

# Unbounded Line Pinning by Pairwise Disjoint Convex Bodies

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19 October 2026

## Abstract

For every integer  $m \geq 3$  we construct  $m+4$  pairwise disjoint compact convex bodies with nonempty interiors in  $\mathbb{R}^3$  and a common transversal line such that every subfamily pinning that line contains at least  $m$  members. Thus isolated line transversals to arbitrary disjoint convex bodies admit no pinning bound depending only on the dimension. Four separated polytopes restrict the nearby transversals to a classical one-parameter family on a hyperbolic paraboloid. The remaining bodies are convex epigraphs of flat oscillatory perturbations: together they exclude every nonzero parameter, while each has an exclusive sequence of excluded parameters tending to zero. All contacts with the pinned line are distinct single points. An explicit extension gives the same conclusion in every dimension at least three. The construction is nonsemialgebraic and does not settle the separate question for pairwise disjoint polytopes.

## 1 The question and the result

A *convex body* here is a compact convex set with nonempty interior. A line is a *transversal* to a family if it intersects every member. The family *pins* the line if that line is an isolated point of the space of its unoriented line transversals, with the usual line-space topology. Isolation, rather than the absence of a path in that space, is essential in the construction below.

Goaoc’s report on Helly numbers and geometric permutations asks whether the local pinning theorem for disjoint balls extends to arbitrary disjoint convex sets [1, p. 2543, Section 3]. For balls in  $\mathbb{R}^d$ , a pinning subfamily of size at most  $2d - 1$  suffices. Our result rules out any bound depending only on  $d$  in the unrestricted convex-body setting.

**Theorem 1.** *For each integer  $m \geq 3$  there is a family  $\mathcal{F}_m$  of  $m+4$  pairwise disjoint convex bodies in  $\mathbb{R}^3$  pinning the  $z$ -axis  $L$ , such that every pinning subfamily contains  $m$  specified members. The intersections of  $L$  with the members are distinct single points. In particular, there are inclusion-minimal pinnings of arbitrarily large cardinality by pairwise disjoint convex bodies in  $\mathbb{R}^3$ .*

The known four-constraint reduction used below is a coordinate realization of the hyperbolic-paraboloid block of Aronov, Cheong, Goaoc and Rote [2, Section 5]; we do not claim that reduction as new. Their bound of six for disjoint polytopes assumes that the pinned line is not coplanar with a face, and their unbounded polytopal examples allow intersections. They distinguish the general disjoint-polytope question in their concluding discussion. Related pinning theorems for semialgebraic ovaloids are given in [3]. Our additional constraints are nonsemialgebraic, so none of these statements is contradicted.

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## 2 Four separated controlling polytopes

Use coordinates  $(x, y, z)$ , put  $\delta = 1/4$ , and let

$$L = \{(0, 0, z) : z \in \mathbb{R}\}, \quad L_t = \{(t, tz, z) : z \in \mathbb{R}\} \quad (t \in \mathbb{R}).$$

For  $r = -4, -3, -2, -1$ , respectively, put  $\sigma_r = -1, +1, -1, +1$  and define

$$P_r = \{(x, y, z) : |x| \leq \delta, |z - r| \leq \delta, |z - r| \leq \sigma_r(y - rx) \leq 1\}. \quad (1)$$

These are bounded full-dimensional convex polytopes, since the absolute-value inequalities are intersections of affine halfspaces. For example,  $(0, \sigma_r/2, r)$  satisfies all inequalities strictly. Their  $z$ -slabs are disjoint, and  $L \cap P_r = \{(0, 0, r)\}$ .

**Lemma 2.** *In a neighborhood of  $L$ , every common transversal to the four  $P_r$  is of the form  $L_t$ . Conversely,  $L_t$  meets all four whenever  $|t| \leq \delta$ .*

*Proof.* The lines near  $L$  have a unique expression

$$x = a + bz, \quad y = c + ez. \quad (2)$$

The conditions  $|e - rb| < 1$  for all four  $r$  define an open neighborhood of  $L$  in this chart. If the line meets  $P_r$  at  $z = r + s$ , then

$$\sigma_r(c + (e - a)r - br^2) \geq |s| - \sigma_r(e - rb)s \geq 0.$$

Set  $q(z) = c + (e - a)z - bz^2$ . The third finite-difference identity is

$$-q(-4) + 3q(-3) - 3q(-2) + q(-1) = 0. \quad (3)$$

Every term on the left is nonnegative, so all four values of  $q$  vanish. Hence  $q$  is the zero polynomial:  $b = c = 0$  and  $e = a$ . The line in (2) is therefore  $L_a$ . Conversely,  $L_t$  meets  $P_r$  at  $(t, rt, r)$  for  $|t| \leq \delta$ .  $\square$

The preceding argument allows intersections anywhere in a polytope, not only in its middle slice. In particular, moving the point of intersection inside its slab cannot evade the conclusion.

## 3 Flat oscillations inside convex epigraphs

Fix  $m \geq 3$  and write

$$\theta_j = \frac{2\pi j}{m} \quad (0 \leq j < m), \quad \eta = \cos \frac{3\pi}{2m}.$$

For  $0 < |x| \leq \delta$  define

$$h_j(x) = e^{-1/x^2}(\cos(1/x - \theta_j) - \eta), \quad g_j(x) = \frac{x^2}{2} + h_j(x), \quad (4)$$

and set  $h_j(0) = g_j(0) = 0$ .

**Lemma 3.** *The functions  $h_j$  are  $C^\infty$  and flat at zero. On  $[-\delta, \delta]$  one has*

$$|h_j''(x)| < \frac{1}{2}, \quad g_j''(x) > \frac{1}{2}, \quad g_j(x) \geq \frac{x^2}{4}.$$

*These bounds hold uniformly in  $m$  and  $j$ .*

*Proof.* Each derivative of  $h_j$  away from zero is a sum of  $e^{-1/x^2}$  times powers of  $x^{-1}$  and bounded trigonometric functions. Such terms tend to zero faster than any power of  $|x|$ . Induction using the difference quotient at zero therefore gives a smooth extension with all derivatives zero there.

For  $x \neq 0$  put  $\phi = 1/x - \theta_j$ . Direct differentiation yields

$$h_j''(x) = e^{-1/x^2} \left[ \left( \frac{4}{x^6} - \frac{6}{x^4} \right) (\cos \phi - \eta) + \left( \frac{4}{x^5} - \frac{2}{x^3} \right) \sin \phi - \frac{\cos \phi}{x^4} \right]. \quad (5)$$

Since  $|\cos \phi - \eta| \leq 2$ , this implies

$$|h_j''(x)| \leq e^{-1/x^2} \left( \frac{8}{|x|^6} + \frac{4}{|x|^5} + \frac{13}{|x|^4} + \frac{2}{|x|^3} \right) \leq 40320e^{-16} < \frac{1}{2}. \quad (6)$$

For the second inequality, each function  $u^k e^{-u^2}$  with  $3 \leq k \leq 6$  is decreasing for  $u \geq 4$ . For the strict final inequality, use  $e^{16} > 16^8/8!$ ,  $8! = 40320$ , and

$$2(40320)^2 = 3251404800 < 4294967296 = 16^8.$$

At zero,  $h_j''(0) = 0$ . Thus  $g_j'' > 1/2$  everywhere. Together with  $g_j(0) = g_j'(0) = 0$ , this gives  $g_j(x) \geq x^2/4$  by convexity of  $g_j(x) - x^2/4$ .  $\square$

For  $0 \leq j < m$ , set  $z_j = j + 1$  and define

$$C_j = \left\{ (x, y, z) : \begin{array}{l} |x| \leq \delta, \quad |z - z_j| \leq \delta, \\ g_j(x) + (z - z_j)^2/2 \leq y - z_j x \leq 1 \end{array} \right\}. \quad (7)$$

The lower inequality describes a convex epigraph in affine coordinates. The other restrictions preserve convexity and make the set bounded. The point  $(0, 1/2, z_j)$  is an interior point. Lemma 3 also shows that  $L \cap C_j = \{(0, 0, z_j)\}$ . Every pair among the  $P_r$  and  $C_j$  has distance at least  $1/2$ , because its members lie in slabs of half-width  $1/4$  about distinct integer  $z$ -coordinates.

**Lemma 4.** For  $|t| \leq \delta$ ,

$$L_t \cap C_j \neq \emptyset \iff h_j(t) \leq 0.$$

*Proof.* At  $z = z_j + s$  on  $L_t$ , the lower inequality in (7) becomes

$$\frac{t^2}{2} + h_j(t) + \frac{s^2}{2} \leq ts, \quad \text{equivalently} \quad h_j(t) + \frac{(s-t)^2}{2} \leq 0.$$

This forces  $h_j(t) \leq 0$ . Conversely, if that condition holds, choose  $s = t$ . Both box bounds in (7) hold, as does the upper bound  $y - z_j x = t^2 \leq 1$ . This proves the equivalence, including  $t = 0$ .  $\square$

## 4 Isolation and indispensable members

*Proof of Theorem 1.* Let

$$\mathcal{F}_m = \{P_{-4}, P_{-3}, P_{-2}, P_{-1}\} \cup \{C_0, \dots, C_{m-1}\}.$$

All its members are pairwise disjoint convex bodies with the stated axis contacts. Shrink the neighborhood in Lemma 2 so that the remaining parameter satisfies  $|t| < \delta$ .

For any real angle  $\alpha$ , some  $\theta_j$  is at circular distance at most  $\pi/m$  from  $\alpha$ . Since cosine is strictly decreasing on  $[0, \pi]$  and  $m \geq 3$ , it follows that

$$\cos(\alpha - \theta_j) \geq \cos \frac{\pi}{m} > \cos \frac{3\pi}{2m} = \eta.$$

For  $t \neq 0$  take  $\alpha = 1/t$ . Then  $h_j(t) > 0$  for at least one  $j$ . Lemma 4 excludes  $L_t$  as a transversal to the full family. Thus  $L = L_0$  is isolated among all nearby common transversals.

To prove indispensability, fix  $j$  and take

$$t_{j,k} = \frac{1}{2\pi k + \theta_j}, \quad k = 1, 2, \dots \quad (8)$$

These parameters lie in  $(0, \delta)$  and tend to zero. At such a parameter,  $h_j(t_{j,k}) > 0$  because  $1 > \eta$ . For  $i \neq j$ , the circular distance between  $\theta_i$  and  $\theta_j$  is at least  $2\pi/m$  and at most  $\pi$ , giving

$$\cos(\theta_j - \theta_i) \leq \cos \frac{2\pi}{m} < \cos \frac{3\pi}{2m} = \eta.$$

Hence  $h_i(t_{j,k}) < 0$  for every  $i \neq j$ . The line  $L_{t_{j,k}}$  meets every member of  $\mathcal{F}_m$  except  $C_j$ , and converges to  $L$  in the line chart. Therefore no subfamily omitting  $C_j$  pins  $L$ .

This applies to each  $j$ , so every pinning subfamily contains at least the  $m$  bodies  $C_j$ . An inclusion-minimal pinning subfamily exists because  $\mathcal{F}_m$  is finite. Its cardinality is at least  $m$ , and  $m$  is arbitrary.  $\square$

*Remark 5.* We have not asserted that all  $m + 4$  members are indispensable. The lower bound  $m$  on every pinning subfamily is sufficient. Also, the witness sequences (8) show failure of isolation after deletion; a continuous motion away from  $L$  is not required.

## 5 Every dimension at least three

**Corollary 6.** *For  $d \geq 3$  and  $m \geq 3$  there are  $m + 3d - 5$  pairwise disjoint convex bodies in  $\mathbb{R}^d$  pinning a line, while every pinning subfamily contains at least  $m$  members.*

*Proof.* The case  $d = 3$  is Theorem 1. For  $d > 3$ , write the additional coordinates as  $w_1, \dots, w_{d-3}$  and replace each three-dimensional body by its product with  $[-\delta, \delta]^{d-3}$ .

For each  $q = 1, \dots, d - 3$ , set

$$s_{q,i} = -(3q + 4) + i \quad (i = 0, 1, 2), \quad (\varepsilon_0, \varepsilon_1, \varepsilon_2) = (1, -1, 1).$$

Adjoin three further bodies  $D_{q,i}$  given by

$$\begin{aligned} |x|, |y| &\leq \delta, & |w_\ell| &\leq \delta \quad (\ell \neq q), \\ |z - s_{q,i}| &\leq \delta, & |z - s_{q,i}| &\leq \varepsilon_i w_q \leq 1. \end{aligned}$$

All these bodies are bounded, convex and full-dimensional. They lie in new, mutually disjoint integer-centered  $z$ -slabs and meet the axis at distinct single points.

A line near  $L$  is a graph over  $z$ , with  $w_q = A_q + B_q z$ . Restrict to  $|B_q| < 1$  for every  $q$ . If it meets  $D_{q,i}$  at  $z = s_{q,i} + u$ , then

$$\varepsilon_i(A_q + B_q s_{q,i}) \geq |u| - \varepsilon_i B_q u \geq 0.$$

The middle value of the affine function  $A_q + B_q z$  at these three equally spaced centers is the mean of its endpoint values. The endpoints are nonnegative and the middle is nonpositive. Consequently all three values are zero, forcing  $A_q = B_q = 0$ . All additional coordinates therefore vanish. The projections of the remaining lines meet the original four controlling polytopes, and the preceding three-dimensional isolation argument applies.

For the deletion witnesses, extend  $L_{t_{j,k}}$  by zero additional coordinates. At any fixed new center  $s_{q,i}$  its coordinates are  $x = t_{j,k}$  and  $y = t_{j,k} s_{q,i}$ , which tend to zero. Thus, for all sufficiently large  $k$ , it meets every  $D_{q,i}$  at its middle slice and satisfies all their box bounds. It also meets all the original product bodies except the deleted  $C_j$  product. The  $m$  indispensable members persist. The total number of bodies is  $m + 4 + 3(d - 3) = m + 3d - 5$ .  $\square$

## 6 Scope and reproducibility

The  $C_j$  have smooth strictly convex lower boundary patches at their contacts with  $L$ . The complete bodies are truncated, and the four controlling bodies are polytopes: no globally smooth or globally strictly convex construction is claimed. Each  $h_j$  has infinitely many sign changes accumulating at zero. Its graph, and hence the corresponding boundary construction, is not semialgebraic. In particular, the theorem does not settle the disjoint-polytope question. The flat oscillations are substantive, not a finite-precision approximation device.

The accompanying standard-library Python program checks the displayed second derivative by formal differentiation, the rational curvature bound, the anchor identities, the square completion, phase-cover bookkeeping for finite samples, and the higher-dimensional controls. Normal and optimized Python runs agree. These are consistency checks; the assertions for all  $m$ , all dimensions and all sufficiently near lines follow from the written analytic proof. In particular, numerical evaluation that rounds  $e^{-1/t^2}$  to zero cannot verify this example.

AI tools assisted with the construction, literature comparison, algebraic checks and preparation of this manuscript. The author is responsible for its claims. This is a self-audited, unrefereed preprint; no external review, proof-assistant certification or absolute priority is asserted. The source question is indexed as OWR-2090-019 in the frozen version 1.6.0 of the UnsolvedMath dataset [4].

## References

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