

Failures of Theorem 1 and Conjecture 3 in Knudson’s Persistence–Gradient Abstract, and a Corrected Form of Conjecture 2 after Kriebel

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Abstract

In an Oberwolfach abstract, Knudson considered the discrete vector field V_P formed by the incident pairs of the persistence pairing P of a simplexwise filtration. He stated that V_P is a discrete gradient (his Theorem 1), conjectured a single gradient path from τ to σ in V_P for every non-incident pair $\{\sigma, \tau\}$ of P (Conjecture 2), and conjectured that a filtration by a function decreasing along the modified Hasse diagram reproduces P (Conjecture 3). Conjecture 2 was first refuted publicly by Kriebel (Zenodo, September 2026) with a filtration of 7 simplices; we credit this and add that there are exactly two such filtrations with 7 simplices, none with fewer, that no filtration with at most 13 simplices has a pair joined by two or more paths, and examples with two and three paths on 17 and 19 simplices. The parts of this note that we did not find elsewhere are the following two. Theorem 1 is false, also in the abstract’s own formulation by the modified Hasse diagram: a filtration of 17 simplices has a closed V -path, and for every filtration of every complex with at most 16 simplices V_P is a gradient (a short lemma and an exhaustive search over 2,484,335,648 filtrations). Conjecture 3 is false even when V_P is acyclic, for every choice of the function and of tie-breaking: a filtration of 17 simplices with a hand-checkable proof; it holds for all 4,005,434 filtrations with at most 11 simplices. Finally, we prove a corrected form of Conjecture 2, which is a consequence of algebraic Morse theory: for the earliest non-incident pair, if the field formed by the earlier pairs is a gradient, the number of gradient paths in it from τ to σ is odd, and σ is the youngest critical simplex reached an odd number of times. This is an unrefereed note.

1 Introduction

Let K be a finite simplicial complex with a filtration $\emptyset = K_{-1} \subset K_0 \subset \cdots \subset K_n = K$ in which each K_{i+1} arises from K_i by adding one simplex. Knudson’s abstract [Knu08] relates the persistence pairing P of Edelsbrunner, Letscher and Zomorodian [ELZ02] to Forman’s discrete Morse theory [For98]. The *incident* pairs of P , those $\{\sigma, \tau\}$ with σ a facet of τ , form a discrete vector field V_P . The abstract makes three claims about it, which we number as follows.

(K1) (Theorem 1, p. 1629.) V_P is a gradient vector field, i.e. has no closed V -path.

(K2) (Conjecture 2, p. 1629.) If $\{\sigma, \tau\} \in P$ and σ is not a face of τ , there is a single gradient path from τ to σ in V_P .

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(K3) (Conjecture 3, p. 1630.) Reverse the arrows of V_P in the Hasse diagram of K to get a directed graph G . For an integer function f decreasing strictly along directed paths of G , the subcomplexes generated by the sublevel sets of f form a filtration (several simplices may enter at once and ties are ordered somehow) with a persistence pairing $P_{V_P, f}$. For the correct choice of f , $P_{V_P, f} = P$.

(K2) and (K3) are the records OWR-2040-002 and OWR-2040-003 of the UnsolvedMath corpus [ula26]. The author’s slides [Knu13] repeat all three, call the picture proof of (K1) a “gross oversimplification”, and list the definition of the pairing P_V of (K3) as work still to be done.

Results. (A) *(K2) is false*; this was first made public by Kriebel [Kri26] with the 7-simplex filtration Π_7 of Proposition 3.1, and we claim no priority for it. We add that exactly two filtrations with 7 simplices violate (K2), Π_7 and a second one, S_7 , which is *not* Kriebel’s; that none exists with fewer; that no filtration with at most 13 simplices has a pair with two or more paths; and examples with two and three paths on 17 and 19 simplices (Section 3). (B) *(K1) is false*: a filtration of 17 simplices has a closed V -path, 17 is smallest possible, and G has a directed cycle (Section 4). (C) *(K3) is false even when V_P is acyclic*, for every f and every tie-breaking: a filtration of 17 simplices with a short hand proof (Section 5); it holds for all filtrations with at most 11 simplices. (D) *A corrected statement is true*: for the earliest non-incident pair, if the field of the earlier pairs is a gradient, the number of gradient paths in it is odd and σ is the youngest critical simplex reached an odd number of times (Section 6); this is a consequence of algebraic Morse theory, and “exactly one” still fails. Every counterexample is given by its filtration and pairing, with the reductions written out. The note concerns statements of a workshop abstract: two are posed as conjectures and the proof sketch of the third is called an oversimplification by its author. We claim no more than correcting the record.

2 Definitions and conventions

Filtrations and the pairing P . Complexes are finite abstract simplicial complexes; $\alpha < \beta$ means that α is a facet (codimension-one face) of β , and $x \leq y$ that x is a face of y . A *filtration* is a total order $\sigma_1 \prec \cdots \prec \sigma_N$ of the simplices in which every simplex follows its facets; $K_i = \{\sigma_1, \dots, \sigma_i\}$; smaller means *older*. Coefficients are in $\mathbb{Z}/2$. Reduce the boundary matrix (indexed by the simplices in the order $\prec$) from left to right: while column j is non-zero and its youngest entry is also the youngest entry of an earlier reduced column R_k , add R_k to column j . If the result $R_j \neq 0$ has youngest entry σ , then $\{\sigma, \sigma_j\} \in P$, σ_j is *negative* and σ *positive*; if $R_j = 0$, σ_j is positive. A positive simplex in no pair is *essential*. This is the pairing of [ELZ02], computed as in [ZC05] and described in [Knu08, pp. 1628–1629]; in a pair (σ, τ) the older simplex comes first and $\dim \sigma = \dim \tau - 1$. Our programs also compute P by the rank formula as a check.

Lemma 2.1. *Let $\sigma_1 \prec \cdots \prec \sigma_N$ be a filtration with reduced columns R_j . (a) The non-zero R_j have pairwise distinct youngest entries. (b) If $\tau = \sigma_j$ is negative, then $R_j \in \partial\tau + \text{span}\{\partial\tau' : \tau' \prec \tau, \dim \tau' = \dim \tau\}$, and every chain in this coset has its youngest simplex at least as young as that of R_j . (c) The youngest simplex of a non-zero cycle is positive. (d) The pairing of K_i consists of the pairs of P inside K_i .*

Proof. (a) holds by construction. (b) Each $R_{j'}$ ($j' < j$) is $\partial\sigma_{j'}$ plus older boundaries, so the spans of the $R_{j'}$ and of the $\partial\sigma_{j'}$ ($j' < j$) coincide; an element of the coset is $R_j + \sum_{j' \in S} R_{j'}$, whose youngest entry is the youngest of the distinct youngest entries of the summands. (c) If $z = \sigma_j + \sum x$ is a cycle with all x older, then $\partial\sigma_j = \sum \partial x$, so $R_j = 0$. (d) Column j depends only on columns $\leq j$. $\square$

Vector fields and paths. A *discrete vector field* V is a set of disjoint pairs $\{\alpha < \beta\}$; α is *lower*, β *upper*, the other simplices are *critical*. A V -*path* is a sequence $\alpha_0 < \beta_0 > \alpha_1 < \beta_1 > \cdots > \alpha_{r+1}$ with all $\{\alpha_i < \beta_i\} \in V$; it is *closed* if $\alpha_{r+1} = \alpha_0$, and V is a *gradient* if there is no closed V -path. We use the standard condition $\alpha_{i+1} \neq \alpha_i$, which the slides [Knu13] print and the definition in [Knu08] omits; read literally, every non-empty V would contain the closed path $\alpha < \beta > \alpha$ and (K1) would fail trivially. None of our counterexamples uses the omission.

Put $V_P = \{\{\sigma < \tau\} : \{\sigma, \tau\} \in P\}$. A pair of P with σ not a face of τ is *non-incident*; both simplices are critical for V_P . Since τ is critical, the abstract's notion of V -path does not literally cover a path “from τ to σ ”. We use the evident reading, which is also that of the example in [Knu13]: a *gradient path from τ to σ in V* is a sequence

$$\tau > \alpha_1 < \beta_1 > \alpha_2 < \beta_2 > \cdots < \beta_r > \sigma, \quad r \geq 1,$$

with all $\{\alpha_i < \beta_i\} \in V$ and $\alpha_{i+1} \neq \alpha_i$. Write $N_V(\tau \rightarrow \sigma)$ for their number. “A single path” means $N_V = 1$; we exhibit $N_V = 0$ (existence fails) and $N_V = 2, 3$ (uniqueness fails).

The graph G and the filtration by f . For a vector field V let $G = G(V)$ be the directed graph on the simplices with an arrow $\beta \rightarrow \alpha$ for every facet pair $\alpha < \beta$ not in V and an arrow $\alpha \rightarrow \beta$ for every $\{\alpha < \beta\} \in V$ (the Hasse diagram with the arrows of V reversed [Knu08]; compare Chari [Cha00]). Directed cycles of G are exactly the closed V -paths. A function f on the simplices is *V -decreasing* if it decreases strictly along every arrow; it exists iff G is acyclic. The subcomplex K_i^f generated by $f^{-1}(-\infty, i]$ consists of the simplices x with $\ell_f(x) \leq i$, where $\ell_f(x) = \min\{f(y) : y \geq x\}$ is the *level* of x . Several simplices may enter at one level, and [Knu08] orders them in some unspecified way in order to speak of a pairing $P_{V,f}$. We read this in the weakest possible way: $P_{V,f}$ is the pairing of *any* filtration L of K in which every K_i^f is an initial segment (a *total refinement* of K^f). So (K3) says: there are a V_P -decreasing f and a total refinement L of K^f with pairing P . Every stronger reading (an injective f , a fixed tie-breaking, the two simplices of a pair adjacent in L) only adds requirements, so a counterexample to this reading is one to all of them.

3 Conjecture (K2): zero, two and three paths

Here simplices are written by their vertices (23 is the edge $\{2, 3\}$) and a filtration as the list of its simplices.

Proposition 3.1. *Let $S_7 = (0, 1, 2, 3, 23, 13, 03)$ and $\Pi_7 = (0, 1, 2, 3, 23, 13, 02)$. For both, the vertex 0 is essential and*

$$P = \{(3, 23), (2, 13), (1, x)\}, \quad x = 03 \text{ resp. } 02, \quad V_P = \{\{3 < 23\}\}.$$

The pair $\{2, 13\}$ has exactly one gradient path, $13 > 3 < 23 > 2$, but there is no gradient path from x to 1, not even a directed path in $G(V_P)$. So (K2) fails for S_7 and for Π_7 .

Proof. The vertices are positive. The reductions are (youngest entry last):

step	σ	$\partial\sigma$	reduction	result
5	23	$\{2, 3\}$	–	pair (3, 23)
6	13	$\{1, 3\}$	$+R(23) \rightarrow \{1, 2\}$	pair (2, 13)
7	03	$\{0, 3\}$	$+R(23) \rightarrow \{0, 2\}; +R(13) \rightarrow \{0, 1\}$	pair (1, 03)
7	02	$\{0, 2\}$	$+R(13) \rightarrow \{0, 1\}$ (Π_7 only)	pair (1, 02)

Only $(3, 23)$ is incident. The facets of 13 are the critical vertex 1 and 3, the lower element of $\{3 < 23\}$; then $23 > 2$ ends at the critical vertex 2: this is the single path to 2. The facets of x are 0, 3 resp. 0, 2; the vertices 0, 2 are critical and 3 leads via 23 to 2, so every maximal path from x ends at 0 or 2, never at 1. In $G(V_P)$ the vertices reachable from x are $\{x, 0, 3, 23, 2\}$ resp. $\{x, 0, 2\}$. $\square$

Remark 3.2. Π_7 is Kriebel's filtration a, b, c, d, cd, bd, ac of a path with $(a, b, c, d) = (0, 1, 2, 3)$ [Kri26]. The star S_7 is the other 7-simplex filtration violating (K2) and is *not* Kriebel's. That deposit treats (K2) only: it makes no minimality statement and does not treat (K1) or (K3). It also shows that the example works over every field, and observes that after reversing the unique path of the earlier pair (c, bd) the missing path from ac to b appears, an instance of the sequential behaviour described by Observation A in Section 6.

Proposition 3.3. *Enumerate all filtrations with n simplices, naming the vertices $0, 1, \dots$ in order of first appearance. Table 1 gives the numbers of filtrations, of those with a non-incident pair, of those with a non-incident pair having no gradient path in V_P , and of those with a non-incident pair having at least two. No filtration with $n \leq 13$ simplices has a closed V_P -path. Hence (a) 7 is the least number of simplices of a counterexample to (K2), and S_7, Π_7 are the only ones with 7 simplices; (b) no filtration with at most 13 simplices has a pair with two or more gradient paths.*

n	filtrations	with a non-incident pair	with a pair of 0 paths	with a pair of ≥ 2 paths
7	522	81	2	0
8	3 713	755	24	0
9	31 773	8 713	441	0
10	317 341	104 668	6 372	0
11	3 651 970	1 421 676	108 130	0
12	47 936 237	21 148 822	1 854 016	0
13	709 593 015	347 498 799	35 028 376	0

Table 1: Exhaustive enumeration (computation). For $n = 1, \dots, 6$ there are 1, 1, 2, 5, 19, 87 filtrations, with 0, 0, 0, 0, 1, 7 having a non-incident pair and none a pair with 0 or ≥ 2 paths.

Method. Every prefix of a filtration is a filtration, so a depth-first search that appends a new vertex or a simplex whose facets are present visits every filtration with n simplices at depth n ; the reduction is updated incrementally. Rows $n \leq 11$ were computed by a Python program and by two independently written C++ programs, rows $n = 12, 13$ by the two C++ programs (see the Verification paragraph in Section 6 for the count at $n = 13$). $\square$

Proposition 3.4. (a) Let $M_{17} = (4, 5, 3, 0, 35, 1, 14, 34, 05, 01, 13, 03, 013, 04, 035, 014, 034)$ (5 vertices, 8 edges, 4 triangles). Then

$$P = \{(3, 35), (1, 14), (5, 34), (0, 05), (03, 013), (13, 035), (04, 014), (01, 034)\},$$

$V_P = \{\{3 < 35\}, \{1 < 14\}, \{0 < 05\}, \{03 < 013\}, \{04 < 014\}\}$ is acyclic, the non-incident pairs are $\{5, 34\}$ and $\{13, 035\}$ (one path each) and $\{01, 034\}$, which has two paths:

$$034 > 04 < 014 > 01, \quad 034 > 03 < 013 > 01.$$

(b) Let $M_{19} = (3, 4, 2, 0, 1, 23, 13, 01, 02, 12, 04, 24, 14, 024, 012, 124, 34, 234, 134)$ (5 vertices, 9 edges, 5 triangles). Then

$$P = \{(2, 23), (1, 13), (0, 01), (4, 04), (24, 024), (12, 012), (14, 124), (34, 234), (02, 134)\},$$

V_P consists of the first eight pairs and is acyclic, and the only non-incident pair $\{02, 134\}$ has three paths:

$$\begin{aligned} 134 > 34 < 234 > 24 < 024 > 02, \quad 134 > 14 < 124 > 24 < 024 > 02, \\ 134 > 14 < 124 > 12 < 012 > 02. \end{aligned}$$

These are the smallest examples with two and three paths that we found. By Proposition 3.3(b) none exists with at most 13 simplices, so the least size for two paths lies in $[14, 17]$ and for three paths in $[14, 19]$.

Proof. The reductions are in Appendix A. (a) The facets of 034 are 03 (lower in $\{03 < 013\}$), 04 (lower in $\{04 < 014\}$) and 34 (critical, the vertex 5 being its partner); the facets of 013 are 01, 13, 03 and of 014 are 14, 01, 04, where 14 is an upper element (a dead end), 13 is critical and 01 is the target. This gives exactly the two paths. The paths for $\{5, 34\}$ and $\{13, 035\}$ are $34 > 3 < 35 > 5$ and $035 > 03 < 013 > 13$, found the same way. (b) The facets of 134 are 13 (upper, a dead end), 14 and 34, lower in $\{14 < 124\}$ and $\{34 < 234\}$; the facets of 124 are 12, 14, 24 and of 234 are 23, 24, 34, where 23 is upper and 12, 24 are lower in $\{12 < 012\}$, $\{24 < 024\}$; 024 has the facet 02 (target) and 012 has 02 (target). This gives the three paths. $\square$

4 Theorem (K1) is false

Proposition 4.1. *Let $T_{17} = (0, 1, 2, 3, 4, 03, 04, 12, 13, 14, 23, 24, 34, 124, 234, 034, 134)$ (5 vertices, 8 edges, 4 triangles). Its pairing is*

$$P = \{(3, 03), (4, 04), (2, 12), (1, 13), (24, 124), (34, 234), (23, 034), (14, 134)\},$$

the vertex 0 is essential and the only non-incident pair is $(23, 034)$. So $V_P \ni \{14 < 134\}, \{34 < 234\}, \{24 < 124\}$ and

$$14 < 134 > 34 < 234 > 24 < 124 > 14$$

is a closed V -path $(34 < 134, 24 < 234, 14 < 124)$. Hence V_P is not a gradient. In $G(V_P)$ this is the directed cycle $14 \rightarrow 134 \rightarrow 34 \rightarrow 234 \rightarrow 24 \rightarrow 124 \rightarrow 14$, so no function f as in (K3) exists: (K1) fails also in the abstract's own formulation.

Proof. The vertices are positive. The remaining reductions are (youngest entry last):

step	σ	$\partial\sigma$	reduction	result
6	03	$\{0, 3\}$	–	pair (3, 03)
7	04	$\{0, 4\}$	–	pair (4, 04)
8	12	$\{1, 2\}$	–	pair (2, 12)
9	13	$\{1, 3\}$	$+R(03) \rightarrow \{0, 1\}$	pair (1, 13)
10	14	$\{1, 4\}$	$+R(04) \rightarrow \{0, 1\}; +R(13) \rightarrow \emptyset$	positive
11	23	$\{2, 3\}$	$+R(03) \rightarrow \{0, 2\}; +R(12) \rightarrow \{0, 1\}; +R(13) \rightarrow \emptyset$	positive
12	24	$\{2, 4\}$	$+R(04) \rightarrow \{0, 2\}; +R(12) \rightarrow \{0, 1\}; +R(13) \rightarrow \emptyset$	positive
13	34	$\{3, 4\}$	$+R(04) \rightarrow \{0, 3\}; +R(03) \rightarrow \emptyset$	positive
14	124	$\{12, 14, 24\}$	–	pair (24, 124)
15	234	$\{23, 24, 34\}$	–	pair (34, 234)
16	034	$\{03, 04, 34\}$	$+R(234) \rightarrow \{03, 04, 23, 24\};$ $+R(124) \rightarrow \{03, 04, 12, 14, 23\}$	pair (23, 034)
17	134	$\{13, 14, 34\}$	$+R(234) \rightarrow \{13, 14, 23, 24\}; +R(124) \rightarrow \{12, 13, 23\};$ $+R(034) \rightarrow \{03, 04, 13, 14\}$	pair (14, 134)

The rest is read off from P . The three spokes 14, 24, 34 at the vertex 4 are positive edges; this is possible because 4 is attached to the rest also by the negative edge 04, which is not on the loop. $\square$

Lemma 4.2. *If some filtration of a complex K has a closed V_P -path, then $|K| \geq 16$. If $|K| = 16$, then K is connected, has exactly 5 vertices and at least three triangles.*

Proof. Take a closed path $\alpha_0 < \beta_0 > \alpha_1 < \cdots > \alpha_m = \alpha_0$ with $\alpha_0, \dots, \alpha_{m-1}$ distinct; the $\beta_i = V_P(\alpha_i)$ are then distinct and all α_i have one dimension k . Let C be the component of K containing the path; the pairing inside C is that of the filtration restricted to C (the boundary matrix is block diagonal). The α_i are positive, the β_i negative. $m \geq 3$: for $m = 2$, $\beta_0 \neq \beta_1$ would both contain the distinct k -faces α_0, α_1 , which span a unique $(k+1)$ -simplex. $k \geq 1$: for $k = 0$ the β_i would be negative edges forming a cycle, but negative edges form a forest (an edge is negative iff it joins two components of the graph built so far). $k \geq 2$: β_0, β_1 are distinct $(k+1)$ -simplices sharing the k -face α_1 ; they generate $3 \cdot 2^{k+1} - 1 \geq 23$ simplices. $k = 1$: the α_i are positive edges, the β_i negative triangles. If C has n_v vertices, exactly $n_v - 1$ of its edges are negative, so C has $\geq n_v - 1 + m$ edges and $\geq m$ triangles. If $m \geq 4$, then $n_v \geq 5$ (for $n_v \leq 4$ at least $n_v - 1 + m \geq 7 > 6$ edges would be needed) and $|C| \geq n_v + (n_v - 1 + m) + m \geq 17$. If $m = 3$, the edges $\alpha_0, \alpha_1, \alpha_2$ pairwise lie in a common triangle, so pairwise meet. They are not the sides of one triangle (then $\beta_0 = \beta_1 = \beta_2$), so $\alpha_i = \{v, u_i\}$ and $\beta_i = \{v, u_i, u_{i+1}\}$, and the three edges $u_i u_{i+1}$ lie in C . They form a cycle, so one of them is positive, and it is not an α_i : C has ≥ 4 positive edges. The first edge at v in the filtration is negative (v is still isolated), hence not a spoke, so v has a neighbour outside $\{u_0, u_1, u_2\}$ and $n_v \geq 5$. Thus $|C| \geq n_v + (n_v - 1) + 4 + 3 = 2n_v + 6 \geq 16$.

So $|C| \geq 16$ always, with equality only for $k = 1$, $m = 3$, $n_v = 5$; if $|K| = 16$ then $C = K$. $\square$

Theorem 4.3. *For every filtration of every simplicial complex with at most 16 simplices, V_P is a gradient. Hence T_{17} is a counterexample to (K1) with the least possible number of simplices.*

Proof. By Lemma 4.2 it suffices to consider complexes on at most five vertices with at least three triangles and at most 16 simplices (only 16 simplices on 5 vertices can occur, but we ran all sizes). There are 17 isomorphism classes, 1, 2, 6, 8 of sizes 13, 14, 15, 16. For each class we enumerated *all* filtrations (linear extensions of the face order), 2,484,335,648 in total, computed P and V_P afresh and tested for a closed V -path: none was found. (Computation by a C++ program; a second, independently written C++ program gave the same total and the same count for every class.) As a positive control, the complex of Proposition 4.1 has 1,412,014,592 filtrations, 437,696 of which have a closed V_P -path. $\square$

Remark 4.4. Some parts of (K1) hold. If all pairs have dimension $(0, 1)$, a closed path would consist of negative edges forming a cycle, so (K1) holds for graphs. Apparent pairs (σ the youngest facet of τ and τ the oldest cofacet of σ) belong to P and form a gradient [Bau21, Lemmas 3.3 and 3.5]. In the reduction basis all pairs of P form an *algebraic* gradient [BR24, Prop. 13], a replacement of (K1) that holds in general. By Lemma 4.2 and Theorem 4.3, a complex with exactly 17 simplices admitting a closed V_P -path has 5 vertices and ≥ 3 triangles; of the 11 isomorphism classes of such complexes only that of T_{17} does (computation over all 30,582,025,120 filtrations of the 11 classes).

5 Conjecture (K3) is false, even when V_P is acyclic

Theorem 5.1. *Let $B_{17} = (4, 1, 0, 14, 3, 13, 01, 2, 23, 02, 04, 12, 24, 124, 024, 123, 012)$ (5 vertices, 8 edges, 4 triangles). Then*

$$P = \{(1, 14), (3, 13), (0, 01), (2, 23), (24, 124), (12, 024), (04, 123), (02, 012)\},$$

the vertex 4 is essential, $V_P = \{\{1 < 14\}, \{3 < 13\}, \{0 < 01\}, \{2 < 23\}, \{24 < 124\}, \{02 < 012\}\}$ is acyclic, and the non-incident pairs are $(12, 024)$ and $(04, 123)$. There is no V_P -decreasing function f

and no total refinement L of K^f whose pairing is P . So $(K3)$ fails in the weakest reading of $P_{V_P, f}$ described in Section 2, hence for every f , every tie-breaking and every stronger reading.

Proof. The pairing. The vertices are positive; the other reductions are (youngest entry last):

step	σ	$\partial\sigma$	reduction	result
4	14	$\{4, 1\}$	–	pair (1, 14)
6	13	$\{1, 3\}$	–	pair (3, 13)
7	01	$\{1, 0\}$	–	pair (0, 01)
9	23	$\{3, 2\}$	–	pair (2, 23)
10	02	$\{0, 2\}$	$+R(23) \rightarrow \{0, 3\}; +R(13) \rightarrow \{1, 0\}; +R(01) \rightarrow \emptyset$	positive
11	04	$\{4, 0\}$	$+R(01) \rightarrow \{4, 1\}; +R(14) \rightarrow \emptyset$	positive
12	12	$\{1, 2\}$	$+R(23) \rightarrow \{1, 3\}; +R(13) \rightarrow \emptyset$	positive
13	24	$\{4, 2\}$	$+R(23) \rightarrow \{4, 3\}; +R(13) \rightarrow \{4, 1\}; +R(14) \rightarrow \emptyset$	positive
14	124	$\{14, 12, 24\}$	–	pair (24, 124)
15	024	$\{02, 04, 24\}$	$+R(124) \rightarrow \{14, 02, 04, 12\}$	pair (12, 024)
16	123	$\{13, 23, 12\}$	$+R(024) \rightarrow \{14, 13, 23, 02, 04\}$	pair (04, 123)
17	012	$\{01, 02, 12\}$	$+R(024) \rightarrow \{14, 01, 04\}; +R(123) \rightarrow \{13, 01, 23, 02\}$	pair (02, 012)

V_P is acyclic. The pairs $\{1 < 14\}, \{3 < 13\}, \{0 < 01\}, \{2 < 23\}$ use the edges 14, 13, 01, 23, which form a tree, so there is no closed path in dimension 0. For $\{24 < 124\}$ and $\{02 < 012\}$: the other facets of 124 are 12 (critical) and 14 (upper), those of 012 are 12 and 01 (upper), so no path continues.

Hand proof. Suppose f is V_P -decreasing and L is a total refinement of K^f with pairing P . We use $\partial 124 = \{12, 14, 24\}$, $\partial 024 = \{02, 04, 24\}$, $\partial 123 = \{12, 13, 23\}$, $\partial 012 = \{01, 02, 12\}$.

F1: 12 precedes 02 and 24 in L . The arrows give $f(02) > f(012) > f(12)$ (the first as $\{02 < 012\} \in V_P$, the second as $\{12 < 012\} \notin V_P$) and $f(024) > f(02)$. The cofaces of 02 are 02, 012, 024, all with f -value $> f(12)$; so $02 \notin K_{f(12)}^f$, while $12 \in K_{f(12)}^f$, an initial segment of L . Hence $12 \prec 02$. Likewise $f(24) > f(124) > f(12)$ and $f(024) > f(24)$, the cofaces of 24 are 24, 124, 024, and $12 \prec 24$.

F2: $024 \prec 123$. As $(04, 123) \in P$, the reduced column of 123 has youngest entry 04 and lies in $\partial 123 + \text{span}\{\partial T : T \prec 123\}$ (Lemma 2.1(b)). The edge 04 lies in $\partial 024$ only and not in $\partial 123$, so $\partial 024$ is used: $024 \prec 123$.

F3: $123 \prec 024$. As $(12, 024) \in P$, the reduced column R of 024 has youngest entry 12 and lies in $\partial 024 + a\partial 124 + g\partial 012 + d\partial 123$, $a, g, d \in \{0, 1\}$, where a coefficient may be 1 only if that triangle precedes 024. All other entries of R are older than 12; by F1 neither 02 nor 24 is. The coefficient of 02 is $1 + g$ and that of 24 is $1 + a$, so $a = g = 1$ and $R = \{01, 04, 14\} + d\{12, 13, 23\}$. Since $12 \in R$, $d = 1$, so $123 \prec 024$.

F2 and F3 contradict each other. □

Remark 5.2. Computer checks, independent of the proof. B_{17} has 1,412,014,592 filtrations (linear extensions of the face order). An unpruned enumeration finds 186 with all pairs of V_P adjacent (the strict reading); none has pairing P . A pruned enumeration finds exactly 1,852 filtrations with pairing P ; none of them satisfies F1 or has the pairs of V_P adjacent. A second example, $A_{17} = (2, 0, 3, 4, 34, 1, 02, 13, 04, 23, 14, 24, 12, 124, 123, 024, 234)$ with $V_P = \{\{0 < 02\}, \{1 < 13\}, \{4 < 34\}, \{12 < 124\}, \{23 < 234\}\}$ acyclic and non-incident pairs $(3, 04), (24, 123), (14, 024)$, has 302 filtrations with pairing P , none with the pairs of V_P adjacent. For A_{17} we have no hand proof, but a computation shows that none of the 302 is a total refinement of some K^f (none satisfies the necessary “block” condition proved in the reproducibility README), so A_{17} also fails in the weak reading. For B_{17} the Morse complex of V_P (critical simplices 4, 04, 12, 024, 123; counting V_P -paths mod 2 gives $\partial_M 024 = 04$ and $\partial_M 123 = 12$) would pair 024 with 04 and 123 with 12, whereas P is crossed; this heuristic is not used in the proof.

Proposition 5.3. *Every filtration with at most 11 simplices has an acyclic V_P and a filtration L of the same complex in which the pairs of V_P are adjacent (so L is a total refinement of some K^f) and whose pairing is P . Thus (K3) holds, even in the strict reading, for the 4,005,434 filtrations with at most 11 simplices (computation, by a Python program and an independently written C++ program). The smallest counterexample we found has 17 simplices; whether there is one with 12 to 16 simplices is open.*

6 A corrected statement

Let τ be the non-incident death of P that is earliest in $\prec$, σ its partner, $k = \dim \sigma$, K' the set of simplices preceding τ , and V' the set of pairs of P whose death precedes τ . By the choice of τ all pairs of V' are incident; V' is the pairing of K' (Lemma 2.1(d)), so its critical simplices in K' are the essential simplices of K' . If V_P is a gradient, so is $V' \subset V_P$. For a critical k -simplex c let $N'(c)$ be the number of gradient paths from τ to c in V' (a path $\tau > c$ of length zero counts).

Theorem 6.1. *Assume V' is a gradient. Then σ is the youngest critical k -simplex c of V' with $N'(c)$ odd. In particular $N_{V'}(\tau \rightarrow \sigma)$ is odd, hence ≥ 1 .*

Lemma 6.2. *Let y_1, y_2 be homologous k -cycles of a filtered complex whose youngest simplices c_1, c_2 are essential. Then $c_1 = c_2$.*

Proof. Let $R = DU$ be the reduction with U upper triangular and invertible, and U_p the column of U at p : $\partial U_p = R_p$, and U_p is p plus older simplices. For positive k -simplices p , U_p is a cycle with youngest simplex p ; these cycles are independent and as many as $\dim Z_k$, hence a basis of Z_k , and the youngest simplex of $\sum \lambda_p U_p$ is the youngest p with $\lambda_p = 1$. If a negative $(k+1)$ -simplex β has R_β with youngest entry p , then $R_\beta = \partial U_\beta = U_p + \sum_{q \prec p} \lambda_q U_q$ is a boundary, so $[U_p]$ lies in the span of the older $[U_q]$. By induction each $[U_p]$ is a combination of $[U_c]$, $c \preceq p$ essential. The number of essential k -simplices equals $\dim H_k$ (positive k -simplices minus negative $(k+1)$ -simplices), so the $[U_c]$ with c essential form a basis of H_k . A cycle with essential youngest simplex c has class $[U_c] + \text{older basis classes}$, so c is the youngest basis element in its expansion; homologous cycles have the same c . $\square$

Proof of Theorem 6.1. A k -simplex of K' is *lower*, *upper* or *critical* for V' ; upper simplices are negative. Let Γ be the directed graph on the k -simplices of K' with an arrow $a \rightarrow a'$ if a is lower with partner β and $a' \neq a$ is a facet of β . A directed cycle of Γ is a closed V' -path, so Γ is acyclic. Define ρ on k -chains by $\rho(a) = a + \partial\beta = \sum_{a \rightarrow a'} a'$ for lower a and $\rho(x) = x$ otherwise; then $x + \rho(x) \in \partial C_{k+1}(K')$. Since Γ is acyclic, $\rho^m = \rho^{m+1} =: \rho^\infty$ for large m , $\rho^\infty(x)$ contains no lower simplex, and the coefficient of a critical c in $\rho^\infty(x)$ is the number of directed paths from x to c in Γ , mod 2.

Let $z = \partial\tau = \sum_{a \prec \tau} a$, a k -cycle of K' . A directed path of Γ from a facet a of τ to c , preceded by the step $\tau > a$, is exactly a gradient path from τ to c in V' . So

$$z' = \rho^\infty(z) = \sum_{c \text{ critical}} N'(c) c + w,$$

with w a combination of upper simplices, and z' is a cycle of K' homologous to z . It is non-zero: otherwise $z = \partial y$ for a chain y of K' , and $\tau + y$ would be a cycle with youngest simplex τ , making τ positive (Lemma 2.1(c)). Its youngest simplex is positive, hence not upper: it is a critical simplex c^* , the youngest critical simplex with $N'(c^*)$ odd, and it is essential in K' .

The reduced column $R_\tau \in z + \partial C_{k+1}(K')$ is a cycle homologous to z whose youngest simplex σ is essential in K' (its partner τ is not in K'). By Lemma 6.2, $\sigma = c^*$. $\square$

Remark 6.3. (1) The map ρ^∞ is the elimination of lower simplices used to build the Morse complex in discrete and algebraic Morse theory [For98, Koz05, Sko06]; in that language the theorem says that over $\mathbb{Z}/2$ the incidence number of τ and σ in the Morse complex of V' is 1, which is what is needed to cancel them there. So Theorem 6.1 is a corollary of that theory and the persistence basis; we give the elementary proof for completeness. In a different setting, Bauer, Lange and Wardetzky show that on combinatorial surfaces a persistence pair can be cancelled along a unique path once its descendants are cancelled [BLW12, Lemma 9], and that this fails on non-manifold 2-complexes and in higher dimensions [BLW12, Sect. 1.1, 6.5] (numbering of arXiv:1001.1269v2). (2) “Odd” cannot be improved to “one”: in M_{19} the only non-incident pair is the earliest one, $V' = V_P$, and $N'(\sigma) = 3$. (3) The statement concerns V' , not V_P : in 60,000 random trials the number of V_P -paths for the earliest pair differed from N' in 23 of the 36,267 trials with a non-incident pair and an acyclic V' , and was even in 14 of them (computation only).

Tested, not proved. Two further observations could not be proved. *Observation A.* Let $\{\sigma, \tau\} \in P$ be non-incident (not necessarily earliest), V^τ the incident pairs of P with death before τ (assumed acyclic) and K^τ the simplices up to and including τ . Column reduction, in the order $\prec$, of the Morse boundary matrix of V^τ on the critical simplices of K^τ pairs τ with σ . *Observation B.* If V_P is acyclic and L is a filtration in which the pairs of V_P are adjacent, then the pairing of L is V_P together with the column-reduction pairing of the Morse boundary matrix of V_P on its critical simplices in the order L . Both were tested on random filtered complexes without counterexample (Verification paragraph); neither is used elsewhere.

Remark 6.4. An argument, not a computation. Suppose V_P were a gradient and every non-incident pair could be cancelled in turn along a unique path in the current field (Forman’s cancellation [For98], which keeps the field a gradient). For a $\mathbb{Z}/2$ -acyclic K the final gradient would have a single critical simplex, a vertex, so K would be collapsible. Triangulations of the dunce hat are contractible, hence $\mathbb{Z}/2$ -acyclic, but not collapsible [Zee63]. So for every filtration of every triangulation of the dunce hat, either V_P has a closed V -path or some non-incident pair cannot be cancelled in the successive sense; this is a form of the known failure of full cancellation outside surfaces [BLW12].

Summary and open questions. (K1) fails at 17 simplices and holds below; (K3) fails at 17 simplices and holds up to 11 (the threshold in between is unknown); (K2) fails at 7. Open: the least number of simplices of a filtration with a pair of two gradient paths ($14 \leq n \leq 17$), the threshold for (K3), and a proof of Observations A and B, which would give a sequential version of (K2).

Verification. The claims were checked as follows. (1) A self-contained Python program by the author, `verify_paper_examples.py` (standard library only, a few seconds), recomputes each explicit example from its filtration: the pairing by column reduction and by the rank formula (they agree), V_P , the non-incident pairs, path counts and lists, the closed path and the directed cycle of T_{17} , the acyclicity of G for B_{17} , the eight cosets of the hand proof, and all filtrations of B_{17} and A_{17} with pairing P (1,852 and 302). (2) Enumerations: Proposition 3.3 by Python ($n \leq 11$) and C++ ($n \leq 13$); Theorem 4.3 by a Python search and two C++ programs; Remark 5.2 by a C++ program (unpruned and pruned); Proposition 5.3 by Python and C++. (3) Two *independent verification runs*, themselves AI-assisted, re-derived the pairings, the closed path of T_{17} , the hand proof of Theorem 5.1 and the proofs of Lemma 4.2 and Theorem 6.1 line by line, and reran the computations with separately written code. The first extended Table 1 to $n = 12, 13$, reproduced the total 2,484,335,648 and the per-class counts of Theorem 4.3, enumerated all 1,412,014,592 filtrations of B_{17} , and tested Theorem 6.1 on 25,677 random cases. The second, with programs written from scratch, reproduced every explicit example, Table 1 for $n \leq 13$, Proposition 5.3, all 2,484,335,648 filtrations of Theorem 4.3, all

30,582,025,120 filtrations of Remark 4.4, and the counts 186, 1,852 and 302 of Remark 5.2, and tested Theorem 6.1 on 92,955 further random cases. No failure was found. The second run corrected the entry for $n = 13$ in Table 1: the program of the first run admits at most 12 vertices and so missed the one filtration consisting of 13 vertices (it counted 709,593,014); the other columns are unaffected. (4) Theorem 6.1 was also tested by the author’s program on 60,000 random filtered complexes: 36,267 had a non-incident pair and an acyclic V' , and the statement held in all ($N'(\sigma) = 1, 3, 5$ in 36,201, 65, 1 cases). Observation A was tested on 43,661 pairs and Observation B on 19,976 random instances, without failure; these are tests only.

Scope and priority. Proposition 3.1 refutes (K2) as stated in [Knu08]; this was first made public by Kriebel [Kri26] for the filtration Π_7 , and that deposit, which we read, treats (K2) alone. The second 7-simplex filtration S_7 , the exhaustive minimality and the examples with two and three paths are ours. The material of Section 4 and Section 5, and the statement of Theorem 6.1 (a consequence of algebraic Morse theory, Remark 6.3), are, as far as we found, not in the literature we could access: Crossref, OpenAlex, DataCite, zbMATH, the arXiv API and a web search gave no statement or refutation of (K1) or (K3), and the only items citing the report that we found were Kriebel’s two Zenodo records (OpenAlex also lists a software paper of 2021, which in fact cites lecture notes with the same title as the report). We could *not* access the manuscript “Persistence and discrete Morse theory” of Knudson and Bauer, which Knudson’s pages list as in preparation [Knu26], nor Knudson’s book [Knu15] (its zbMATH review describes the applications chapter as treating evasiveness and a homology reduction algorithm), nor Bauer’s 2011 thesis [Bau11], nor textbook treatments of persistence and discrete Morse theory (e.g. [Sco19, Koz21]), and we did not read [MN13] beyond its abstract. Any of these could contain overlapping material. Remark 6.4 is an argument, not a computation. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A Reductions for the examples with two and three paths

Elements of each set are listed from oldest to youngest. For M_{17} (steps 5–17; vertices are positive):

step	σ	$\partial\sigma$	reduction	result
5	35	$\{5, 3\}$	–	pair (3, 35)
7	14	$\{4, 1\}$	–	pair (1, 14)
8	34	$\{4, 3\}$	$+R(35) \rightarrow \{4, 5\}$	pair (5, 34)
9	05	$\{5, 0\}$	–	pair (0, 05)
10	01	$\{0, 1\}$	$+R(14) \rightarrow \{4, 0\}; +R(05) \rightarrow \{4, 5\}; +R(34) \rightarrow \emptyset$	positive
11	13	$\{3, 1\}$	$+R(14) \rightarrow \{4, 3\}; +R(35) \rightarrow \{4, 5\}; +R(34) \rightarrow \emptyset$	positive
12	03	$\{3, 0\}$	$+R(05) \rightarrow \{5, 3\}; +R(35) \rightarrow \emptyset$	positive
13	013	$\{01, 13, 03\}$	–	pair (03, 013)
14	04	$\{4, 0\}$	$+R(05) \rightarrow \{4, 5\}; +R(34) \rightarrow \emptyset$	positive
15	035	$\{35, 05, 03\}$	$+R(013) \rightarrow \{35, 05, 01, 13\}$	pair (13, 035)
16	014	$\{14, 01, 04\}$	–	pair (04, 014)
17	034	$\{34, 03, 04\}$	$+R(014) \rightarrow \{14, 34, 01, 03\}; +R(013) \rightarrow \{14, 34, 13\};$ $+R(035) \rightarrow \{35, 14, 34, 05, 01\}$	pair (01, 034)

and for M_{19} (steps 6–19):

step	σ	$\partial\sigma$	reduction	result
6	23	$\{3, 2\}$	–	pair (2, 23)
7	13	$\{3, 1\}$	–	pair (1, 13)
8	01	$\{0, 1\}$	$+R(13) \rightarrow \{3, 0\}$	pair (0, 01)
9	02	$\{2, 0\}$	$+R(01) \rightarrow \{3, 2\}; +R(23) \rightarrow \emptyset$	positive
10	12	$\{2, 1\}$	$+R(13) \rightarrow \{3, 2\}; +R(23) \rightarrow \emptyset$	positive
11	04	$\{4, 0\}$	$+R(01) \rightarrow \{3, 4\}$	pair (4, 04)
12	24	$\{4, 2\}$	$+R(23) \rightarrow \{3, 4\}; +R(04) \rightarrow \emptyset$	positive
13	14	$\{4, 1\}$	$+R(13) \rightarrow \{3, 4\}; +R(04) \rightarrow \emptyset$	positive
14	024	$\{02, 04, 24\}$	–	pair (24, 024)
15	012	$\{01, 02, 12\}$	–	pair (12, 012)
16	124	$\{12, 24, 14\}$	–	pair (14, 124)
17	34	$\{3, 4\}$	$+R(04) \rightarrow \emptyset$	positive
18	234	$\{23, 24, 34\}$	–	pair (34, 234)
19	134	$\{13, 14, 34\}$	$+R(234) \rightarrow \{23, 13, 24, 14\}; +R(124) \rightarrow \{23, 13, 12\};$ $+R(012) \rightarrow \{23, 13, 01, 02\}$	pair (02, 134)

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