

Common Support Lines of Three Disjoint Convex Sets Have Zero Volume

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Abstract

Let A, B, C be pairwise disjoint convex subsets of $\mathbb{R}^3$. We prove that the union of the lines supporting all three sets has Lebesgue outer measure zero. No smoothness, strict convexity, boundedness, closedness, or dimension assumption on the sets is required. Near a line with two distinct single-point contacts, we construct a bi-Lipschitz two-parameter chart of common support lines using local convex graphs and a contraction argument. At density points of the parameters giving a third contact, the three support-normal constraints force the normals to be parallel. The resulting exceptional lines are covered by Ivanov's theorem on extended boundary segments. This establishes the measure-zero assertion in Goaoc's 2008 problem; the stronger assertion about a countable cover by two-dimensional manifolds is not addressed.

1 Statement and background

A line supports a convex set if it meets the set and is contained in a plane bounding a closed halfspace that contains the set. For three sets, write $S(A, B, C)$ for the union of all their common support lines. In the problem session of the 2008 Oberwolfach workshop *Discrete Geometry*, Goaoc asked whether this union is contained in a countable union of two-dimensional manifolds, and, more weakly, whether it is contained in a null set [1, p. 2552, Conjectures 3 and 4]. The empty-interior result discussed there is also stated in Demouth's thesis [2, Chapter 4, Theorem 3 and Lemma 20]. Empty interior does not imply measure zero.

Theorem 1. *If $A, B, C \subset \mathbb{R}^3$ are pairwise disjoint convex sets, then $S(A, B, C)$ has three-dimensional Lebesgue outer measure zero.*

In particular, the assertion is about full lines, not only the contact points or the directions of the lines. We first work with compact full-dimensional convex bodies. Countable localization and enlargement then remove those restrictions.

We use the following known theorem. For a convex body K , let $V(K)$ be the union of all full lines containing a nontrivial segment of ∂K .

Theorem 2 (Ivanov). *For a compact full-dimensional convex body $K \subset \mathbb{R}^3$, $V(K)$ has σ -finite $\mathcal{H}^2$ measure. Consequently $V(K)$ has Lebesgue outer measure zero.*

The full-line assertion is explicitly stated and attributed to Ivanov [3] in Makai and Soltan's proof of Lemma 2 [4]. That statement in the author version, PDF page 6, is the source used here for this input; we do not give a new proof of Ivanov's theorem. The corresponding assertion about *directions* of boundary segments alone would not suffice.

We also use Rademacher's theorem, the Lebesgue density theorem, the contraction mapping theorem, and preservation of Lebesgue-null sets by Lipschitz maps between subsets of equal-dimensional Euclidean spaces. For standard measure-theoretic background see [5].

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2 Local coordinates for two contacts

A contact is called simple if the intersection of the support line with the body is a singleton.

Lemma 3. *Let K_1, K_2 be disjoint compact full-dimensional convex bodies. Near any common support line with two simple contacts, their common support lines have a bi-Lipschitz chart on an open subset of $\mathbb{R}^2$. The inverse chart is a fixed linear projection from four affine coordinates on line space.*

Proof. Let the two contacts be a_1, a_2 , and put $e = a_2 - a_1$. Choose w_i with $a_i + w_i \in \text{int } K_i$ so that e, w_1, w_2 are linearly independent. This is possible because each set of allowed w_i is open and nonempty. In the oblique coordinates

$$a_1 + te + xw_1 + yw_2,$$

the reference line has $x = y = 0$, and the contact parameters are 0, 1.

Near the first contact, K_1 is locally given by $x \geq f_1(t, y)$, with f_1 convex and Lipschitz. Near the second it is K_2 that is locally given by $y \geq f_2(t, x)$, with the same properties. To justify these assertions also at corners, project along w_i onto the indicated complementary plane. An interior ball about $a_i + w_i$ projects to a neighborhood of the projection of a_i . The lower fiber endpoint is a finite convex function there, and hence is locally Lipschitz. Its value at the contact is zero: a support plane at a_i has outward normal with strictly negative scalar product with w_i , excluding $a_i - sw_i$ for $s > 0$. The interior ball makes the upper fiber endpoint uniformly positive nearby. Restricting to heights near zero gives exactly the local epigraph description.

Fix $M \geq 1$ bounding both local Lipschitz constants, and choose

$$0 < \delta < \frac{1}{4(M+1)}.$$

Shrink δ further if needed so that the rectangles around

$$I_1 = [-\delta, \delta], \quad I_2 = [1 - \delta, 1 + \delta]$$

are in the graph domains. The functions $f_1(t, 0)$ and $f_2(t, 0)$ are nonnegative there and vanish only at 0 and 1, respectively. Their endpoint values are therefore positive.

Write nearby lines as

$$\ell_{u,z}(t) = a_1 + te + ((1-t)u_1 + tz_2)w_1 + ((1-t)z_1 + tu_2)w_2, \quad (1)$$

where $u = (u_1, u_2)$ and $z = (z_1, z_2)$. Define

$$F_1(u, z) = \min_{t \in I_1} \{f_1(t, (1-t)z_1 + tu_2) - (1-t)u_1 - tz_2\}, \quad (2)$$

$$F_2(u, z) = \min_{t \in I_2} \{f_2(t, (1-t)u_1 + tz_2) - (1-t)z_1 - tu_2\}. \quad (3)$$

Both gaps are locally Lipschitz. If u_1 increases by $h > 0$, F_1 decreases by between $(1 - \delta)h$ and $(1 + \delta)h$. The analogous statement holds for F_2 and u_2 . Changing u_2 changes F_1 by at most $M\delta$ times the change, and changing u_1 changes F_2 by the same bound. These estimates hold for the minima since they hold uniformly for every t .

Thus the equations $F_i = 0$ can be solved uniquely for their diagonal variables:

$$u_1 = \phi_1(u_2, z), \quad u_2 = \phi_2(u_1, z).$$

Continuity and the strict diagonal monotonicity prove existence of these roots in a product neighborhood of zero. Explicitly, fix a small interval for the solved variable; the signs at its

endpoints remain opposite when the other arguments are sufficiently small. The roots are Lipschitz in all their arguments, with cross Lipschitz constant at most

$$q = \frac{M\delta}{1-\delta} < \frac{1}{4}.$$

For a fixed sufficiently small z , the map

$$(u_1, u_2) \mapsto (\phi_1(u_2, z), \phi_2(u_1, z))$$

is a contraction in the maximum norm on a small closed square. To see that it maps this square into itself, use $|\phi_i(v, z)| \leq q|v| + C|z|$ and then shrink the z neighborhood. Its unique fixed point $u(z)$ satisfies

$$\|u(z) - u(z')\|_\infty \leq \frac{C}{1-q}|z - z'|.$$

The resulting map $z \mapsto (u_1(z), z_1, z_2, u_2(z))$ is Lipschitz, and projection to the z coordinates is its inverse.

It remains to check that these are precisely the nearby common support lines. Endpoint positivity in each interval persists when the parameters are small, so a zero minimum is attained in the interval's interior. The line meets the body there and does not enter its interior locally. If it entered the interior elsewhere, segments joining that interior point to the contact would give interior points arbitrarily close to the contact, a contradiction. Separating the open convex interior from the line gives a supporting plane containing the line.

Conversely, compactness and simplicity of the original contacts force all contacts of sufficiently nearby support lines into the graph neighborhoods: a contrary sequence would produce another contact of the reference line. Such a line has both gaps zero, and is therefore in the chart. The nearby lines themselves may have segment contacts; this does not affect the chart construction. $\square$

The space of affine lines is second countable. Thus countably many of these local charts cover all common support lines of two fixed bodies whose contacts are simple.

3 The third contact at density points

We record the linear algebra used in the argument.

Lemma 4. *Let ν_1, ν_2, ν_3 be nonzero covectors on a two-dimensional vector space W , and let t_1, t_2, t_3 be distinct real numbers. The three covectors on $W \times W$*

$$((1 - t_i)\nu_i, t_i\nu_i)$$

have rank three unless the ν_i are all parallel. In the latter case their rank is two.

Proof. Replacing (p, r) by $(p, r - p)$ changes the rows to $(\nu_i, t_i\nu_i)$. If two covectors are independent, relabel them as ν_1, ν_2 . For a dependence with coefficients c_i , subtract t_3 times the first vector equation from the second to get

$$c_1(t_1 - t_3)\nu_1 + c_2(t_2 - t_3)\nu_2 = 0.$$

Hence $c_1 = c_2 = 0$ and then $c_3 = 0$. If all covectors are parallel, nonzero multiples of two rows with distinct t_i give rank exactly two. $\square$

Fix three disjoint compact full-dimensional bodies K_1, K_2, K_3 . In a chart of Lemma 3, write

$$\ell_z(t) = a_1 + te + (1-t)p(z) + tr(z), \quad p(z), r(z) \in W = \text{span}(w_1, w_2). \quad (4)$$

The map $\Psi(z) = (p(z), r(z))$ is Lipschitz and has a linear projection π with $\pi \circ \Psi = \text{id}$. Its differential therefore has rank two wherever it exists.

Let E consist of the parameters for which ℓ_z also supports K_3 and all three contacts are simple. This set is Borel. For a compact body, meeting the body and missing its interior are closed conditions in these line coordinates. The lines whose intersection contains two points at distance at least $1/m$ also form a closed set, by compactness. Excluding the union of these sets gives the single-contact condition.

Proposition 5. *For almost every $z \in E$, the line ℓ_z is contained in*

$$\bigcup_{1 \leq i < j \leq 3} V(\text{conv}(K_i \cup K_j)).$$

Proof. Take a density-one point z of E where Ψ is differentiable. These conditions hold at almost every point of E . Denote the three contact parameters by t_i , which are distinct by disjointness. Choose an outward normal $n_i \neq 0$ to a supporting plane of K_i containing ℓ_z . Each n_i annihilates the direction of ℓ_z .

For each $h \in \mathbb{R}^2$, density one gives sequences in E of the form $z_s = z + sh + o(|s|)$, with s tending to zero through either sign. Indeed, a missing ball of radius $\varepsilon|s|$ about $z + sh$ would contradict density one in a ball of comparable radius about z . Let $t_{i,s}$ be the contact parameter with K_i . Compactness and uniqueness of the base contact imply $t_{i,s} \rightarrow t_i$. The fixed base supporting plane gives

$$n_i \cdot (\ell_{z_s}(t_{i,s}) - \ell_z(t_i)) \leq 0.$$

In the expansion, the term $(t_{i,s} - t_i)$ times the base line direction is annihilated by n_i . The remaining inequality is

$$s n_i \cdot ((1 - t_i)Dp(z)h + t_i Dr(z)h) + o(|s|) \leq 0.$$

Here the mixed error is $o(1)O(|s|) = o(|s|)$, using only convergence of the contacts and Lipschitz continuity of Ψ . Both signs of s therefore give

$$n_i \cdot ((1 - t_i)Dp(z)h + t_i Dr(z)h) = 0. \quad (5)$$

In particular, we have not differentiated the contact-point map.

Let $\nu_i = n_i|_W$. These are nonzero. Restriction to W is injective on the annihilator of the current line direction, since that direction is transverse to W . By (5), the three rows in Lemma 4 annihilate the rank-two image of $D\Psi$. They cannot have rank three, whose kernel would have dimension one. The ν_i , and hence the n_i , are all parallel.

Their supporting planes coincide because they all contain ℓ_z . At least two of the three bodies lie on the same side of this plane. For that pair K_i, K_j , the segment joining their distinct contact points lies in the boundary of $\text{conv}(K_i \cup K_j)$. Its supporting line is ℓ_z , so the whole line belongs to $V(\text{conv}(K_i \cup K_j))$. $\square$

4 Nullity for bodies and removal of restrictions

Proof of Theorem 1 for compact full-dimensional bodies. Set

$$N = \bigcup_{i=1}^3 V(K_i) \cup \bigcup_{1 \leq i < j \leq 3} V(\text{conv}(K_i \cup K_j)).$$

This is null by Theorem 2. A common support line with a nontrivial contact segment lies entirely in N . All remaining common support lines are covered by countably many two-contact charts.

In one such chart, Proposition 5 puts the lines from almost every parameter in E entirely in N . The remaining parameter set E_0 has planar measure zero. The evaluation map

$$(z, t) \mapsto \ell_z(t)$$

is Lipschitz on bounded parameter pieces and bounded t intervals. Each $E_0 \times [-m, m]$ is three-dimensionally null, and its image is therefore null. Countable exhaustion of the chart and of the t axis, and then the countable union over charts, proves the assertion. $\square$

Proof for general convex sets. First suppose K_1, K_2, K_3 are nonempty disjoint compact convex sets, possibly of lower dimension. Choose $\varepsilon > 0$ so their closed ε neighborhoods are disjoint. For each i , take all closed rational balls in that neighborhood. Their convex hulls with K_i form a countable family of compact convex bodies with nonempty interior. Each hull stays in the same neighborhood.

If a line supports K_i in a halfspace H_i , one of these balls lies in $\text{int } H_i$. Choose a point just inside H_i and at distance less than ε from a contact point; a sufficiently small rational ball can be placed nearby inside both open conditions. The convex hull of that ball with K_i is still in H_i and still meets the line. Thus the common support lines of the compact sets are covered by the common support lines of countably many enlarged triples. Each such triple is disjoint, so the full-dimensional case applies.

Now let A_i be arbitrary pairwise disjoint convex sets. Empty sets give an empty union. For any common support line choose contacts $x_i \in A_i$. These points are distinct. Choose pairwise disjoint closed rational balls B_i with $x_i \in \text{int } B_i$. The sets

$$\overline{A_i} \cap B_i$$

are nonempty disjoint compact convex sets. The same line supports them, since the original supporting halfspace contains $\overline{A_i}$. There are only countably many triples of rational balls. The compact case therefore supplies a countable null cover of all original common support lines. The conclusion is outer-measure zero, without a measurability assumption on the original sets. $\square$

5 Scope and reproducibility

Theorem 1 establishes Conjecture 4 on the cited source page. It does not prove Conjecture 3: the exceptional image of a planar-null parameter set times a line need not be covered by countably many two-dimensional manifolds merely by the argument above.

The accompanying Python diagnostic uses only the standard library. It checks the rank dichotomy on 30 720 exact rational cases, a coincident-contact regression, and 289 nonsmooth two-contact models. These finite checks are diagnostics for the formulas, not proofs of the analytic lemmas. The proof depends on Ivanov's theorem in the form quoted by Makai and Soltan; the empty-interior theorem and the standard measure-theoretic inputs are credited above.

This is an AI-assisted, self-audited, unrefereed preprint. AI assistance was used in the argument development, source comparison, drafting and computational diagnostics. No independent review or proof-assistant certification is claimed. The literature comparison was bounded and does not certify absolute priority.

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