

Counterexamples to Conforti's Subtree Conjecture for Mixed-Integer Bipartite Covers

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Abstract

For a bipartite graph $G = (U \cup V, E)$, a set $I \subseteq U \cup V$ and rationals b_{ij} , let $S(G, I) = \{x \in \mathbb{R}^{U \cup V} : x_i + x_j \geq b_{ij} \ (ij \in E), \ x_i \in \mathbb{Z} \ (i \in I)\}$, and let k be the least positive integer with kb integral. In the 2008 Oberwolfach report on combinatorial optimization, Conforti conjectured that $\text{conv } S(G, I)$ is the intersection of the hulls $\text{conv } S(T, I \cap V(T))$ over the subtrees T of G whose integral vertices are exactly their leaves. This would place the membership problem for $\text{conv } S(G, I)$ in coNP. He noted that the case $k = 2$ follows from work of Conforti, Gerards and Zambelli, and that the conjecture was open for every $k \geq 3$. We show that it fails for every $k \geq 3$. For each such k we give two unicyclic counterexamples. One has six vertices. The other has seven, and in it every integral vertex is a pendant vertex with a continuous neighbour, so the standard normalisation of such sets (splitting integral vertices) does not remove it. In both, an explicit point lies in $\text{conv } S(T, I \cap V(T))$ for every subtree T of G , but violates a facet-defining inequality of $\text{conv } S(G, I)$ by $(k - 2)/(2k - 3)$. The proofs are short and by hand. We also describe exact validity certificates based on an extended formulation, and use them to certify a further normalised counterexample for $k = 3$. The complexity of the membership problem remains open. This is an unrefereed note.

1 Introduction

Let $G = (U \cup V, E)$ be a bipartite graph, let $I \subseteq U \cup V$ be a set of *integral* vertices, and let $b \in \mathbb{Q}^E$. The other vertices are called *continuous*. Conforti, Wolsey and Zambelli [CWZ10] study the set of mixed-integer covers

$$S(G, I) = \{x \in \mathbb{R}^{U \cup V} : x_i + x_j \geq b_{ij} \ (ij \in E), \ x_i \in \mathbb{Z} \ (i \in I)\}.$$

It is equivalent to the network-dual sets of Conforti, Di Summa, Eisenbrand and Wolsey [CDEW09], which contain several mixing-type sets from production planning. Let k be the least positive integer such that $kb_{ij} \in \mathbb{Z}$ for all $ij \in E$; we call it the *denominator parameter*. The papers [CDEW09, CWZ10] give extended formulations of $\text{conv } S(G, I)$ whose size grows with k . For $k = 2$, Conforti, Gerards and Zambelli [CGZ07] describe the hull in the original space; their setting also includes the bounds $x \geq 0$.

In his abstract in the 2008 Oberwolfach report [Con08], Conforti asks for the complexity of optimizing a linear function over $S(G, I)$, equivalently of deciding membership in $\text{conv } S(G, I)$ (his Problem 1). As Conforti notes, this problem is in NP. For a subgraph H of G put $I_H = V(H) \cap I$. Let $\mathcal{T} = \mathcal{T}(G, I)$ be the family of subtrees T of G such that I_T is exactly the set of leaves of T . Here $S(T, I_T) \subseteq \mathbb{R}^{V(T)}$ is defined by the edges of T , and we identify $\text{conv } S(T, I_T)$ with the cylinder $\{x \in \mathbb{R}^{U \cup V} : x|_{V(T)} \in \text{conv } S(T, I_T)\}$.

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Conjecture 1.1 (Conforti [Con08, Conjecture 2]). $\text{conv } S(G, I) = \bigcap_{T \in \mathcal{T}} \text{conv } S(T, I_T)$.

If this were true, a single tree T with $\bar{x} \notin \text{conv } S(T, I_T)$ would certify $\bar{x} \notin \text{conv } S(G, I)$. Optimization over $S(T, I_T)$ is polynomial for trees [CDEW09], so Problem 1 would be in coNP [Con08]. Conforti states that the case $k = 2$ follows from the description in [CGZ07], and that the conjecture is open for every $k \geq 3$. In [Con08] the conjecture is presented as suggested by a characterisation of the facet-defining inequalities of $\text{conv } S(G, I)$ when G is a tree. Descriptions of the same shape, as intersections of the hulls of relaxations, are known for special sets: for the continuous mixing set with flows [CWZ10, Theorem 23], and for two classes of network-dual sets whose hulls are intersections of hulls of mixing sets [DW11]. The conjecture appears twice in the UnsolvedMath dataset [ula26], as records OWR-2489-004 and OWR-2489-009.

We show that the conjecture fails for every $k \geq 3$, also in normalised form. Let $P(G) = \{x : x_i + x_j \geq b_{ij} \ (ij \in E)\}$ be the linear relaxation.

Theorem 1.2. *Let $k \geq 3$. The instances (G_k, I) of Section 3 and $(\widehat{G}_k, \widehat{I})$ of Section 4 have denominator parameter exactly k . There are explicit points $x^{(k)}$ and $\widehat{x}^{(k)}$ with the following properties.*

- (a) $x^{(k)} \in P(G_k)$, and $x^{(k)} \in \text{conv } S(T, I_T)$ for every subtree T of G_k . But $x^{(k)} \notin \text{conv } S(G_k, I)$: the inequality

$$F_k(x) = k(x_p + x_q + x_m + x_z) + x_t + x_w \geq k$$

defines a facet of $\text{conv } S(G_k, I)$, and $F_k(x^{(k)}) = k - (k - 2)/(2k - 3)$.

- (b) $\widehat{x}^{(k)} \in P(\widehat{G}_k)$, and $\widehat{x}^{(k)} \in \text{conv } S(T, \widehat{I}_T)$ for every subtree T of $\widehat{G}_k$. But the inequality

$$H_k(x) = k(x_p + x_q + x_m + 2x_z + x_s) + x_t + x_w \geq k$$

defines a facet of $\text{conv } S(\widehat{G}_k, \widehat{I})$, and $H_k(\widehat{x}^{(k)}) = k - (k - 2)/(2k - 3)$.

In particular $\text{conv } S \subsetneq P \cap \bigcap_{T \in \mathcal{T}} \text{conv } S(T, I_T)$ for both instances, so Conjecture 1.1 fails whether or not P is included on the right-hand side (Section 2.1).

The graph G_k has six vertices; its only cycle is a 4-cycle through an integral vertex. The graph $\widehat{G}_k$ has seven vertices. Its continuous vertices induce a 4-cycle, and each of its three integral vertices is a pendant vertex with a continuous neighbour (Figure 1). The second example matters because of normalisation. Splitting an integral vertex into copies, one for each incident edge, does not change the convex hull up to identifying the copies (Remark 2.5, following [CDEW09, DW11]). Splitting the integral vertex z of G_k gives a tree whose integral vertices are its leaves, and for such a tree the conjecture is trivial. So G_k refutes Conjecture 1.1 as stated, but not its restriction to normalised instances, and that restriction would still give the coNP certificate. The graph $\widehat{G}_k$ refutes both. Hence the certificate proposed by Conforti, a single subtree of G or of a normalisation of G , does not exist in general. This does not decide whether Problem 1 is in coNP; the complexity question (record OWR-2489-008 of [ula26]) remains open.

For $k = 1$ the relaxation $P(G)$ is integral, so $\text{conv } S(G, I) = P(G)$ and the identity holds once $P(G)$ is included on the right-hand side. For $k = 2$ we rely on Conforti's statement (Section 6). So the cases $k \geq 3$ left open in [Con08] all have a negative answer.

2 Preliminaries

2.1 Reading of the conjecture

Throughout, $\text{conv } S(H, I_H)$ for a subgraph H is read as a cylinder in $\mathbb{R}^{U \cup V}$, as above.

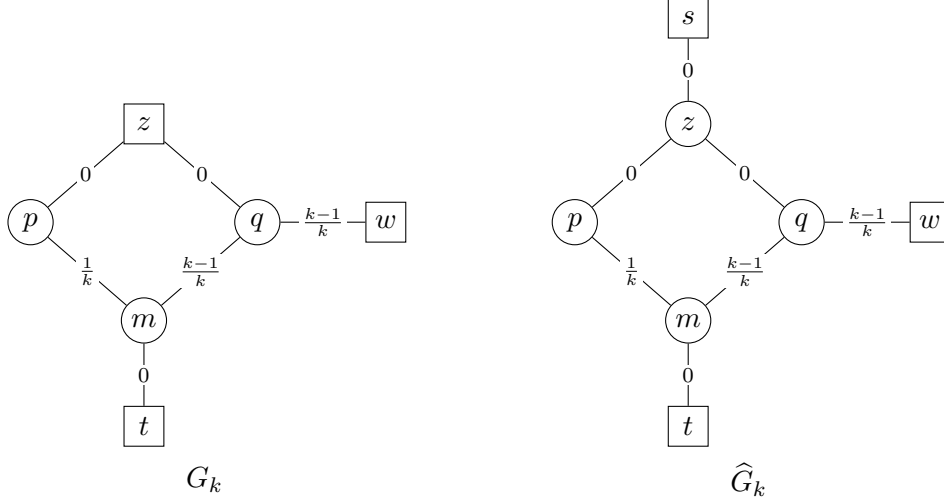


Figure 1: The instances G_k and $\hat{G}_k$. Squares are integral vertices, circles are continuous vertices, and each edge ij is labelled with b_{ij} . The vertices p, q, t (and s) lie in U , and m, w, z lie in V .

Lemma 2.1. *If $H \subseteq H'$ are subgraphs of G , then $\text{conv } S(H', I_{H'}) \subseteq \text{conv } S(H, I_H)$. In particular $\text{conv } S(G, I) \subseteq P(G) \cap \bigcap_{T \in \mathcal{T}} \text{conv } S(T, I_T)$.*

Proof. The constraints of $S(H, I_H)$ are among those of $S(H', I_{H'})$, and $I_H \subseteq I_{H'}$. So the restriction to $V(H)$ maps $S(H', I_{H'})$ into $S(H, I_H)$, and hence $\text{conv } S(H', I_{H'})$ into $\text{conv } S(H, I_H)$. Edges are subgraphs, which gives $P(G)$. $\square$

So only the inclusion “ $\supseteq$ ” in Conjecture 1.1 is in question. Read literally, it can fail for a trivial reason: an edge of G that lies in no member of $\mathcal{T}$ contributes no constraint on the right-hand side. For example, for $G = K_2$ and $I = \emptyset$ no member of $\mathcal{T}$ has an edge. (A single continuous vertex is a subtree without leaves, so it lies in $\mathcal{T}$, but it imposes no constraint.) So the right-hand side is $\mathbb{R}^2$. The intended statement is presumably

$$\text{conv } S(G, I) = P(G) \cap \bigcap_{T \in \mathcal{T}} \text{conv } S(T, I_T). \quad (1)$$

Our points lie in $P(G)$, so they refute both readings. (In our examples every edge lies in a member of $\mathcal{T}$ anyway; Remarks 3.6 and 4.6.)

If G is connected, the vector ℓ with $\ell_u = 1$ ($u \in U$) and $\ell_v = -1$ ($v \in V$) satisfies $\ell_i + \ell_j = 0$ on every edge. Hence $S(G, I) + \mathbb{Z}\ell = S(G, I)$, and $\text{conv } S(G, I) + \mathbb{R}\ell = \text{conv } S(G, I)$ [CWZ10, Remark 2]. We call ℓ the lineality direction.

Lemma 2.2. *Let G be connected with exactly one cycle C . If $x \in \text{conv } S(G - e, I)$ for every edge e of C , then $x \in P(G)$ and $x \in \text{conv } S(T, I_T)$ for every subtree T of G , in particular for every $T \in \mathcal{T}$.*

Proof. A subtree T does not contain all edges of C . So $T \subseteq G - e$ for some edge e of C , and Lemma 2.1 gives $x \in \text{conv } S(T, I_T)$. Every edge of G lies in $G - e$ for some edge e of C , since C has at least four edges. So $x \in P(G)$. $\square$

2.2 Forests

The next two facts explain why a counterexample must contain a cycle.

Lemma 2.3. *Let ℓ_0 be a continuous vertex of degree one in G , with neighbour n . Then $\text{conv } S(G, I) = \text{conv } S(G - \ell_0, I) \cap \{x : x_{\ell_0} + x_n \geq b_{\ell_0 n}\}$.*

Proof. The inclusion “ $\subseteq$ ” is Lemma 2.1. Conversely, let $\bar{x}$ lie in the right-hand side, and write $\bar{x}|_{V(G) \setminus \{\ell_0\}} = \sum_i \lambda_i x^i$ as a convex combination of points $x^i \in S(G - \ell_0, I)$. Extend x^i by $x_{\ell_0}^i = \bar{x}_{\ell_0} + \bar{x}_n - x_n^i$. Then $x_{\ell_0}^i + x_n^i = \bar{x}_{\ell_0} + \bar{x}_n \geq b_{\ell_0 n}$, so $x^i \in S(G, I)$, and $\sum_i \lambda_i x_{\ell_0}^i = \bar{x}_{\ell_0}$. $\square$

Proposition 2.4. *If G is a forest, then (1) holds.*

Proof. Both sides of (1) are products over the components of G , so we may assume that G is a tree. We use induction on $|E|$. If $|E| = 0$, both sides equal $\mathbb{R}^{V(G)}$. Suppose first that some integral vertex j has degree $d \geq 2$, and let $G_1, \dots, G_d$ be the branches of G at j , each containing j . Every continuous vertex has all its edges in one branch. By the decomposition lemma of Conforti, Wolsey and Zambelli [CWZ10, Lemma 14], applied $d - 1$ times, $\text{conv } S(G, I) = \bigcap_i \text{conv } S(G_i, I_{G_i})$. Each G_i satisfies (1) by induction. A member of $\mathcal{T}(G_i, I_{G_i})$ is a member of $\mathcal{T}(G, I)$, and $P(G) = \bigcap_i P(G_i)$. So the right-hand side of (1) is contained in $\text{conv } S(G, I)$, and equality follows from Lemma 2.1. Next, if some continuous vertex has degree one, we use Lemma 2.3 and induction in the same way. In the remaining case every leaf of G is integral and every other vertex is continuous, so $G \in \mathcal{T}(G, I)$ and (1) is trivial. $\square$

Both sides of (1) are also products over the components of G in general. So every counterexample to (1) has a component with a cycle. Our examples have only one cycle.

2.3 Normalisation

Remark 2.5 (Normalisation). We use two reductions. Self-contained proofs of the two identities below are given in Corollary 5.6.

(i) *Edges between integral vertices.* Let $uv \in E$ with $u, v \in I$. On $S(G, I)$ the sum $x_u + x_v$ is an integer, and

$$\text{conv } S(G, I) = \text{conv } S(G - uv, I) \cap \{x : x_u + x_v \geq \lceil b_{uv} \rceil\}.$$

So removing uv alone does not preserve membership: the constraint of uv has to be kept in the rounded form $x_u + x_v \geq \lceil b_{uv} \rceil$. For instance, let G be a single edge uv with $u, v \in I$ and $b_{uv} = \frac{1}{2}$. Then $(\frac{3}{10}, \frac{3}{10})$ lies in $\text{conv } S(G - uv, I) = \mathbb{R}^2$, but not in $\text{conv } S(G, I) = \{x : x_u + x_v \geq 1\}$.

(ii) *Splitting an integral vertex.* Let j be an integral vertex of degree $d \geq 2$, with neighbours $n_1, \dots, n_d$. Replace j by new integral vertices $j_1, \dots, j_d$ on the side of j , with j_i adjacent only to n_i and $b_{j_i n_i} = b_{jn_i}$. Call the result (G', I') , and put $L = \{x : x_{j_1} = \dots = x_{j_d}\}$. Copying x_j into every x_{j_i} maps $S(G, I)$ bijectively onto $S(G', I') \cap L$, and

$$\text{conv}(S(G', I') \cap L) = \text{conv } S(G', I') \cap L.$$

So splitting preserves membership exactly: $\bar{x} \in \text{conv } S(G, I)$ if and only if the copied point $\bar{x}'$ lies in $\text{conv } S(G', I')$.

In the variables $y_u = x_u$ ($u \in U$) and $y_v = -x_v$ ($v \in V$), the constraints added in (i) and (ii) are difference constraints with integral right-hand sides between integer variables. So both identities are special cases of Proposition 3 of [DW11], a consequence of the extended formulation of [CDEW09]. Di Summa and Wolsey [DW11, Section 2] use them, as their steps (i) and (ii), to show that for describing such hulls one may assume that no constraint involves only integer variables and that every integer variable occurs in at most one constraint. We call an instance *normalised* if every integral vertex has degree one and a continuous neighbour. Apply (i) to every edge between two integral vertices, delete the integral vertices that become isolated (this does not change the hull, read as a cylinder), and then

apply (ii). This gives, in polynomial time, a normalised instance (G', I') such that $\bar{x} \in \text{conv } S(G, I)$ if and only if $\bar{x}' \in \text{conv } S(G', I')$ and $\bar{x}_u + \bar{x}_v \geq \lceil b_{uv} \rceil$ for every removed edge uv . So if Conjecture 1.1 (or (1)) held for all normalised instances, Problem 1 would still be in coNP: a violated rounded inequality or a single subtree of G' would certify non-membership. Theorem 1.2(b) rules this out.

3 The six-vertex counterexample

Definition 3.1. For $k \geq 2$ let G_k have $U = \{p, q, t\}$, $V = \{m, w, z\}$, integral vertices $I = \{t, w, z\}$, and the following edges and right-hand sides:

$$b_{pm} = \frac{1}{k}, \quad b_{pz} = 0, \quad b_{qm} = \frac{k-1}{k}, \quad b_{qz} = 0, \quad b_{qw} = \frac{k-1}{k}, \quad b_{tm} = 0.$$

The denominator parameter is k . The only cycle of G_k is $C = pmqz$. We write points as (p, q, t, m, w, z) for $(x_p, x_q, x_t, x_m, x_w, x_z)$, and put

$$\lambda = \frac{k-2}{2k-3}, \quad \mu = \frac{1}{2k-3}, \quad x^{(k)} = \left(0, \frac{(k-1)(k-2)}{k(2k-3)}, -\frac{k-2}{2k-3}, \frac{k^2-k-1}{k(2k-3)}, \frac{k-1}{2k-3}, 0\right).$$

Note that $2\lambda + \mu = 1$, $\lambda = (k-2)\mu$, and $\lambda + \mu = (k-1)/(2k-3)$. For $k = 3$ we have $\lambda = \mu = \frac{1}{3}$ and $x^{(3)} = (0, \frac{2}{9}, -\frac{1}{3}, \frac{5}{9}, \frac{2}{3}, 0)$.

Lemma 3.2. For every $k \geq 2$, $F_k(x) = k(p+q+m+z) + t + w \geq k$ holds on $S(G_k, I)$.

Proof. Let $x \in S(G_k, I)$. Replacing x by $x + z\ell$, where $\ell = (1, 1, 1, -1, -1, -1)$ is the lineality direction, keeps x in $S(G_k, I)$ since $z \in \mathbb{Z}$, makes the z -coordinate 0, and does not change F_k , because $F_k(\ell) = 0$. So we may assume $z = 0$. The variables p, q and t have positive coefficients in F_k and occur only in the constraints of the edges at them. Hence $p \geq \max\{0, \frac{1}{k} - m\}$, $q \geq \max\{0, \frac{k-1}{k} - m, \frac{k-1}{k} - w\}$, and $t \geq \lceil -m \rceil$ because $t \in \mathbb{Z}$. With $\sigma = km$ this gives $F_k(x) \geq \Phi(\sigma, w)$, where

$$\Phi(\sigma, w) = \max\{0, 1 - \sigma\} + \max\{0, k - 1 - \sigma, k - 1 - kw\} + \lceil -\sigma/k \rceil + \sigma + w.$$

Recall that $w \in \mathbb{Z}$. We use that $\lceil -\sigma/k \rceil = 0$ for $0 < \sigma < k$, and $\lceil -\sigma/k \rceil \geq -\sigma/k$ always.

Case $w \geq 1$. Then $k - 1 - kw < 0$, so $\Phi = \psi(\sigma) + w$ with $\psi(\sigma) = \max\{0, 1 - \sigma\} + \max\{0, k - 1 - \sigma\} + \lceil -\sigma/k \rceil + \sigma$. If $\sigma \leq 0$, then $\psi = k - \sigma + \lceil -\sigma/k \rceil \geq k$. If $0 < \sigma < k$, then $\psi \geq \max\{0, k - 1 - \sigma\} + \sigma = \max\{k - 1, \sigma\} \geq k - 1$. If $\sigma \geq k$, then $\psi \geq \sigma - \sigma/k \geq k - 1$. So $\Phi \geq k - 1 + w \geq k$.

Case $w \leq 0$. The second maximum is at least $k - 1 - kw$, so $\Phi \geq (k - 1) - (k - 1)w + \chi(\sigma) \geq k - 1 + \chi(\sigma)$, where $\chi(\sigma) = \max\{0, 1 - \sigma\} + \sigma + \lceil -\sigma/k \rceil$. If $\sigma \leq 0$, then $\chi = 1 + \lceil -\sigma/k \rceil \geq 1$. If $0 < \sigma < k$, then $\chi = \max\{1, \sigma\} \geq 1$. If $\sigma \geq k$, then $\chi \geq \sigma - \sigma/k \geq k - 1 \geq 1$. So $\Phi \geq k$. $\square$

Lemma 3.3. Let $k \geq 3$. For every edge e of C , $x^{(k)} \in \text{conv } S(G_k - e, I)$; explicit convex combinations are given in Table 1.

Proof. The weights are positive and sum to 1. Every listed point has $t, w, z \in \mathbb{Z}$, and a direct check shows that it satisfies the five constraints of $G_k - e$. For example, a_3 gives $p + m = 1$, $p + z = 0$, $q + m = q + w = \frac{k-1}{k}$ and $t + m = 0$, and violates only $q + z \geq 0$. It remains to compute the weighted sums; we use $\lambda = (k-2)\mu$ repeatedly. In all four rows $t = -\lambda$, $w = \lambda + \mu$ and $z = 0$. In the rows qm , pz and pm we get $q = \lambda \frac{k-1}{k}$, and in the row qz we get $q = (\lambda + \mu) \frac{k-2}{k} = \lambda \frac{k-1}{k}$. In the rows qz and qm we get $m = \lambda + \frac{\lambda + \mu}{k}$, and in the rows pz and pm we get $m = \lambda + \mu \frac{k-1}{k}$, which is the same number. In the row pz , $p = \lambda - \mu \frac{k-2}{k} - \lambda \frac{k-1}{k} = \frac{\lambda - (k-2)\mu}{k} = 0$, and in the other rows $p = 0$ trivially. Finally $\lambda \frac{k-1}{k} = \frac{(k-1)(k-2)}{k(2k-3)}$ and $\lambda + \frac{\lambda + \mu}{k} = \frac{k(k-2) + k-1}{k(2k-3)} = \frac{k^2 - k - 1}{k(2k-3)}$. $\square$

edge e	point of $S(G_k - e, I)$, as (p, q, t, m, w, z)	weight	satisfies e ?
qz	$a_1 = (0, \frac{k-1}{k}, 0, \frac{1}{k}, 0, 0)$	λ	yes
	$a_2 = (0, \frac{k-2}{k}, 0, \frac{1}{k}, 1, 0)$	μ	yes
	$a_3 = (0, -\frac{1}{k}, -1, 1, 1, 0)$	λ	no
qm	$b_1 = (0, 0, 0, \frac{1}{k}, 1, 0)$	$\lambda + \mu$	no
	$b_2 = (0, \frac{k-1}{k}, -1, 1, 0, 0)$	λ	yes
pz	$c_1 = (1, \frac{k-1}{k}, 0, 0, 0, 0)$	λ	yes
	$c_2 = (-\frac{k-2}{k}, 0, 0, \frac{k-1}{k}, 1, 0)$	μ	no
	$c_3 = (-\frac{k-1}{k}, 0, -1, 1, 1, 0)$	λ	no
pm	$d_1 = (0, \frac{k-1}{k}, 0, 0, 0, 0)$	λ	no
	$d_2 = (0, 0, 0, \frac{k-1}{k}, 1, 0)$	μ	yes
	$d_3 = (0, 0, -1, 1, 1, 0)$	λ	yes

Table 1: Convex combinations for Lemma 3.3. For each edge e of the cycle C , the listed points lie in $S(G_k - e, I)$, and the weighted sum is $x^{(k)}$. The last column says whether the point also satisfies the constraint of e , that is, whether it lies in $S(G_k, I)$.

Theorem 3.4. *Let $k \geq 3$. Then $x^{(k)} \in P(G_k)$ and $x^{(k)} \in \text{conv } S(T, I_T)$ for every subtree T of G_k , but*

$$F_k(x^{(k)}) = \frac{2(k-1)^2}{2k-3} = k - \frac{k-2}{2k-3} < k.$$

Hence $x^{(k)} \notin \text{conv } S(G_k, I)$, and (G_k, I) is a counterexample to Conjecture 1.1 and to (1).

Proof. The first two claims follow from Lemmas 2.2 and 3.3. By Lemma 3.2, $F_k \geq k$ is valid for $\text{conv } S(G_k, I)$. Finally, in the row qz of Table 1 we have $F_k(a_1) = F_k(a_2) = k$ and $F_k(a_3) = k - 1$. Hence $F_k(x^{(k)}) = \lambda k + \mu k + \lambda(k - 1) = k - \lambda$, because $2\lambda + \mu = 1$. $\square$

Here a_1 and a_2 lie in $S(G_k, I)$, while a_3 violates only $q + z \geq 0$. Its weight λ is positive exactly when $k \geq 3$.

Lemma 3.5. *For $k \geq 3$ the inequality $F_k \geq k$ defines a facet of $\text{conv } S(G_k, I)$.*

Proof. Increasing a coordinate (by an integer, if it is integral) preserves membership in $S(G_k, I)$, so $\text{conv } S(G_k, I)$ is full-dimensional. The points a_1, a_2, d_2, d_3 of Table 1 and $P_1 = (\frac{1}{k}, \frac{k-1}{k}, 0, 0, 0, 0)$ lie in $S(G_k, I)$ and satisfy $F_k = k$, and $F_k(\ell) = 0$. Consider the vectors

$$\begin{aligned} P_1 - a_1 &= (\frac{1}{k}, 0, 0, -\frac{1}{k}, 0, 0), & a_2 - a_1 &= (0, -\frac{1}{k}, 0, 0, 1, 0), & d_2 - a_1 &= (0, -\frac{k-1}{k}, 0, \frac{k-2}{k}, 1, 0), \\ d_3 - a_1 &= (0, -\frac{k-1}{k}, -1, \frac{k-1}{k}, 1, 0), & \ell &= (1, 1, 1, -1, -1, -1). \end{aligned}$$

Only ℓ has a nonzero z -coordinate. Among the others, only $d_3 - a_1$ has a nonzero t -coordinate, and then only $P_1 - a_1$ has a nonzero p -coordinate. The last two, $a_2 - a_1$ and $d_2 - a_1$, are linearly independent because $k \geq 3$. So the five vectors are linearly independent. Since $\text{conv } S(G_k, I) + \mathbb{R}\ell = \text{conv } S(G_k, I)$, the face of $\text{conv } S(G_k, I)$ on which $F_k = k$ contains $a_1 + \mathbb{R}\ell$, and so it has dimension at least 5. It is a proper face, as F_k is not constant on $\text{conv } S(G_k, I)$ (for instance $F_k(a_1 + e_p) = 2k$, where e_p is the unit vector of p). Hence it is a facet. $\square$

Remark 3.6. A direct enumeration shows that G_k has 39 subtrees with at least one edge, and 7 of them lie in $\mathcal{T}(G_k, I)$. The maximal members are $T_1 = G_k - qz$ and $T_2 = \{qm, qw, qz, tm\}$, and every edge lies in one of them. So Conjecture 1.1 for G_k asserts $\text{conv } S(G_k, I) = \text{conv } S(T_1, I_{T_1}) \cap \text{conv } S(T_2, I_{T_2})$, and the rows qz and pm of Table 1 already refute it, because $T_2 \subseteq G_k - pm$. For $k = 2$ we have $\lambda = 0$ and $x^{(2)} = (0, 0, 0, \frac{1}{2}, 1, 0) \in S(G_2, I)$, so there is no violation. The graph G_k is not normalised: splitting z as in Remark 2.5 gives the tree with edges $pz_1, pm, qm, qz_2, qw, tm$, whose leaves z_1, z_2, t, w are exactly its integral vertices.

4 The normalised counterexample

Definition 4.1. For $k \geq 2$ let $\widehat{G}_k$ be obtained from G_k by declaring z continuous and adding a new integral vertex $s \in U$ adjacent only to z , with $b_{sz} = 0$. So $\widehat{I} = \{s, t, w\}$, $U = \{p, q, t, s\}$ and $V = \{m, w, z\}$. The instance is normalised: s, t, w have degree one, with continuous neighbours z, m, q . The only cycle is again $C = pmqz$, now on continuous vertices. We write points as (p, q, t, s, m, w, z) and put $\widehat{x}^{(k)} = (0, q^{(k)}, t^{(k)}, 0, m^{(k)}, w^{(k)}, 0)$, where $q^{(k)}, t^{(k)}, m^{(k)}, w^{(k)}$ are the coordinates of $x^{(k)}$. Note that $H_k(x) = F_k(p, q, t, m, w, z) + k(z + s)$.

Lemma 4.2. *For every $k \geq 2$, $H_k(x) \geq k$ holds on $S(\widehat{G}_k, \widehat{I})$.*

Proof. Let $x \in S(\widehat{G}_k, \widehat{I})$ and put $\delta = z + s \geq 0$. Let $\bar{x} = (p + \delta, q + \delta, t, m, w, -s)$, written as (p, q, t, m, w, z) . Then $\bar{x} \in S(G_k, I)$: its z -coordinate $-s$ is an integer, $\bar{x}_p + \bar{x}_z = p + z \geq 0$, $\bar{x}_q + \bar{x}_z = q + z \geq 0$, the constraints of pm , qm and qw only became weaker, and tm is unchanged. Moreover $F_k(\bar{x}) = k(p + q + 2\delta + m - s) + t + w = k(p + q + m + 2z + s) + t + w = H_k(x)$. Now apply Lemma 3.2. $\square$

Lemma 4.3. *Let $k \geq 3$. For every edge e of C , $\widehat{x}^{(k)} \in \text{conv } S(\widehat{G}_k - e, \widehat{I})$.*

Proof. The linear map $\iota(p, q, t, m, w, z) = (p, q, t, -z, m, w, z)$ maps $S(G_k - e, I)$ into $S(\widehat{G}_k - e, \widehat{I})$: the edges of $\widehat{G}_k - e$ are those of $G_k - e$ together with sz , the image satisfies $s + z = 0$, and $s = -z$ is an integer. Hence $\iota(\text{conv } S(G_k - e, I)) \subseteq \text{conv } S(\widehat{G}_k - e, \widehat{I})$, and $\iota(x^{(k)}) = \widehat{x}^{(k)}$ because $x_z^{(k)} = 0$. Now use Lemma 3.3. Explicitly, every point in Table 1 has $z = 0$ and is extended by $s = 0$. $\square$

Theorem 4.4. *Let $k \geq 3$. Then $\widehat{x}^{(k)} \in P(\widehat{G}_k)$ and $\widehat{x}^{(k)} \in \text{conv } S(T, \widehat{I}_T)$ for every subtree T of $\widehat{G}_k$, but $H_k(\widehat{x}^{(k)}) = F_k(x^{(k)}) = k - \frac{k-2}{2k-3} < k$. Hence $(\widehat{G}_k, \widehat{I})$ is a normalised counterexample to Conjecture 1.1 and to (1).*

Proof. Combine Lemmas 2.2, 4.2 and 4.3, and use $H_k(\widehat{x}^{(k)}) = F_k(x^{(k)})$, which holds because $z + s = 0$ at $\widehat{x}^{(k)}$. $\square$

Lemma 4.5. *For $k \geq 3$ the inequality $H_k \geq k$ defines a facet of $\text{conv } S(\widehat{G}_k, \widehat{I})$.*

Proof. As in Lemma 3.5, $\text{conv } S(\widehat{G}_k, \widehat{I})$ is full-dimensional, that is, 7-dimensional. The map ι also maps $S(G_k, I)$ into $S(\widehat{G}_k, \widehat{I})$, and $H_k(\iota(x)) = F_k(x)$. So the images under ι of the five points used in Lemma 3.5 lie in $S(\widehat{G}_k, \widehat{I})$ and satisfy $H_k = k$. So does $y = (-\frac{1}{k}, -\frac{1}{k}, -1, 0, 1, 1, \frac{1}{k})$, as a direct check shows. The lineality direction is $\widehat{\ell} = (1, 1, 1, 1, -1, -1, -1)$, and $H_k(\widehat{\ell}) = 0$. Consider the six vectors $\widehat{\ell}$, $y - \iota(a_1)$, and the images under ι of the four vectors $P_1 - a_1, a_2 - a_1, d_2 - a_1, d_3 - a_1$ of Lemma 3.5. Only $\widehat{\ell}$ has a nonzero s -coordinate, and among the remaining vectors only $y - \iota(a_1)$ has a nonzero z -coordinate. The last four are linearly independent, because those of Lemma 3.5 are and ι is injective. So the six vectors are linearly independent, and we conclude as in Lemma 3.5. $\square$

Remark 4.6. (a) *The family.* $\widehat{G}_k$ has 67 subtrees with at least one edge, and 9 of them lie in $\mathcal{T}(\widehat{G}_k, \widehat{I})$. The maximal members are $\widehat{G}_k - qz$, $\widehat{G}_k - qm$ and $\{qm, qw, qz, tm, sz\}$. In $\widehat{G}_k - qm$ the continuous vertex z has degree three. This tree corresponds to $G_k - qm$, which is not in $\mathcal{T}(G_k, I)$ because z is integral there. The row qm of Table 1 is used exactly here.

(b) *Nonnegativity.* All points in Table 1 have coordinates at least -1 . Shift every coordinate by $+1$, so that b_{ij} becomes $b_{ij} + 2$, and add the bounds $x \geq 0$ to $S(G, I)$ and to every $S(T, I_T)$. The shifted points certify membership in the bounded tree hulls, and the shifted inequalities stay valid on the smaller sets. So both examples also refute the bounded variant with $x \geq 0$ (the setting of [CGZ07], but with $k \geq 3$).

5 Exact validity certificates and a further example

Validity of an inequality for $\text{conv } S(G, I)$ can be certified exactly by nonnegative multipliers on a lifted system. Up to a change of variables, the system is the extended formulation of [CDEW09] in the form described by Conforti, Wolsey and Zambelli [CWZ10, Section 2] (see the paragraph after Proposition 5.5). For the certificates we need only the easy direction. We use them to check Lemmas 3.2 and 4.2 for small k by computer, independently of the proofs above, and to certify another counterexample. We also give a short self-contained proof that the projection of the lifted system is $\text{conv } S(G, I)$ (Proposition 5.5), and derive from it the two identities used in Remark 2.5.

Fix (G, I, b) and a positive integer k such that kb is integral (for instance the denominator parameter), and put $\beta_{uv} = kb_{uv} \in \mathbb{Z}$. The variables are $g_v(r)$ for $v \in U \cup V$ and $r \in \{0, \dots, k-1\}$. For an arbitrary integer $r' = r + fk$ with $0 \leq r < k$ we write $g_v(r') = g_v(r) - f$. The *lifted system* Λ consists of the rows

$$\begin{aligned} \text{(M)} \quad & g_v(r) - g_v(r+1) \geq 0 & (v \in U \cup V, 0 \leq r \leq k-2), \\ \text{(W)} \quad & g_v(k-1) - g_v(0) \geq -1 & (v \in U \cup V, v \notin I), \\ \text{(W)} \quad & g_v(k-1) - g_v(0) \geq 0 & (v \in I), \\ \text{(E)} \quad & g_u(r) - g_v(r - \beta_{uv}) \geq 0 & (uv \in E, u \in U, v \in V, 0 \leq r \leq k-1). \end{aligned}$$

With the convention above, each row has the form $g(a) - g(a') \geq h$ with $h \in \mathbb{Z}$. Let $\pi(g)_u = \frac{1}{k} \sum_r g_u(r)$ for $u \in U$ and $\pi(g)_v = -\frac{1}{k} \sum_r g_v(r)$ for $v \in V$.

Lemma 5.1. *Let $x \in S(G, I)$ with kx integral. Put $y_u = x_u$ for $u \in U$ and $y_v = -x_v$ for $v \in V$, and $g_v(r) = \lceil y_v - r/k \rceil$ for all integers r . Then g is integral, is consistent with the convention above, satisfies all rows of Λ , and $\pi(g) = x$.*

Proof. Clearly $g_v(r+k) = g_v(r) - 1$. Write $ky_v = ak + c$ with $a, c \in \mathbb{Z}$ and $0 \leq c < k$. For $0 \leq r \leq k-1$ we get $g_v(r) = a + 1$ if $r < c$ and $g_v(r) = a$ if $r \geq c$. So $\sum_{r=0}^{k-1} g_v(r) = ka + c = ky_v$, the rows (M) hold, and $g_v(k-1) \geq g_v(0) - 1$. If $v \in I$, then $c = 0$ and g_v is constant on $\{0, \dots, k-1\}$. For an edge uv with $u \in U$, the constraint $x_u + x_v \geq b_{uv}$ reads $y_u - y_v \geq \beta_{uv}/k$. Hence $g_u(r) = \lceil y_u - r/k \rceil \geq \lceil y_v - (r - \beta_{uv})/k \rceil = g_v(r - \beta_{uv})$. $\square$

Lemma 5.2 ([CWZ10, Remark 1]). *Every point of $S(G, I)$ is a convex combination of points $x \in S(G, I)$ with kx integral.*

Proof. Fix $\bar{x} \in S(G, I)$ and let L be the set of continuous vertices. The vectors $\zeta = kx_L$ with $(\bar{x}_I, x_L) \in S(G, I)$ form a polyhedron. Its constraint matrix consists of rows of the incidence matrix of a bipartite graph, so it is totally unimodular, and its right-hand side is integral. So this polyhedron is integral [Sch86, Ch. 19] and is the convex hull of its integral points [Sch86, Sec. 16.3]. Apply this to $\zeta = k\bar{x}_L$. $\square$

Proposition 5.3. *Let $c \in \mathbb{R}^{U \cup V}$, and let (π_ρ) be nonnegative numbers indexed by the rows ρ of Λ , written as $g(a_\rho) - g(a'_\rho) \geq h_\rho$. Suppose that, for every v and r , the net multiplier $\sum_{\rho: a_\rho=(v,r)} \pi_\rho - \sum_{\rho: a'_\rho=(v,r)} \pi_\rho$ equals c_v/k if $v \in U$ and $-c_v/k$ if $v \in V$. Then $c^\top x \geq \alpha := \sum_\rho \pi_\rho h_\rho$ for all $x \in \text{conv } S(G, I)$.*

Proof. For $x \in S(G, I)$ with kx integral, take g from Lemma 5.1. Then $c^\top x = \sum_{u \in U} \frac{c_u}{k} \sum_r g_u(r) - \sum_{v \in V} \frac{c_v}{k} \sum_r g_v(r) = \sum_\rho \pi_\rho (g(a_\rho) - g(a'_\rho)) \geq \alpha$. By Lemma 5.2 this extends to $S(G, I)$, and then to its convex hull. $\square$

Such multipliers are a flow in the network whose arcs are the rows of Λ . They can be found by linear programming and then checked in exact arithmetic. By Proposition 5.5 below and linear programming duality, every valid inequality has such a certificate. For example, for $k = 3$ the inequality $F_3 \geq 3$ is certified by multiplier 1 on the seven rows $E(pm, 0)$, $E(pz, 1)$, $E(pz, 2)$, $E(qm, 0)$, $E(qw, 1)$, $E(qz, 2)$ and $E(tm, 0)$, multiplier $\frac{1}{3}$ on $M(t, 1)$ and $M(w, 0)$, multiplier $\frac{2}{3}$ on $W(t)$ and $W(w)$, and multiplier 1 on $W(z)$. Here $E(uv, r)$ is the row (E) for the edge uv and index r . The right-hand sides are 1 for $E(pm, 0)$, $E(qm, 0)$ and $E(qw, 1)$ and 0 otherwise, so $\alpha = 3$. We computed such certificates for $F_k \geq k$ on G_k and for $H_k \geq k$ on $\widehat{G}_k$ for all $3 \leq k \leq 20$; each has $\alpha = k$ and was verified exactly. This is a computer check of Lemmas 3.2 and 4.2 for these k .

Remark 5.4 (A further normalised counterexample). An independent verification run found, by a random search over normalised instances, the following instance N with $k = 3$. Its continuous vertices $u_1, u_2, u_3 \in U$ and $v_1, v_2 \in V$ induce $K_{2,3}$, with

$$(b_{u_i v_j}) = \begin{pmatrix} 2/3 & 2/3 \\ 2/3 & 1 \\ 4/3 & 2/3 \end{pmatrix} \quad (\text{rows } u_1, u_2, u_3; \text{ columns } v_1, v_2).$$

The integral vertices $l_1, l_2 \in V$ and $l_3, l_4, l_5 \in U$ are pendant vertices attached to u_1, u_2, v_1, v_2 , with right-hand sides $0, 0, 1, \frac{4}{3}, 1$. Take the objective c and the point $\bar{x}$ given in the order $(u_1, u_2, u_3, v_1, v_2; l_1, \dots, l_5)$ by

$$c = (8, 4, 2, 15, 7; 1, 2, 3, 4, 4), \quad \bar{x} = (\frac{2}{3}, \frac{8}{9}, \frac{4}{3}, 0, \frac{2}{9}; -\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, \frac{4}{3}, 1).$$

Then $\bar{x} \in P(N)$ and $c^\top \bar{x} = \frac{223}{9}$. The family $\mathcal{T}(N)$ has 116 members, and for each of them we give an explicit convex combination of points of $S(T, I_T)$ equal to $\bar{x}|_{V(T)}$. These were found by linear programming and threshold rounding $g \mapsto \lceil g - \theta \rceil$ in the lifted system of T . A certificate as in Proposition 5.3 with 26 nonzero multipliers proves $c^\top x \geq 25$ on $\text{conv } S(N)$. So N is a further normalised counterexample for $k = 3$; unlike $\widehat{G}_k$, its continuous part $K_{2,3}$ has more than one cycle. Another independent verification run found explicit convex combinations showing that $\bar{x}$ lies in $\text{conv } S(T, I_T)$ for every one of the 788 subtrees T of N , so N also refutes the reading over all subtrees. All these certificates are part of the release, as data or as code that recomputes them, and are checked exactly (see the Verification paragraph).

Finally, we show that the projection of the lifted system is $\text{conv } S(G, I)$, and derive the two identities of Remark 2.5. Let $Q = Q(G, I)$ be the set of real vectors g that satisfy all rows of Λ .

Proposition 5.5. $\pi(Q) = \text{conv } S(G, I)$.

Proof. By Lemma 5.1, $\pi(Q)$ contains every point $x \in S(G, I)$ with kx integral. As $\pi(Q)$ is convex, Lemma 5.2 gives $\text{conv } S(G, I) \subseteq \pi(Q)$. Conversely, each row of the constraint matrix of Λ has at most one entry 1 and at most one entry -1 , and all other entries are 0. So the matrix is totally unimodular, and as the right-hand sides are integers, Q is integral [Sch86, Ch. 19] and is the convex hull of its

integral points [Sch86, Sec. 16.3]. It therefore suffices to show that $\pi(g) \in S(G, I)$ for every integral $g \in Q$. For each vertex v the rows (M) and (W) give $g_v(0) \geq g_v(1) \geq \dots \geq g_v(k-1) \geq g_v(0) - 1$, with $g_v(k-1) = g_v(0)$ if $v \in I$. Hence there are integers a_v and $0 \leq c_v < k$, with $c_v = 0$ if $v \in I$, such that $g_v(r) = a_v + 1$ for $0 \leq r < c_v$ and $g_v(r) = a_v$ for $c_v \leq r < k$. With $y_v = a_v + c_v/k$ this means that $g_v(r) = \lceil y_v - r/k \rceil$ for $0 \leq r < k$, and then for all integers r by the convention. Moreover $\pi(g)_u = y_u$ for $u \in U$ and $\pi(g)_v = -y_v$ for $v \in V$, and these are integers for $v \in I$. For an edge uv with $u \in U$, take $r = c_u$, so that $g_u(r) = a_u = y_u - r/k$. The row $E(uv, r)$ gives

$$y_u - \frac{r}{k} = g_u(r) \geq g_v(r - \beta_{uv}) = \left\lceil y_v - \frac{r - \beta_{uv}}{k} \right\rceil \geq y_v - \frac{r}{k} + b_{uv},$$

that is, $\pi(g)_u + \pi(g)_v = y_u - y_v \geq b_{uv}$. So $\pi(g) \in S(G, I)$. $\square$

Under the change of variables $\mu_u^t = g_u(k-1-t)$ for $u \in U$ and $\mu_v^t = -g_v(t)$ for $v \in V$, the rows (E) of Λ become the constraints (2) and (3) of the system of [CWZ10, Theorem 3], and $\pi(g)$ becomes the point x of their constraints (1). The rows (M) and (W) together are equivalent to their constraints (4) together with $\mu_i^t - \mu_i^{t-1} \geq 0$ ($1 \leq t \leq k-1$) and $\mu_i^0 - \mu_i^{k-1} \geq -1$. The system with these added constraints is the extended formulation of [CDEW09] as described at the end of Section 2 of [CWZ10].

Corollary 5.6. (a) *If $uv \in E$ and $u, v \in I$, then*

$$\text{conv } S(G, I) = \text{conv } S(G - uv, I) \cap \{x : x_u + x_v \geq \lceil b_{uv} \rceil\}.$$

(b) *In the notation of Remark 2.5(ii), $\text{conv}(S(G', I') \cap L) = \text{conv } S(G', I') \cap L$.*

Proof. We use the lifted systems of $G - uv$ and G' with the same k as for G ; this is allowed, as their right-hand sides are among those of G . If g satisfies the rows (M) and (W) at an integral vertex v , then g_v is constant on $\{0, \dots, k-1\}$, say equal to γ_v , and $\pi(g)_v = \pm \gamma_v$.

(a) Name the ends so that $u \in U$. Every point of $S(G, I)$ satisfies $x_u + x_v \geq \lceil b_{uv} \rceil$, as $x_u + x_v$ is an integer, so “ $\subseteq$ ” follows from Lemma 2.1. Conversely, let x lie in the right-hand side. By Proposition 5.5 there is $g \in Q(G - uv, I)$ with $\pi(g) = x$, so $x_u = \gamma_u$ and $x_v = -\gamma_v$. By the convention, $g_v(r - \beta_{uv}) = \gamma_v - \lfloor (r - \beta_{uv})/k \rfloor$, so the row $E(uv, r)$ reads $x_u + x_v \geq \lceil (\beta_{uv} - r)/k \rceil$. For $0 \leq r \leq k-1$ the right-hand side is at most $\lceil \beta_{uv}/k \rceil = \lceil b_{uv} \rceil$, so g satisfies these rows. Hence $g \in Q(G, I)$, and $x \in \text{conv } S(G, I)$ by Proposition 5.5.

(b) The inclusion “ $\subseteq$ ” is clear. Conversely, let $x' \in \text{conv } S(G', I') \cap L$, and take $g' \in Q(G', I')$ with $\pi(g') = x'$. The copies j_i are integral, and as $x' \in L$ the constants γ_{j_i} are all equal, say to γ . Define g for G by $g_j(r) = \gamma$ ($0 \leq r < k$) and $g_w = g'_w$ for $w \neq j$. Each row of Λ for G at j arises from a row of Λ for G' at a copy j_i by replacing g'_{j_i} with g_j , and the other rows are rows of Λ for G' . So $g \in Q(G, I)$, and x' is the copied point of $\pi(g)$. By Proposition 5.5, $\pi(g) \in \text{conv } S(G, I)$. Copying is linear and maps $S(G, I)$ onto $S(G', I') \cap L$, so it maps $\text{conv } S(G, I)$ onto $\text{conv}(S(G', I') \cap L)$, and x' lies in the latter set. $\square$

6 Concluding remarks

Status in terms of k . For $k = 1$ the matrix is totally unimodular and b is integral, so $\text{conv } S(G, I) = P(G)$. Thus (1) holds, but the literal identity of Conjecture 1.1 can fail, as for K_2 with $I = \emptyset$. For $k = 2$, Conforti states that the conjecture follows from the description of Conforti, Gerards and Zambelli [CGZ07]. Their setting includes the bounds $x \geq 0$, and their odd I -paths may end at a continuous vertex, where the bound is used. We have not re-derived the case $k = 2$ for $S(G, I)$

without bounds. In random experiments with $k = 2$ we found no counterexample. For every $k \geq 3$ the conjecture fails (Theorem 1.2).

Complexity. Theorem 1.2 shows that non-membership in $\text{conv } S(G, I)$ cannot always be certified by one subtree, of G or of a normalisation of G . It says nothing about other certificates, and the complexity of Problem 1 of [Con08] remains open. A natural question is whether some other family of small subgraphs suffices, for example subgraphs with at most one cycle, and whether optimization over them is polynomial.

How the examples were found. We computed $\min c^\top x$ over $\text{conv } S(G, I)$ and over the right-hand side of (1) by linear programming, using the lifted system of Section 5 for G and for every tree of $\mathcal{T}$. A random search over bipartite graphs with at most eight vertices first found gaps on seven and eight vertices, for $k \in \{3, 5, 6\}$. Greedy deletion of edges and integrality constraints, and lowering of right-hand sides, reduced the one with $k = 3$ to six vertices. A scan over the remaining right-hand sides gave the family G_k . The normalised family $\widehat{G}_k$ was then obtained by replacing z with a continuous vertex and a pendant integral vertex, as suggested by Remark 2.5. In random experiments we found no counterexample with at most five vertices ($k \leq 6$), and none among normalised instances whose continuous part is a 4-cycle with two pendant integral vertices ($k \leq 5$). These experiments are evidence only; we do not know the least number of vertices of a counterexample.

Verification. The proofs of Theorem 1.2 in Sections 2 to 4 are by hand. The claims were also checked by computer as follows.

1. A program by the author, `check_certificates.py` (standard library only, exact rational arithmetic, a few seconds), does the following.
 - For $3 \leq k \leq 200$ it checks Table 1 and its image under ι , the linear relaxations, the values of F_k and H_k , and the face points and ranks of Lemmas 3.5 and 4.5.
 - It enumerates all subtrees and the families $\mathcal{T}$ of G_k , $\widehat{G}_k$ and N by brute force. It tests the reduction in the proof of Lemma 4.2 on random points.
 - It rebuilds the rows of Λ from the instance data and verifies the certificates of Proposition 5.3 for F_k and H_k ($3 \leq k \leq 20$) and for N . It also tests Lemma 5.1 on all $(1/k)$ -integral points of S in a box, for G_3 , $\widehat{G}_3$, G_4 and N .
 - It verifies the 116 convex combinations for N .

The certificates were produced by a separate program, `make_certificates.py`, using linear programming.

2. The programs used in the search checked the family G_k with their own code. They produced exact convex combinations of $x^{(k)}$ for all 39 subtrees ($3 \leq k \leq 12$) and evaluated the function Φ of the proof of Lemma 3.2 exactly on a fine grid ($k \leq 8$). The mixed-integer solver CBC gave $\min F_k = k$ over $S(G_k, I)$ for $2 \leq k \leq 8$. The linear-programming solver HiGHS, in floating point and for the fixed objective F_3 , gave the value 3 over $\text{conv } S(G_k, I)$ and $3 - 1/k$ over the right-hand side of (1), for $3 \leq k \leq 12$.
3. Two independent verification runs, which were AI-assisted, checked the results with their own code.
 - The first run re-derived Lemma 3.2 by hand and re-enumerated the subtrees and the family of G_k . With separate code it rechecked the convex combinations for T_1 and T_2 for $3 \leq k \leq 300$ and the 39 subtree certificates for $k = 3$, and it confirmed $\min F_k = k$ by exact enumeration on a bounded box for $k \leq 7$. It also found the instance N of Remark 5.4 and pointed out the role of normalisation, which led to Section 4.

- The second run checked the proofs by hand and wrote exact code from the text of the paper before reading the programs above. By exact enumeration on the $(1/k)$ -grid in a rigorously bounded box it found $\min F_k = k$ over $S(G_k, I)$ for $2 \leq k \leq 16$, $\min H_k = k$ over $S(\widehat{G}_k, \widehat{I})$ for $2 \leq k \leq 14$, and $\min c^\top x = 25$ over $S(N)$. Independently of Λ , with the extended formulation of [CWZ10, Theorem 3] and an exact simplex method, it found explicit convex combinations placing $x^{(k)}$ in the hulls of all 39 subtrees of G_k ($3 \leq k \leq 7$), $\widehat{x}^{(k)}$ in those of all 67 subtrees of $\widehat{G}_k$ ($3 \leq k \leq 6$), and $\bar{x}$ in those of all 788 subtrees of N . In the same way it confirmed that none of these points lies in the hull of its whole instance. By exact linear programming, $x^{(k)}$ minimises F_k over $P(G_k) \cap \bigcap_{T \in \mathcal{T}} \text{conv } S(T, I_T)$ for $k = 3, 4, 5$, and $\widehat{x}^{(k)}$ minimises H_k over the corresponding set for $\widehat{G}_k$ for $k = 3, 4$. It also rechecked Table 1, the facet ranks, the families and the certificates of the release with its own code, and it reran the release.

Scope and priority. Theorem 1.2 answers Conforti’s question [Con08, Conjecture 2] negatively for every $k \geq 3$. The answer does not depend on the reading: it holds for the literal statement, with the linear relaxation added, for normalised instances, for the intersection over all subtrees, and with the bounds $x \geq 0$. We read Conforti’s abstract in the Oberwolfach report, the revised preprint (November 2009) of [CWZ10], the discussion-paper version (October 2010) of [DW11], obtained from the CORE website (cdn.uclouvain.be), a preprint (February 2010) of [CCZ10], and a preprint of [CGZ07]. The revised preprint of [CWZ10] does not restate the conjecture. Its final section proposes a different conjecture, on the facets of $\text{conv } S(G, I)$ when G is a tree (Conjecture 26 there). Di Summa and Wolsey [DW11, Section 5] give an example indicating that mixing sets do not suffice to describe such hulls. Their example is a tree whose integral vertices are its leaves, so it is consistent with Conjecture 1.1. The preprint of [CCZ10] discusses the set $S(G, I)$ and its extended formulation (Section 8 there) without mentioning the conjecture. We also searched arXiv, zbMATH, Crossref and OpenAlex, including the works citing [CGZ07, CDEW09, CWZ10] and the Oberwolfach report, and made three web searches. We found no counterexample and no other discussion of the conjecture. We did not see the published versions of [CWZ10], [DW11] and [CCZ10], nor the textbook [CCZ14]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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