

Five Possible Centers for a Cube Move of a 10-Realization

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Abstract

Melotti, Ramassamy and Thévenin proposed that a cube move of an α -realization, for $\alpha > 1$, has at most three possible central points. We give a counterexample for $\alpha = 10$. Two construction curves with one common focus have at least five distinct intersection points. All foci and the outer vertices of the associated cube configuration are rational. The proof reduces the intersection equations to one continuous function and certifies five sign changes using rational enclosures of fifth roots. A reflection construction supplies an explicit initial center and checks the noncollinearity conditions of the cube move. The sign changes also persist for exponents in a neighborhood of 10. We make no assertion that five is the maximum, and the result concerns realizations in which edges may cross.

1 The question and the counterexample

For $\alpha > 0$, a quadrilateral $ABCD$ is an α -quad when

$$|A - B|^\alpha + |C - D|^\alpha = |A - D|^\alpha + |B - C|^\alpha. \quad (1)$$

An α -realization requires this identity on each quadrilateral face; edges may cross. This is Definition 2.4 of [1], distinguished there from an α -embedding. The term α -immersion is used for the same setting in the Oberwolfach report [2, p. 1785].

A cube move fixes six distinct boundary vertices $A_1, \dots, A_6$. An input center A_0 makes

$$A_1 A_2 A_3 A_0, \quad A_3 A_4 A_5 A_0, \quad A_5 A_6 A_1 A_0$$

into α -quads; an output center A_7 must do the same for

$$A_2 A_3 A_4 A_7, \quad A_4 A_5 A_6 A_7, \quad A_6 A_1 A_2 A_7.$$

The two triples A_1, A_3, A_5 and A_2, A_4, A_6 are required to be noncollinear [1, Definition 2.5]. Existence of an output center is known for $\alpha > 1$. The proposed upper bound of three appears after Theorem 4 in [2, p. 1785] and as the construction-curve intersection bound in [1, Remark 4.8]. The latter curves are level sets of $|X - U|^\alpha - |X - V|^\alpha$.

Theorem 1. *There is a cube move of a 10-realization with rational boundary vertices, noncollinear triples of each color, an explicit rational input center, and at least five distinct output centers. In particular, three is not a universal upper bound for $\alpha > 1$.*

Set

$$U = (0, 0), \quad V = (1, 0), \quad W = (3, 1/20), \quad P = (1/3, 5/6), \quad (2)$$

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and define the positive rational constants

$$b = |P - V|^{10} - |P - U|^{10} = \left(\frac{41}{36}\right)^5 - \left(\frac{29}{36}\right)^5, \quad (3)$$

$$c = |P - W|^{10} - |P - U|^{10} = \left(\frac{27809}{3600}\right)^5 - \left(\frac{29}{36}\right)^5. \quad (4)$$

We first prove that the two equations

$$|X - V|^{10} - |X - U|^{10} = b, \quad |X - W|^{10} - |X - U|^{10} = c \quad (5)$$

have at least five distinct solutions $X \in \mathbb{R}^2$. Both curves have the common focus U , and their other foci are not collinear with U .

2 A scalar equation and an exact certificate

For $s \geq 0$, use the positive real fifth roots to define

$$B(s) = (s^5 + b)^{1/5}, \quad C(s) = (s^5 + c)^{1/5}, \quad (6)$$

$$x(s) = \frac{1 + s - B(s)}{2}, \quad y(s) = 10 \left(\frac{3601}{400} + s - C(s) - 6x(s) \right), \quad (7)$$

$$H(s) = x(s)^2 + y(s)^2 - s. \quad (8)$$

All these functions are continuous on $[0, \infty)$.

Lemma 2. *The map $s \mapsto (x(s), y(s))$ is a bijection from the nonnegative zeros of H to the solutions of (5).*

Proof. Suppose $X = (x, y)$ solves (5) and let $s = |X - U|^2 = x^2 + y^2$. Since squared distances are nonnegative,

$$|X - V|^2 = B(s), \quad |X - W|^2 = C(s).$$

Subtracting s from these two squared distances gives

$$1 - 2x = B(s) - s, \quad \frac{3601}{400} - 6x - \frac{y}{10} = C(s) - s.$$

These equations uniquely give (7), and then $H(s) = 0$. Conversely, if $H(s) = 0$, the point defined in (7) has squared distance s from U , $B(s)$ from V , and $C(s)$ from W . Taking fifth powers gives (5). Finally, different zeros have different squared distances from U , so their corresponding points are distinct. $\square$

Lemma 3. *The signs of H at*

$$3/4, \quad 9/10, \quad 6/5, \quad 3/2, \quad 2, \quad 4000$$

are, respectively, $+, -, +, -, +, -$.

Proof. The following is a rational certificate. Put $Q = 10^{12}$. In each row of Table 1, direct integer arithmetic checks

$$k_b^5 \leq Q^5(s^5 + b) < (k_b + 1)^5, \quad k_c^5 \leq Q^5(s^5 + c) < (k_c + 1)^5. \quad (9)$$

Denominators can be cleared in these inequalities, since all quantities are rational. Thus $B(s) \in [k_b/Q, (k_b + 1)/Q]$ and $C(s) \in [k_c/Q, (k_c + 1)/Q]$. Substitution in (7)–(8), with outward interval

s	k_b	k_c	Interval containing $H(s)$
3/4	1126507380543	7724716497882	[1.821692, 1.821693]
9/10	1167306009984	7724736335988	[-0.724507, -0.724506]
6/5	1323778659904	7724842931926	[1.272250, 1.272251]
3/2	1557684102781	7725129660815	[-1.032219, -1.032218]
2	2019333014540	7726499765940	[9.395978, 9.395979]
4000	4000000000000000	40000000000000021	[-396.749376, -396.749375]

Table 1: An exact certificate for five sign changes.

arithmetic, gives the last column. Its finite decimal endpoints denote exact rationals. None of the intervals contains zero.

For specificity, the interval rules used here are $[a, b] + [c, d] = [a + c, b + d]$ and $[a, b] - [c, d] = [a - d, b - c]$, together with scalar multiplication and the range of the square function on an interval. In particular an interval containing zero has square interval $[0, \max(a^2, b^2)]$; no monotonicity of squaring across zero is assumed. These rules, the displayed integers, and (9) suffice to check the certificate without numerical root finding. $\square$

By continuity, there is a zero in each of the five disjoint open intervals between consecutive sample values. Lemma 2 therefore proves the required five intersections. One is the known point P , whose parameter is $s = 29/36 \in (3/4, 9/10)$. No statement about the absence of additional zeros is needed.

3 Realizing the intersections as a cube move

Let R_{AB} denote reflection in the line through distinct points A, B . Choose the boundary vertices

$$\begin{aligned}
A_1 &= R_{WU}(P) = \left(\frac{3899}{10803}, -\frac{17755}{21606} \right), & A_2 &= U, \\
A_3 &= R_{UV}(P) = \left(\frac{1}{3}, -\frac{5}{6} \right), & A_4 &= V, \\
A_5 &= R_{VW}(P) = \left(\frac{1805}{4803}, -\frac{8315}{9606} \right), & A_6 &= W.
\end{aligned} \tag{10}$$

They are pairwise distinct. With $\det(B - A, C - A)$ as the orientation determinant, direct calculation gives

$$\det(A_3 - A_1, A_5 - A_1) = \frac{71690}{51886809}, \quad \det(V - U, W - U) = \frac{1}{20}. \tag{11}$$

Both same-color triples are therefore noncollinear.

The circumcenter of A_1, A_3, A_5 is

$$A_0 = \left(\frac{2550}{7169}, -\frac{30396}{35845} \right). \tag{12}$$

It differs from every boundary vertex. Reflection preserves distances to points on the reflecting line, so

$$|U - A_1| = |U - P| = |U - A_3|, \quad |V - A_3| = |V - P| = |V - A_5|, \quad |W - A_5| = |W - P| = |W - A_1|.$$

Also $|A_0 - A_1| = |A_0 - A_3| = |A_0 - A_5|$. The two pairs of equal lengths in each input face imply (1) for every positive α , in particular for 10. Thus A_0 is an explicit valid input center.

For an output center X , the face UA_3VX is a 10-quad exactly when

$$|X - V|^{10} - |X - U|^{10} = |A_3 - V|^{10} - |A_3 - U|^{10} = b.$$

Likewise, WA_1UX is a 10-quad exactly when the second equation of (5) holds. Subtracting the two equations gives

$$|X - W|^{10} - |X - V|^{10} = c - b = |A_5 - W|^{10} - |A_5 - V|^{10},$$

which is precisely the condition for VA_5WX . Consequently all five intersections give output centers of this same cube move.

We finally check that none of these intersections is a boundary vertex. The points U, V, W fail (5) by direct substitution. Each of A_1, A_3, A_5 is a reflection of P preserving its distances to two of U, V, W . If it also satisfied both differences in (5), its distance to the third focus would equal that of P . Distances to three noncollinear points determine a point uniquely, by subtracting their squares. This would make the reflection equal to P , contrary to P lying on none of the three reflecting lines. All output faces thus have four distinct vertices. This proves Theorem 1.

4 Persistence and scope

Corollary 4. *The same rational boundary hexagon has at least five possible output centers for every α in some open interval containing 10.*

Proof. Keep U, V, W, P and the reflected boundary vertices fixed. Replace b, c by $b_\alpha = |P - V|^\alpha - |P - U|^\alpha$ and $c_\alpha = |P - W|^\alpha - |P - U|^\alpha$, both positive near 10. Replace the radicals in (6) by $(s^{\alpha/2} + b_\alpha)^{2/\alpha}$ and $(s^{\alpha/2} + c_\alpha)^{2/\alpha}$, leaving (7) and (8) unchanged. The scalar equivalence holds by the same squared-distance subtraction. At the six fixed sample values the functions depend continuously on α . The six strict signs therefore persist in a common neighborhood of 10, giving five distinct zeros. The reflection and circumcenter arguments still give the input realization and exclude boundary vertices; the focus vertices remain excluded by continuity. Shrink the neighborhood to lie in $(1, \infty)$. $\square$

The result refutes the proposed universal three-center bound, which is the question indexed as OWR-2654831-006 in the frozen UnsolvedMath v1.6.0 collection. It does not settle the largest possible number of centers, or uniqueness in settings requiring embedded quads. The known existence theorem for $\alpha > 1$ and the results for s -embeddings are unaffected.

The accompanying standard-library Python file `certify_five_intersections.py` checks the displayed certificate, reflection formulas, input quads, and exclusions using integer and rational arithmetic. It passes with and without Python optimization; its 67 checks do not constitute proof-assistant verification. The argument above uses the intermediate value theorem, not the numerical search that suggested the example. This unrefereed manuscript was developed with AI assistance for exploration, drafting, and arithmetic checking, and was self-audited. No independent review or absolute priority certification is claimed.

References

- [1] P. Melotti, S. Ramassamy and P. Thévenin, *Cube moves for s -embeddings and α -realizations*, Ann. Inst. H. Poincaré D Comb. Phys. Interact. **10** (2023), 781–817. [doi:10.4171/AIHPD/163](https://doi.org/10.4171/AIHPD/163).
- [2] P. Melotti (joint work with S. Ramassamy and P. Thévenin), *Cube flips in s -embeddings and α -immersions*, Oberwolfach Rep. **17** (2020), report 35, 1783–1785 (published 2021). [doi:10.4171/OWR/2020/35](https://doi.org/10.4171/OWR/2020/35).