

Sharp Coefficient Ratios and Optimal Constants for Dilated Ehrhart Series

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Abstract

For each dimension and dilation factor $n \geq d + 1$, we prove that the ratios of the coefficients of a dilated Ehrhart h^* -polynomial to its last coefficient are maximized by the unimodular simplex. The same inequality holds for every nonnegative numerator of degree at most d under the Veronese operator. Combining this extremal statement with known real-rootedness and alternating-inequality results gives a finite, explicit characterization of every admissible pair of constants in the Beck–Stapledon theorem and its Ehrhart corollary. The smallest dilation threshold when the multiplier is free is $d + 2$. The smallest eventual integer multiplier is one more than the largest Eulerian number of order d , but these two minima need not occur together. Exact polynomial certificates give the least threshold at that multiplier through dimension eight. The real-rootedness and eventual alternating inequalities themselves are prior results; the main contribution is the sharp ratio reduction and the resulting complete finite optimization.

1 The constants and the result

Fix an integer $d \geq 1$. For $h(t) = \sum_{j=0}^d h_j t^j$, define $\mathcal{U}_n h(t) = \sum_{i=0}^d H_i(n) t^i$ by

$$\frac{h(t)}{(1-t)^{d+1}} = \sum_{r \geq 0} g(r) t^r, \quad \frac{\mathcal{U}_n h(t)}{(1-t)^{d+1}} = \sum_{r \geq 0} g(nr) t^r.$$

For a d -dimensional lattice polytope P , this gives $\mathcal{U}_n h_P^* = h_{nP}^*$. Throughout, a nonnegative numerator has $h_j \geq 0$ and $h_0 > 0$.

We use the following unambiguous definition of the strict alternating inequalities:

$$H_i(n) < H_{d-i}(n) \quad \left(0 \leq i \leq \left\lfloor \frac{d-1}{2} \right\rfloor \right), \quad (1)$$

$$H_{d+1-k}(n) < H_k(n) \quad \left(1 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor \right). \quad (2)$$

In particular the first inequality is $H_0(n) < H_d(n)$, including when $d = 1$.

Call a positive integer pair (m, N) *Ehrhart-admissible* if, for every d -dimensional lattice polytope and every integer $n \geq N$, its dilated h^* -polynomial has d distinct negative real roots, positive strictly log-concave coefficients, satisfies (1)–(2), and satisfies $H_i(n) < m H_d(n)$ for all i . The roots must also have the Eulerian limits in the Beck–Stapledon statement. That asymptotic assertion is already known [1].

Beck and Stapledon asked for the optimal dimension-dependent integers m_d, n_d [1, Theorem 1.2, Corollary 1.3, and Section 5]; the Oberwolfach formulation is [2, p. 2499]. The frozen

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dataset record OWR-2090-008 [6] refers to this question. The two integers have different roles: m_d is a coefficient multiplier and n_d is a dilation threshold.

Let $\Delta_d = \text{conv}(0, e_1, \dots, e_d)$, and put

$$A_{d,i}(x) = \sum_{k=0}^i (-1)^k \binom{d+1}{k} \binom{(i-k)x+d}{d} \quad (0 \leq i \leq d), \quad (3)$$

$$B_d(x) = \binom{x-1}{d}. \quad (4)$$

Here binomial expressions in x are polynomials. For positive integral n , $A_{d,i}(n) = h_i^*(n\Delta_d)$; $A_{d,0} = 1$ and $A_{d,d} = B_d$. Write $E(d, i)$ for the Eulerian number counting permutations with $i - 1$ descents, and set

$$M_d = \max_{1 \leq i \leq d} E(d, i).$$

Theorem 1. *For every nonnegative numerator h , every integer $n \geq d+1$, and every $0 \leq i \leq d$,*

$$\frac{H_i(n)}{H_d(n)} \leq \frac{A_{d,i}(n)}{B_d(n)}.$$

Each bound is sharp. Indeed, all bounds are simultaneously attained by $h = 1$, the h^* -polynomial of the unimodular simplex.

Here and below the positivity of $B_d(n)$ follows from $n \geq d + 1$.

Theorem 2. *A pair (m, N) is Ehrhart-admissible if and only if*

$$N \geq d + 2 \quad \text{and} \quad mB_d(n) > A_{d,i}(n) \quad \text{for every integer } n \geq N, \quad 0 \leq i \leq d. \quad (*)$$

The infinitely many scalar comparisons in $(*)$ reduce to the explicit finite formulas in Section 4. In particular:

1. *the least possible N , with m free, is $d + 2$;*
2. *the least possible m , with N free, is $M_d + 1$.*

Every pair satisfying $(*)$ also gives the conclusions of the general-numerator Beck–Stapledon theorem: the root and ratio conclusions hold for all nonnegative numerators, and the strict alternating inequalities hold whenever

$$\sum_{j=0}^k h_j \geq \sum_{j=d+1-k}^d h_j \quad (1 \leq k \leq \lfloor d/2 \rfloor). \quad (\text{H})$$

Thus the same optimal pairs work simultaneously for the theorem and its Ehrhart corollary. If only the root, positivity, log-concavity, and ratio conclusions are required for arbitrary nonnegative numerators, the exact criterion is instead $(*)$ with $N \geq d + 1$. Necessity follows from $H_d(d) = 0$ for the simplex, and sufficiency follows from the proofs below. There is a small literal edge case in the Oberwolfach wording: its additional hypothesis requires at least one strict inequality. When $d = 1$ that list is empty, so its general-numerator theorem only requires the root/ratio conclusions, with least pair $(2, 2)$. Its Ehrhart corollary still requires the full strict chain, with least pair $(2, 3)$.

This does not assert that the optimal threshold for real-rootedness alone is $d+2$. Jochemko proved a much stronger root bound [3, Theorem 1.1 and Corollary 4.4]. Higashitani established degree-sensitive strict alternating inequalities [4, Theorem 1.2], and Jochemko later obtained real-rooted symmetric decompositions [5]. The least threshold $d + 2$ for the whole strict chain already follows from these prior results and the unimodular-simplex obstruction. The multiplier $M_d + 1$ was explicitly chosen in the proof of [1, Theorem 1.2]; we establish its necessity and determine the tradeoff with the dilation threshold. The sharp extremal ratio in Theorem 1 supplies that reduction.

2 A log-concave kernel and sharp ratios

Put $S_n(t) = 1 + t + \dots + t^{n-1}$, and write

$$S_n(t)^{d+1} = \sum_{\ell=0}^D c_\ell t^\ell, \quad D = (d+1)(n-1),$$

with $c_\ell = 0$ outside this interval. The standard coefficient extraction formula [1, Lemma 3.2] gives

$$H_i(n) = \sum_{j=0}^d h_j c_{in-j}. \quad (5)$$

We recall two elementary facts about this kernel. Its coefficients are positive on their support, symmetric, and log-concave. For completeness, convolution preserves log-concavity for nonnegative sequences with no internal zeros: their Toeplitz matrices have nonnegative minors of order two, and the Cauchy–Binet formula preserves this property under matrix multiplication, which represents convolution. Applying this to the constant sequence of n ones proves the assertion. The Toeplitz-minor criterion itself follows by multiplying the nonincreasing adjacent ratios of a log-concave sequence; zeros outside the interval of support cause no negative minors.

The coefficients are also strictly increasing to the middle, with only the possible equality of the two central terms. One can verify this without a strict convolution theorem. For two factors the sequence is triangular. If $c^{(r)}$ denotes the coefficients of S_n^r , then

$$c_k^{(r)} - c_{k-1}^{(r)} = c_k^{(r-1)} - c_{k-n}^{(r-1)}.$$

For $r \geq 3$ and $1 \leq k \leq \lfloor r(n-1)/2 \rfloor$, the index k lies in the preceding support and is strictly closer to its middle than $k-n$, or the latter index is negative. Indeed, $2k-n < (r-1)(n-1)$. Induction proves the strict assertion, also recorded in [1, Lemma 4.3].

Lemma 3. *For a positive log-concave sequence on an interval, and integers $a \geq b$ in that interval, the ratio c_{a+j}/c_{b+j} is nonincreasing as j increases while the denominator remains positive. A numerator beyond the upper support is interpreted as zero.*

Proof. When both numerators are positive, c_{k+1}/c_k is nonincreasing in k . Hence

$$\frac{c_{a+j+1}}{c_{a+j}} \leq \frac{c_{b+j+1}}{c_{b+j}}.$$

Cross multiplication gives the claim. After the numerator becomes zero it stays zero. $\square$

Proof of Theorem 1. Set $b = n - d - 1 \geq 0$. By symmetry and the elementary coefficients below degree n ,

$$c_{dn-j} = c_{b+j} = \binom{n-1+j}{d} > 0 \quad (0 \leq j \leq d).$$

For a fixed i , put $a = D - in$. Then $a - b = (d-i)n \geq 0$, and symmetry gives $c_{in-j} = c_{a+j}$. Lemma 3 therefore implies

$$\frac{c_{in-j}}{c_{dn-j}} \leq \frac{c_{in}}{c_{dn}} = \frac{A_{d,i}(n)}{B_d(n)}.$$

For $i = 0$, the same conclusion holds with zero numerators for $j > 0$. Multiplying by the nonnegative weights $h_j c_{dn-j}$ and summing proves the inequality. Formula (5) with $h = 1$ gives equality. $\square$

3 The remaining conclusions and their sharp threshold

Proposition 4. *For every nonnegative numerator and every integer $n \geq d+1$, $\mathcal{U}_n h$ has degree d , distinct negative real roots, and positive strictly log-concave coefficients. If $n \geq d+2$, it satisfies (1). Under (H), it also satisfies (2) for $n \geq d+2$.*

Proof. The degree follows from (5), since $H_d(n) \geq h_0 B_d(n) > 0$. Jochemko's theorem [3, Theorem 1.1], with denominator exponent $d+1$, applies because $n > d \geq \deg h$ and $h_0 > 0$. It gives distinct negative real roots. Positivity and strict log-concavity follow from their elementary symmetric functions and Newton's inequalities. In degree one the log-concavity conditions are empty.

For the alternating conclusions assume $n \geq d+2$. For (1), fix $0 \leq i \leq \lfloor (d-1)/2 \rfloor$ and $0 \leq j \leq d$. Reflecting the coefficient in $H_{d-i}(n)$ replaces its index by

$$v = (i+1)n - (d+1) + j, \quad u = in - j.$$

Then $v - u = n - d - 1 + 2j > 0$, $0 < v < D$, and $u + v = (2i+1)n - (d+1) < D$. If $u \geq 0$, strict unimodality and symmetry give $c_v > c_u$; if $u < 0$, the same inequality follows from positivity. Summing against h_j proves (1).

For the other half, decompose

$$h(t) = a(t) + b(t), \quad a(t) = t^d a(t^{-1}), \quad b(t) = t^{d+1} b(t^{-1}),$$

where

$$a_j = \sum_{\ell=0}^j h_\ell - \sum_{\ell=d+1-j}^d h_\ell \quad (0 \leq j \leq d).$$

Here $b_0 = 0$, $a_0 = h_0 > 0$, and (H) together with symmetry gives $a_j \geq 0$ for every j . Write $p(t) = a(t)S_n(t)^{d+1} = \sum_{\ell} p_\ell t^\ell$. Its symmetry degree is $(d+1)n - 1$. The contribution from bS_n^{d+1} , after extracting powers divisible by n , is symmetric of degree $d+1$. Consequently, for $1 \leq k \leq \lfloor d/2 \rfloor$,

$$H_k(n) - H_{d+1-k}(n) = p_{kn} - p_{kn-1} = \sum_{j=0}^d a_j (c_{kn-j} - c_{kn-1-j}).$$

Since $2kn \leq dn < (d+1)(n-1) = D$, every summand is nonnegative and the $j=0$ summand is positive. This proves (2). This decomposition argument is the one underlying [1, Section 4]. $\square$

For Ehrhart numerators, (H) follows from the usual Hibi inequalities; see [1, Example 2.7] or [4, inequality (1.4)]. For the simplex, $H_0(d+1) = H_d(d+1) = 1$. Thus any threshold for the full strict chain must satisfy $N \geq d+2$.

Proof of the equivalence in Theorem 2. Necessity follows by taking $P = \Delta_d$, including the last obstruction at $n = d+1$. For sufficiency, apply Proposition 4 and Theorem 1. The convergence of the roots to the roots of the Eulerian polynomial is the established Beck–Stapledon asymptotic theorem, which is unchanged by the finite cutoff. The same arguments apply to all nonnegative numerators, imposing (H) for the alternating inequalities. $\square$

4 Finite formulas for every optimal choice

We give finite formulas rather than leave an optimization over all dilations implicit. All polynomials used below have rational coefficients and are explicitly given by (3)–(4).

For a nonconstant polynomial $q(x) = \sum_{k=0}^{\ell} q_k x^k$, $q_{\ell} \neq 0$, define the integer root bound

$$\mathcal{C}(q) = 1 + \left\lceil \max_{0 \leq k < \ell} \left| \frac{q_k}{q_{\ell}} \right| \right\rceil.$$

Set $\mathcal{C}(q) = 0$ for a constant or zero polynomial. Every root of a nonconstant q has modulus strictly less than $1 + \max_{k < \ell} |q_k/q_{\ell}|$, the standard Cauchy bound. For example, this follows by comparing the leading term with the geometric sum of the other terms when the modulus is at least that quantity (the monomial case is immediate).

Let

$$Q_{d,i} = A'_{d,i} B_d - A_{d,i} B'_d, \quad K_d = \max\{d+2, \mathcal{C}(Q_{d,0}), \dots, \mathcal{C}(Q_{d,d})\}.$$

For an integer $N \geq d+1$, put $L = \max\{N, K_d\}$, and define

$$S_d(N) = \max \left(\{M_d\} \cup \left\{ \frac{A_{d,i}(n)}{B_d(n)} : N \leq n \leq L, n \in \mathbb{Z}, 0 \leq i \leq d \right\} \right). \quad (6)$$

Theorem 5. *The least admissible multiplier at a prescribed threshold $N \geq d+2$ is*

$$m_d(N) = 1 + \lfloor S_d(N) \rfloor.$$

No multiplier $m \leq M_d$ is eventually admissible. For $m \geq M_d + 1$, put

$$P_{d,m,i} = m B_d - A_{d,i}, \quad C_d(m) = \max\{d+2, \mathcal{C}(P_{d,m,0}), \dots, \mathcal{C}(P_{d,m,d})\}.$$

The least admissible threshold at that multiplier is the finite integer

$$N_d(m) = \max \left(\{d+2\} \cup \{n+1 : d+2 \leq n \leq C_d(m), n \in \mathbb{Z}, P_{d,m,i}(n) \leq 0 \text{ for some } i\} \right). \quad (7)$$

Proof. The leading coefficient of $A_{d,i}$, for $i \geq 1$, is $E(d,i)/d!$, whereas that of B_d is $1/d!$. This is the usual Eulerian limit [1, Lemma 3.3]. Thus $A_{d,i}(x)/B_d(x) \rightarrow E(d,i)$ as $x \rightarrow \infty$; the limit for $i = 0$ is zero.

Beyond K_d , each nonconstant ratio has derivative of constant nonzero sign, and any remaining ratio is constant. Its supremum on the integer tail is therefore bounded by its value at L and its limit. This proves that (6) is exactly the supremum of all the required ratios.

We must check strict inequalities carefully: an integer supremum that is not attained would require separate treatment. For $d = 1$, the supremum is 1, attained by $A_{1,1}/B_1$. For $d \geq 2$, choose $i < d$ with $E(d,i) = M_d$. The two leading terms of (x^+_d) , followed by [1, Lemma 3.3], give

$$\frac{A_{d,i}(x)}{B_d(x)} = E(d,i) + \frac{d(d+1)}{2x} (E(d,i) + E(d-1,i) - E(d-1,i-1)) + O(x^{-2}). \quad (8)$$

Eulerian numbers outside their natural range are zero. The recurrence

$$E(d,i) = iE(d-1,i) + (d+1-i)E(d-1,i-1)$$

shows that the bracket in (8) is $(i+1)E(d-1,i) + (d-i)E(d-1,i-1) > 0$. Thus a ratio exceeds M_d arbitrarily far out. Since every ratio has limit at most M_d , the supremum is attained at a finite integer. The strict integer multiplier is therefore $1 + \lfloor S_d(N) \rfloor$. This also

proves that $m \leq M_d$ is impossible; for $d = 1$, the strict comparison $H_d < mH_d$ alone excludes $m = 1$.

If $m \geq M_d + 1$, each $P_{d,m,i}$ has degree d and positive leading coefficient. Beyond its Cauchy root bound it is positive, so every failing integer $n \geq d + 2$ lies in the finite interval used in (7). The last failure plus one, together with the necessary cutoff $d + 2$, is precisely the least threshold. This also proves existence of a threshold for $m = M_d + 1$. Finally, (6) is finite when $N = d + 2$, proving existence of a multiplier at the least dilation cutoff. $\square$

These formulas characterize all admissible pairs, not only two separately minimized constants. They use finite arithmetic and coefficient bounds in one variable; they do not require enumeration of lattice polytopes or an oracle for Ehrhart realizability. They make no efficiency claim about the size of the Cauchy bounds. For the general-numerator root/ratio conclusions alone, the same multiplier formula is valid for $N \geq d + 1$. Its least threshold is given by (7) after replacing both occurrences of $d + 2$ in that formula by $d + 1$, retaining the same root bound $C_d(m)$. Thus this version of the constants is also determined.

5 Exact small-dimensional values

Table 1 illustrates the distinction between the two optimizations.

d	$M_d + 1$	$N_d(M_d + 1)$	$m_d(d + 2)$
1	2	3	2
2	2	9	5
3	5	28	18
4	12	147	157
5	67	999	1313
6	303	7196	19217
7	2417	67665	250778
8	15620	606406	5048201

Table 1: The optimal eventual multiplier, its least dilation threshold, and the optimal multiplier at the smallest dilation threshold.

For $d = 2$,

$$A_{2,1}(n) = \frac{n^2 + 3n - 4}{2}, \quad B_2(n) = \frac{n^2 - 3n + 2}{2}.$$

At $m = 2$, the difficult difference is $2B_2 - A_{2,1} = (n - 1)(n - 8)/2$, giving threshold 9. At $N = 4$, the ratio is 4, so strict inequality requires $m \geq 5$; this multiplier works for all $n \geq 4$.

The accompanying standalone Python verifier builds the integer polynomials $d!A_{d,i}$ and $d!B_d$ directly from products of linear factors. For each displayed pair (m, N) , it expands $d!P_{d,m,i}(N + y)$. Every constant coefficient is positive and every other coefficient is nonnegative, proving the inequalities for all real $y \geq 0$, not just sampled integers. For $N_d(M_d + 1) > d + 2$, an exact failure at $N - 1$ proves minimality. For $m_d(d + 2)$, the exact simplex ratio at $n = d + 2$ proves minimality. In dimension one the obstruction at $n = 2$ is the strict alternating inequality. An additional finite check independently compares coefficient convolution with the forward-difference formula in dimensions 1 through 12. Those finite checks supplement the dimension-uniform proofs above.

AI tools assisted with literature searches, proof development, exact computation, and drafting. The accompanying computations verify the finite certificates; the general statements are justified by the written proofs and the cited theorems.

References

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