{R_{t_a}(u')R_{t_a}(v)}{\zeta - t_a} + \frac{R_{t_b}(u)R_{t_b}(v')}{\zeta - t_b}.$$

Here $R_{t_a}(v) = R_{t_b}(v')/(t_a - t_b)$ and $R_{t_b}(u) = R_{t_a}(u')/(t_b - t_a)$. By the partial fraction identity

$$\frac{1}{(\zeta - t_a)(t_a - t_b)} + \frac{1}{(\zeta - t_b)(t_b - t_a)} = \frac{1}{(\zeta - t_a)(\zeta - t_b)}$$

the right side equals $R_{t_a}(u')R_{t_b}(v')/((\zeta - t_a)(\zeta - t_b)) = R_\zeta(u)R_\zeta(v)$. This proves (6).

Now take $\zeta = z$ and our nonempty words $u = u'a$ and $v = v'b$. Multiply (6) by z and use $z/(z - t) = 1 + t/(z - t)$. This gives

$$\sum_{\sigma \in u \sqcup v} O(\sigma) = \frac{z O(u) O(v)}{(z - t_a)(z - t_b)} - \sum_{\sigma \in u \sqcup v} \frac{O(\sigma) t_{c(\sigma)}}{z - t_{c(\sigma)}},$$

where $c(\sigma)$ is the last letter of σ . For a nonzero $f = g/h \in F$ let $\deg f = \deg g - \deg h$; then a nonzero sum of elements of degree at most -1 has degree at most -1 . Every nonzero term on the right has degree at most -1 , while the left side is a constant, and a nonzero constant has degree 0. Hence both sides are 0. $\square$

Corollary 3.2. *In $K\langle C \rangle$, $\sum_{\sigma} \text{PT}(1\sigma)\sigma = \sum_Q \text{PT}(12Q)\ell[2Q]$, where σ runs over the orderings of $\{2, \dots, n\}$ and Q over the orderings of $\{3, \dots, n\}$.*

Proof. Apply Lemma 2.2 with $x = 2$ to the linear form $\varphi(\sigma) = \text{PT}(1\sigma)$, which vanishes on Sh_C by Lemma 3.1. $\square$

4 Central commutators and traceless matrices

Proposition 4.1. *Let $\mathcal{A}$ be an associative K -algebra with unit, and let $y_2, \dots, y_n \in \mathcal{A}$ be such that every commutator $[y_b, y_c]$ commutes with every y_d ($2 \leq b, c, d \leq n$). If $n \geq 4$, then*

$$\sum_{\sigma} \text{PT}(1\sigma) y_{\sigma_2} y_{\sigma_3} \cdots y_{\sigma_n} = 0.$$

For $n = 3$ the sum equals $\text{PT}(123)[y_2, y_3]$.

Proof. Let $\Phi: K\langle C \rangle \rightarrow \mathcal{A}$ be the algebra homomorphism with $\Phi(c) = y_c$ for $c \in C = \{2, \dots, n\}$. The sum is $\Phi(\sum_{\sigma} \text{PT}(1\sigma)\sigma) = \sum_Q \text{PT}(12Q)\Phi(\ell[2Q])$ by Corollary 3.2. Since Φ preserves commutators, $\Phi(\ell[w])$ is the left-normed commutator of $y_{w_1}, \dots, y_{w_k}$. If $k \geq 3$, its innermost part $[[y_{w_1}, y_{w_2}], y_{w_3}]$ vanishes by assumption, so $\Phi(\ell[w]) = 0$. For $n \geq 4$ every word $2Q$ has length $n - 1 \geq 3$. For $n = 3$ the right side is $\text{PT}(123)\Phi(\ell[23]) = \text{PT}(123)[y_2, y_3]$. $\square$

Lemma 4.2. *Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and, for a column vector $X \in K^2$, let $M_X = JXX^\top$. Then $X^\top JY = [XY]$ and $\text{tr } M_X = 0$, and for all $X_1, \dots, X_n \in K^2$*

$$\text{tr}(M_{X_1} M_{X_2} \cdots M_{X_n}) = [X_1 X_2][X_2 X_3] \cdots [X_{n-1} X_n][X_n X_1].$$

Proof. $X^\top JY = x_1y_2 - x_2y_1 = [XY]$, and $\text{tr}(JXX^\top) = X^\top JX = [XX] = 0$. By associativity, $M_{X_1} \cdots M_{X_n} = JX_1(X_1^\top JX_2)(X_2^\top JX_3) \cdots (X_{n-1}^\top JX_n)X_n^\top$, and $\text{tr}(JX_1X_n^\top) = X_n^\top JX_1 = [X_nX_1]$. $\square$

Lemma 4.3. *Let R be a commutative ring and let $P, Q \in M_2(R)$ have trace 0. Then $PQ + QP = \text{tr}(PQ)I$.*

Proof. Write $P = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}$ and $Q = \begin{pmatrix} p' & q' \\ r' & -p' \end{pmatrix}$. A direct computation gives $PQ + QP = (2pp' + qr' + rq')I$, and $\text{tr}(PQ) = 2pp' + qr' + rq'$. $\square$

5 Proofs of the results

Proof of Theorem 1.4. (a) Let $\Lambda(V)$ be the exterior algebra of the vector space V with basis $e_2, \dots, e_n$, and let $\mathcal{A} = M_2(K) \otimes_K \Lambda(V)$ with the product $(P \otimes \omega)(P' \otimes \omega') = PP' \otimes \omega\omega'$. Put $y_c = M_c \otimes e_c$. Since $e_c e_b = -e_b e_c$, Lemma 4.3 gives

$$[y_b, y_c] = (M_b M_c + M_c M_b) \otimes e_b e_c = \text{tr}(M_b M_c) (I \otimes e_b e_c).$$

This element is central in $\mathcal{A}$, because I is central in $M_2(K)$ and $e_b e_c$, having even degree, is central in $\Lambda(V)$. Moreover $y_{\sigma_2} \cdots y_{\sigma_n} = M_{\sigma_2} \cdots M_{\sigma_n} \otimes e_{\sigma_2} \cdots e_{\sigma_n} = \text{sgn}(\sigma) M_{\sigma_2} \cdots M_{\sigma_n} \otimes e_2 \cdots e_n$. For $n \geq 4$, Proposition 4.1 gives

$$\left(\sum_{\sigma} \text{sgn}(\sigma) \text{PT}(1\sigma) M_{\sigma_2} \cdots M_{\sigma_n} \right) \otimes e_2 \cdots e_n = 0,$$

and since $e_2 \cdots e_n \neq 0$, the matrix vanishes. For $n = 3$ the two terms are $\text{PT}(123)M_2M_3$ and $-\text{PT}(132)M_3M_2 = \text{PT}(123)M_3M_2$, because $\text{PT}(132) = -\text{PT}(123)$; their sum is $\text{PT}(123) \text{tr}(M_2M_3)I$ by Lemma 4.3.

(b) Multiply the sum in (a) by M_1 on the left and take the trace. For $n = 3$ this gives $\text{PT}(123) \text{tr}(M_2M_3) \text{tr}(M_1) = 0$. $\square$

Proof of Theorem 1.2. Put $B_i = A'_i$ and $M_i = JA_i A_i^\top$ for $1 \leq i \leq n$. These matrices have trace 0 by Lemma 4.2. For every ordering σ of $\{2, \dots, n\}$, Lemma 4.2 gives

$$\alpha(A_1, A_{\sigma_2}, \dots, A_{\sigma_n} \mid A'_1, A'_{\sigma_2}, \dots, A'_{\sigma_n}) = \text{tr}(M_1 M_{\sigma_2} \cdots M_{\sigma_n}) \text{PT}(1\sigma).$$

The permutations π with $\pi_1 = 1$ are exactly the permutations 1σ , and $\text{sgn}(1\sigma) = \text{sgn}(\sigma)$. Hence $S_n = \sum_{\sigma} \text{sgn}(\sigma) \text{PT}(1\sigma) \text{tr}(M_1 M_{\sigma_2} \cdots M_{\sigma_n})$, which is 0 by Theorem 1.4(b). $\square$

Proof of Corollary 1.3. Let L be the field of rational functions over $\mathbb{Q}$ in $4n$ variables, and let $A_i, A'_i \in L^2$ be the points whose coordinates are these variables. Then $D_n \neq 0$ in L , so Theorem 1.2 over L gives $P_n = D_n S_n = 0$ in L . Thus P_n is the zero polynomial in $\mathbb{Z}[\text{coordinates}] \subset L$, and it vanishes under every specialization to a commutative ring. $\square$

Conversely, if R is a commutative ring and the brackets $[A'_i A'_j]$ ($i \neq j$) are units of R , then S_n is defined in R and $S_n = D_n^{-1} P_n = 0$.

Remark 5.1. For $n \geq 4$, Theorem 1.4(a) with $M_i = JA_i A_i^\top$ gives more than Theorem 1.2. Write the matrix product as $JA_{\sigma_2}(\prod_{k=2}^{n-1} [A_{\sigma_k} A_{\sigma_{k+1}}])A_{\sigma_n}^\top$, multiply by $X^\top$ on the left and by JY on the right. This gives, for all $X, Y \in K^2$,

$$\sum_{\sigma} \text{sgn}(\sigma) \text{PT}(1\sigma) [XA_{\sigma_2}][A_{\sigma_2} A_{\sigma_3}] \cdots [A_{\sigma_{n-1}} A_{\sigma_n}][A_{\sigma_n} Y] = 0 \quad (n \geq 4).$$

Theorem 1.2 is the case $X = Y = A_1$. For $n = 3$ the left side equals $-\text{PT}(123)[A_2 A_3]^2[XY]$, by the three-term Grassmann–Plücker relation; it vanishes for $X = Y$ but not in general.

For a subset $T \subseteq \{2, \dots, n\}$ and an ordering σ of $\{2, \dots, n\}$, let $\sigma|_T$ be the subsequence of the letters in T , and let $\text{sgn}(\sigma|_T)$ be its sign as a permutation of T . Since the letter 1 is the smallest and comes first, $\text{sgn}((1\sigma)|_U) = \text{sgn}(\sigma|_{U \setminus \{1\}})$ for every $U \subseteq \{1, \dots, n\}$.

Proposition 5.2. *Let $n \geq 4$. Then $\sum_{\sigma} \text{sgn}(\sigma|_T) \text{PT}(1\sigma) = 0$ for every $T \subseteq \{2, \dots, n\}$. For $n = 3$ the sum is 0 for $T \neq \{2, 3\}$ and equals $2\text{PT}(123)$ for $T = \{2, 3\}$.*

Proof. Let $\mathcal{A}$ be the exterior algebra with generators e_c ($c \in T$), and put $y_c = e_c$ for $c \in T$ and $y_c = 1$ for $c \notin T$. The commutators are $[e_b, e_c] = 2e_b e_c$, $[e_b, 1] = 0$ and $[1, 1] = 0$, and they are central. The product $y_{\sigma_2} \cdots y_{\sigma_n}$ is $\text{sgn}(\sigma|_T) e_T$, where $e_T \neq 0$ is the product of the e_c ($c \in T$) in increasing order. Now apply Proposition 4.1. $\square$

Remark 5.3. Proposition 5.2 is not new in substance. Let $C = \{2, \dots, n\}$ and $T \neq C$. If $|T| \leq 1$, then all the signs $\text{sgn}(\sigma|_T)$ are 1, and $\sum_{\sigma} \sigma = x \sqcup \sum_{\rho} \rho$ for any $x \in C$, where ρ runs over the orderings of $C \setminus \{x\}$. If $|T| \geq 2$, then

$$\sum_{\sigma} \text{sgn}(\sigma|_T) \sigma = \left(\sum_{\tau} \text{sgn}(\tau) \tau \right) \sqcup \left(\sum_{\rho} \rho \right),$$

where τ runs over the orderings of T and ρ over the orderings of $C \setminus T$, because every ordering σ of C arises exactly once, from the pair $(\sigma|_T, \sigma|_{C \setminus T})$. In both cases the right side lies in Sh_C , so for $T \neq C$ the identity follows directly from Lemma 3.1, for every $n \geq 3$. The case $T = C$ is $\sum_{\sigma} \text{sgn}(\sigma) \text{PT}(1\sigma) = 0$ for $n \geq 4$. Its proof applies to every linear form on W_C that vanishes on Sh_C (Section 6), so it shows that $\sum_{\sigma} \text{sgn}(\sigma) \sigma$ lies in Sh_C for $|C| \geq 3$, that is, by Lemmas 2.1 and 2.2, that it is orthogonal to all Lie polynomials in W_C . Over fields of characteristic 0 this is equivalent to the classical fact that the sign representation of S_{n-1} does not occur in the multilinear part of the free Lie algebra on $n-1 \geq 3$ generators [Kly74]. Kol and Shir record the cyclic analogue, checked by inspection for $n \leq 9$ [KS14, Section 3.2]. The proof above works in every characteristic.

6 Concluding remarks

The proof uses the brackets $[A'_i A'_j]$ only through the shuffle identity of Lemma 3.1. So Theorem 1.4, Remark 5.1 and Proposition 5.2 remain true when $\text{PT}(1\sigma)$ is replaced by $\varphi(\sigma)$ for any linear form φ on W_C that vanishes on Sh_C ; by Lemma 2.1(a) and Lemma 2.2 these are exactly the coefficient functions $\sigma \mapsto \langle \Gamma, \sigma \rangle$ of the Lie polynomials $\Gamma \in W_C$. The numerators enter only through Lemma 4.3. Neither structure alone suffices: the controls in the Verification paragraph replace one of them by random data, and the signed sums are then nonzero for $n \equiv 1, 2 \pmod{4}$. The random denominators in these controls are not constant, and they must not be: for constant denominators the signed sum of the numerators is $\text{tr}(M_1 s_{n-1}(M_2, \dots, M_n))$ with $M_i = J A_i A_i^{\top}$, where s_k is the standard polynomial of degree k , and this vanishes for $n \geq 4$. For $n \geq 5$ this follows from the Amitsur–Levitzki theorem [AL50], which gives $s_k = 0$ on 2×2 matrices for $k \geq 4$; for $n = 4$ it holds because s_3 takes scalar values on traceless 2×2 matrices and $\text{tr } M_1 = 0$.

Verification. The proofs above are complete and self-contained; no step depends on a computation. The claims were also checked by computer as follows.

1. A program by the author, `check_bkrg.py` (standard library only, a few seconds), does the following.

- It evaluates S_n exactly, in rational arithmetic, at random integer points for $3 \leq n \leq 11$: by a recursion over subsets for all these n , and also term by term for $n \leq 8$. It evaluates S_n modulo the primes $2^{61} - 1$ and $2^{31} - 1$ for $3 \leq n \leq 15$, three random instances each, and over the fields $\mathbb{F}_p$, $p \in \{3, 5, 7, 11, 13\}$, for $3 \leq n \leq \min(p + 1, 12)$. All values are 0.
 - Controls are nonzero: the case $n = 2$; the unsigned sum ($3 \leq n \leq 14$); and, for $n \equiv 1, 2 \pmod{4}$ and $n \leq 14$, random symmetric weights, the brackets $[A_i A_j]$ over a random antisymmetric denominator, and a random antisymmetric numerator over the brackets $[A'_i A'_j]$. For $n \equiv 0, 3 \pmod{4}$ the last three vanish, as Remark 1.5(b) predicts.
 - It checks Theorem 1.4 exactly for $3 \leq n \leq 9$, with nonzero controls for general matrices; Remark 5.1 for $n \leq 8$; Proposition 5.2 for $n \leq 8$ and all T ; Lemma 3.1 for all pairs (u, v) with $n \leq 6$ and for 400 random pairs with $n = 7$; Corollary 3.2 for $n \leq 7$; Lemma 4.2 for $n \leq 7$, on at most 50 orderings for each n ; and $P_n = 0$ at integer points with $A'_1 = A'_2$ for $4 \leq n \leq 7$.
2. The programs used in the search, with their own code, evaluated S_n modulo large primes for $3 \leq n \leq 14$ and $n = 17, 18$, and checked Lemma 3.1 and the vanishing of the signed numerators on random left-normed brackets for $4 \leq n \leq 7$.
 3. Two independent AI-assisted verification runs re-derived each step of the proof and checked the claims with separate code.
 - The first evaluated S_n modulo $2^{61} - 1$ for $3 \leq n \leq 21$ (a C program, with nonzero controls), checked that $\sum_{\sigma} \text{PT}(1\sigma) \sigma$ is a Lie polynomial by the Dynkin–Specht–Wever criterion for $n \leq 8$, checked Theorem 1.4(b) for random traceless matrices for $n \leq 8$, and checked Proposition 5.2 for $n \leq 7$.
 - The second evaluated S_n exactly at random integer points for $2 \leq n \leq 12$, modulo $2^{61} - 1$ for $2 \leq n \leq 26$, modulo $2^{31} - 1$ for $n \leq 18$ and modulo $10^9 + 7$ for $n \leq 16$, and over thirteen finite fields $\mathbb{F}_q$ with $q \leq 32$, among them fields of characteristic 2 and 3. It expanded P_n over $\mathbb{Z}$ for $3 \leq n \leq 7$ and found the zero polynomial. It checked Lemma 3.1 for all pairs (u, v) with $n \leq 7$, Corollary 3.2 for $n \leq 7$, Theorem 1.4 for $n \leq 9$, Remark 5.1 and Proposition 5.2 for $n \leq 8$, Remark 1.6 for $n \leq 9$, and Lemma 2.2 over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$ and $\mathbb{F}_7$ for $|C| \leq 6$. Its controls behaved as predicted (over very small fields some random controls vanish by chance). Its reruns of the programs of items 1 and 2 and of the first run reproduced their recorded outputs.

Scope and priority. Theorem 1.2 proves [BKR03, Conjecture 6.1], which is Problem 6 of [BKSZ09] and record OWR-3385-008 of [ula26], for every $n \geq 3$ and every field; for $n \leq 2$ the statement is void or false (Remark 1.5). The bracket polynomial form holds over every commutative ring (Corollary 1.3). We read the problem in [BKSZ09] and the conjecture in the version of [BKR03] linked from the third author’s publication list. We searched arXiv, zbMATH, Crossref and Semantic Scholar for the problem and for works citing [BKR03], and made one web search. Of the citing works listed by zbMATH (only [AD12]) and by Semantic Scholar (eight), none that we could see mentions a proof. In the literature on scattering amplitudes we searched arXiv abstracts and read the relevant sections of [KS14, FMM23, MS23]. As explained in the introduction, the shuffle identity, Ree’s theorem and the Del Duca–Dixon–Maltoni decomposition are standard there, but we did not find the identities of Theorems 1.2 and 1.4. Proposition 5.2 is not new in substance (Remark 5.3). We did not have access to [Ree58], [Reu93], [KK89], [Kly74], [AL50] or [Ros76]. Our references to the first four follow [FMM23, MS23, KS14]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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