

Polynomial Multiples of MacLane's Universal Functions Need Not Be Universal: A Negative Answer to a Question of Costakis

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Abstract

An entire function f is universal in the sense of MacLane if its derivatives $f, f', f'', \dots$ are dense in the space $H(\mathbb{C})$ of entire functions with the topology of locally uniform convergence; $\text{HC}(D)$ denotes the set of these functions. G. Costakis proved (Oberwolfach Report 6/2008, Theorem 6) that there is a residual set $G \subset \text{HC}(D)$ such that $pf \in \text{HC}(D)$ for every $f \in G$ and every non-zero polynomial p , and asked (Question 5 there) whether $pf \in \text{HC}(D)$ for every $f \in \text{HC}(D)$ and every non-constant polynomial p . We show that the answer is no. Theorem A gives an explicit $f_A = \sum_m a_m z^m / m!$ in $\text{HC}(D)$, built from the Taylor data of an enumeration of the polynomials with non-zero Gaussian-rational coefficients placed in sparse blocks at positions $n_j \geq 2/\delta_j$, where δ_j bounds the moduli of the data in block j from below. Then $(zf_A)^{(n)}(0) = na_{n-1}$ is 0 or of modulus at least 2 for every n , so no derivative of zf_A lies in the open set $\{h : |h(0) - 1| < 1\}$, and $zf_A \notin \text{HC}(D)$; the same f_A works for every monomial cz^d with $c \neq 0$, $d \geq 1$. Theorem B gives, with a ramp of large Taylor coefficients in front of each block, one $f_B \in \text{HC}(D)$ with $pf_B \notin \text{HC}(D)$ for every non-constant polynomial p : beyond a threshold depending on p , $(pf_B)^{(n)}(0)$ is 0 or of modulus at least 2, and finitely many derivatives do not affect density. Hence Costakis's Theorem 6 cannot be extended from a residual set to all of $\text{HC}(D)$. The question and the residual result are Costakis's, and the construction is the classical block construction of MacLane-type universal functions; the only new observation is that the right-hand side of the Leibniz formula $(pf)^{(n)}(0) = \sum_i c_i n(n-1) \cdots (n-i+1) a_{n-i}$ for $p = \sum_i c_i z^i$ can be controlled on such functions (with the ramps for Theorem B). Gardiner (2014) answered the analogous questions for universal Taylor series negatively. The argument is elementary; we found no printed answer to the question, but a folklore version may exist. Exact computations illustrate the constructions. This is an unrefereed note.

1 Introduction

1.1 The question

Let $H(\mathbb{C})$ be the space of entire functions with the topology of uniform convergence on compact subsets of $\mathbb{C}$; it is a separable Fréchet space, and point evaluations $h \mapsto h(z_0)$ are continuous on it. Let $D: H(\mathbb{C}) \rightarrow H(\mathbb{C})$, $Df = f'$, be the differentiation operator. Following MacLane [Mac52] and Costakis [Cos08, Definition 4], an entire function f is *universal with respect to derivatives* if the sequence $\{f^{(n)} : n = 0, 1, 2, \dots\}$ of its derivatives is dense in $H(\mathbb{C})$, and the set of these functions is denoted by $\text{HC}(D)$. In the language of linear dynamics, $\text{HC}(D)$ is the set of hypercyclic vectors of D [Cos08, p. 330]. MacLane proved that $\text{HC}(D)$ is not empty [Mac52]; the first universal entire function, for translations instead of derivatives, is due to Birkhoff [Bir29].

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In his report “Which maps preserve universal functions?” [Cos08], Costakis asks which maps between spaces of holomorphic functions preserve universality. For $\text{HC}(D)$ he shows that $f + e^{az} \in \text{HC}(D)$ for every $f \in \text{HC}(D)$ if $|a| \leq 1$, while there is $g \in \text{HC}(D)$ with $g + e^{az} \notin \text{HC}(D)$ for every a with $|a| > 1$ [Cos08, Theorem 5]. On the multiplication by polynomials he writes that he does not have a complete answer, and he states the following result.

Theorem 1.1 (Costakis [Cos08, Theorem 6]). *There is a residual subset G of $H(\mathbb{C})$ such that $G \subset \text{HC}(D)$ and $pf \in \text{HC}(D)$ for every non-zero polynomial p and every $f \in G$.*

The proof is not given in [Cos08]; it is said there to use in an essential way a result of Costakis’s preprint *Common Cesàro hypercyclic vectors*, which appeared as [Cos10] (see Section 1.3). Costakis then asks the following question [Cos08, Question 5, p. 330], which we paraphrase.

Question 1.2. *Is $pf \in \text{HC}(D)$ for every $f \in \text{HC}(D)$ and every non-constant polynomial p ?*

This is record OWR-1787-006 of the UnsolvedMath dataset [ula26], whose statement consists of Question 1.2 and the definition of $\text{HC}(D)$. For a constant $p = c \neq 0$ the answer is trivially yes, since $(cf)^{(n)} = cf^{(n)}$. Questions 1–4 of [Cos08] concern universal Taylor series in the unit disc (record OWR-1787-005 of [ula26]); see Section 1.3.

The question is not settled by the general theory of hypercyclic operators. If S is a continuous linear operator on $H(\mathbb{C})$ with dense range and $SD = DS$, then S maps $\text{HC}(D)$ into itself: the orbit $\{D^n S f\} = \{S D^n f\}$ is the image of a dense set under S , hence dense in $S(H(\mathbb{C}))$ and therefore in $H(\mathbb{C})$. This applies to $S = p(D)$ for every non-zero polynomial p , since $p(D)$ maps the polynomials onto the polynomials; that $p(T)x$ is hypercyclic for every hypercyclic vector x of an operator T and every non-zero polynomial p is a theorem of Bourdon, recalled in [Shk10, Section 1]. The multiplication operator $f \mapsto pf$, however, does not commute with D : $D(pf) - pDf = p'f$.

1.2 Results

We write the Taylor expansion of an entire function as $f(z) = \sum_{m \geq 0} a_m z^m / m!$, so that $a_m = f^{(m)}(0)$, and we put $a_m := 0$ for $m < 0$. For a polynomial $p(z) = \sum_{i=0}^d c_i z^i$ the Taylor data of pf at the origin are (Lemma 2.1)

$$(pf)^{(n)}(0) = \sum_{i=0}^d c_i n^i a_{n-i} \quad (n \geq 0), \quad \text{where } n^i := n(n-1) \cdots (n-i+1); \quad (1)$$

in particular $(zf)^{(n)}(0) = na_{n-1}$. If $zf \in \text{HC}(D)$, then the numbers $(zf)^{(n)}(0) = na_{n-1}$ are dense in $\mathbb{C}$, because evaluation at 0 is continuous and onto $\mathbb{C}$; if $f \in \text{HC}(D)$, only the numbers a_n have to be dense. A function in $\text{HC}(D)$ whose non-zero Taylor data satisfy $|(m+1)a_m| \geq 2$ separates the two properties, and the classical construction of universal functions from sparse blocks of Taylor data produces such functions.

Theorem A. *There is an entire function $f_A \in \text{HC}(D)$, constructed explicitly in Section 3, such that for every $n \geq 0$ either $(zf_A)^{(n)}(0) = 0$ or $|(zf_A)^{(n)}(0)| \geq 2$. Consequently no derivative of zf_A lies in the non-empty open set $\{h \in H(\mathbb{C}) : |h(0) - 1| < 1\}$, and $zf_A \notin \text{HC}(D)$. More generally, $cz^d f_A \notin \text{HC}(D)$ for every $c \in \mathbb{C} \setminus \{0\}$ and every integer $d \geq 1$.*

So the answer to Question 1.2 is negative, already for $p(z) = z$. With a ramp of large Taylor coefficients in front of each block, one function serves for all non-constant polynomials.

Theorem B. *There is an entire function $f_B \in \text{HC}(D)$, constructed explicitly in Section 4, such that $pf_B \notin \text{HC}(D)$ for every non-constant polynomial p . More precisely, for every non-constant polynomial p there is an integer $n_*(p)$ such that for every $n \geq n_*(p)$ either $(pf_B)^{(n)}(0) = 0$ or $|(pf_B)^{(n)}(0)| \geq 2$.*

The finitely many derivatives $(pf_B)^{(n)}$ with $n < n_*(p)$ do not matter (Lemma 2.2). By Theorem B, Theorem 1.1 cannot be extended from a residual set to all of $\text{HC}(D)$: the function f_B belongs to $\text{HC}(D)$ but to no set G as in Theorem 1.1. Theorem B also refutes the weaker statement that every $f \in \text{HC}(D)$ has some non-constant polynomial multiple in $\text{HC}(D)$.

What is new and what is not. The question and Theorem 1.1 are Costakis's [Cos08]. The functions f_A and f_B are built by the classical construction of universal functions with respect to derivatives, in which the Taylor data of a dense sequence of polynomials are placed in sparse blocks (compare MacLane's theorem [Mac52], and Birkhoff's theorem for translations [Bir29]); we do not claim this construction as new. What we add is that the right-hand side of (1) can be controlled on such functions. The argument is elementary. No printed answer to Question 1.2 was found in the sources we searched (see "Scope and priority" in Section 6), but a folklore version of the argument may exist.

1.3 Related results

Universal Taylor series. Questions 1–3 of [Cos08] ask whether $(z-a)f$ for $|a| \geq 1$, pf for non-constant polynomials p , and f' belong to Nestoridis's class $U(\mathbb{D}, 0)$ of universal Taylor series in the unit disc $\mathbb{D}$ whenever f does; Question 4 asks for an inner function in the class $U_1(\mathbb{D}, 0)$. For factors $z-a$ with a in the unit disc, both classes are preserved, and $U_1(\mathbb{D}, 0)$ even for a in the closed disc [Cos08, Theorem 3]. Gardiner [Gar14, p. 330] constructed $f \in U(\mathbb{D}, 0)$ with $(z-1)f \notin U(\mathbb{D}, 0)$, which answers Questions 1 and 2 in the negative; he noted that the question for derivatives remains open. Theorem A shows that for $\text{HC}(D)$ even the factor z can destroy universality.

Ideals and algebras. Shkarin [Shk10, final section] observed that $\text{HC}(D) \cup \{0\}$ contains no non-trivial ideal of $H(\mathbb{C})$: for $f \neq 0$ put $f^*(z) := \overline{f(\bar{z})}$; then ff^* is real on the real axis, the entire functions that are real on $\mathbb{R}$ form a closed, nowhere dense subset of $H(\mathbb{C})$ that D maps into itself, and so $ff^* \notin \text{HC}(D)$. Thus every $f \in \text{HC}(D)$ has an entire multiple outside $\text{HC}(D)$. The multiplier f^* depends on f and is transcendental (a polynomial has only finitely many non-zero derivatives, so no member of $\text{HC}(D)$ is a polynomial, and f^* is a polynomial only if f is), so this observation does not decide Question 1.2, which concerns polynomial multipliers. Shkarin also constructed an f with $g \circ f \in \text{HC}(D)$ for every non-constant polynomial g [Shk10, Theorem 1.4]; this concerns compositions, not products with polynomials.

Weighted orbits. The preprint on which the proof of Theorem 1.1 relies was published, under the same title, as [Cos10]. There $\text{HC}(\{T_n\})$ denotes the set of vectors x for which $\{T_1x, T_2x, \dots\}$ is dense [Cos10, p. 205], and Theorem 1.7 of [Cos10] states that $\bigcap_{b \in \mathbb{R}} \text{HC}(\{n^b D^n\})$ is residual in $H(\mathbb{C})$. Weighted sequences of derivatives enter naturally, since $(zf)^{(n)} = zf^{(n)} + nf^{(n-1)}$. The function f_A of Theorem A is not in $\text{HC}(\{nD^n\})$ (Remark 3.3), so it lies outside the residual set of [Cos10, Theorem 1.7], in accordance with the fact that Theorem 1.1 concerns a residual set only.

Growth. Costakis's Theorem 5(ii) is also a result of non-preservation; its proof uses a member of $\text{HC}(D)$ of optimal growth, whose existence is due to Grosse-Erdmann [GE90] and Shkarin [Shk93], as stated in [Cos08, p. 330].

2 Preliminaries

For $h \in H(\mathbb{C})$ and $R > 0$ we write $\|h\|_R := \max_{|z| \leq R} |h(z)|$. The sets $\{g \in H(\mathbb{C}) : \|g - h\|_R < \varepsilon\}$ with $R, \varepsilon > 0$ form a base of neighbourhoods of h . For integers $n \geq 0$ and $i \geq 0$, $n^{\underline{i}} = n(n-1) \cdots (n-i+1)$ is the falling factorial, with $n^{\underline{0}} = 1$; if $n < i$, one of its factors is $n - n = 0$, so $n^{\underline{i}} = 0$.

Lemma 2.1. Let $f(z) = \sum_{m \geq 0} a_m z^m / m!$ be an entire function, put $a_m := 0$ for $m < 0$, and let $p(z) = \sum_{i=0}^d c_i z^i$ be a polynomial.

(a) For every $n \geq 0$, $f^{(n)}(z) = \sum_{k \geq 0} a_{n+k} z^k / k!$, the series converging absolutely and locally uniformly on $\mathbb{C}$.

(b) For every $n \geq 0$, $(pf)^{(n)}(0) = \sum_{i=0}^d c_i n^i a_{n-i}$.

Proof. (a) A power series with infinite radius of convergence may be differentiated termwise, and the n -th derivative of $z^m / m!$ is $z^{m-n} / (m-n)!$ for $m \geq n$ and 0 for $m < n$.

(b) The power series of pf is the Cauchy product of those of p and f , so the coefficient of z^n in pf is $\sum_{0 \leq i \leq \min\{d, n\}} c_i a_{n-i} / (n-i)!$. The number $(pf)^{(n)}(0)$ is $n!$ times this coefficient, and $n! / (n-i)! = n^i$ for $i \leq n$; the terms of (b) with $n < i \leq d$ vanish because $a_{n-i} = 0$. $\square$

Lemma 2.2. Let $g \in H(\mathbb{C})$, $w_0 \in \mathbb{C} \setminus \{0\}$ and $N \geq 0$. If $g^{(n)}(0) \in \{0\} \cup \{w \in \mathbb{C} : |w| \geq 2|w_0|\}$ for every $n \geq N$, then $g \notin \text{HC}(D)$.

Proof. If $w = 0$ or $|w| \geq 2|w_0|$, then $|w - w_0| \geq |w_0|$. The set $U := \{h \in H(\mathbb{C}) : |h(0) - w_0| < |w_0|\}$ is open, since evaluation at 0 is continuous, and it is infinite, since it contains the constant functions $w_0 + t$ with $0 \leq t < |w_0|$. The set $F := \{g^{(n)} : 0 \leq n < N\}$ is finite and hence closed, so $U \setminus F$ is open and not empty. Suppose that $g \in \text{HC}(D)$. Then $g^{(n)} \in U \setminus F$ for some n . Since $g^{(n)} \notin F$, we have $n \geq N$, and then $|g^{(n)}(0) - w_0| \geq |w_0|$ contradicts $g^{(n)} \in U$. $\square$

Lemma 2.3. Let $\mathcal{P}$ be the set of polynomials $q(z) = \sum_{k=0}^m q_k z^k$ ($m \geq 0$) whose coefficients $q_0, \dots, q_m$ all lie in $\mathbb{Q}(i) \setminus \{0\}$. Then $\mathcal{P}$ is countable and dense in $H(\mathbb{C})$.

Proof. $\mathcal{P}$ is countable because it is a set of finite sequences of elements of the countable set $\mathbb{Q}(i) \setminus \{0\}$. Let $h \in H(\mathbb{C})$, $R > 0$ and $\varepsilon > 0$. The Taylor polynomials of h converge to h uniformly on $\{|z| \leq R\}$, so there is a polynomial $T(z) = \sum_{k=0}^m t_k z^k$ with $\|h - T\|_R < \varepsilon/2$. Since $\mathbb{Q}(i) \setminus \{0\}$ is dense in $\mathbb{C}$, we can choose $q_k \in \mathbb{Q}(i) \setminus \{0\}$ with $|t_k - q_k| R^k < \varepsilon / (2(m+1))$ for $0 \leq k \leq m$. Then $q := \sum_{k=0}^m q_k z^k \in \mathcal{P}$ and $\|h - q\|_R \leq \|h - T\|_R + \sum_{k=0}^m |t_k - q_k| R^k < \varepsilon$. $\square$

The data. We fix an enumeration $(g_j)_{j \geq 1}$ of $\mathcal{P}$ in which every element of $\mathcal{P}$ occurs infinitely often. For instance, if $(q_k)_{k \geq 1}$ is any enumeration of $\mathcal{P}$, let $g_j := q_{v(j)+1}$, where $2^{v(j)}$ is the largest power of 2 that divides j ; the programs of Section 6 use another explicit enumeration. For $j \geq 1$ let $d_j := \deg g_j$ and $b_{j,k} := g_j^{(k)}(0)$ for $0 \leq k \leq d_j$. Thus $b_{j,k}$ is $k!$ times the coefficient of z^k in g_j , so $b_{j,k} \in \mathbb{Q}(i) \setminus \{0\}$, and $g_j(z) = \sum_{k=0}^{d_j} b_{j,k} z^k / k!$. We put

$$M_j := \max \left\{ 1, \max_{0 \leq k \leq d_j} (|\operatorname{Re} b_{j,k}| + |\operatorname{Im} b_{j,k}|) \right\}, \quad \delta_j := \min_{0 \leq k \leq d_j} \max \{ |\operatorname{Re} b_{j,k}|, |\operatorname{Im} b_{j,k}| \}.$$

These are positive rational numbers, $M_j \geq 1$, and

$$\delta_j \leq |b_{j,k}| \leq M_j \quad (0 \leq k \leq d_j). \quad (2)$$

Only (2) and $M_j \geq 1$ are used below; the choice makes all data rational. We also set $n_0 := 0$ and $d_0 := 0$.

Finally, for a real number $r > 0$ the ratio of consecutive terms of the sequence $r^u / u!$ ($u = 0, 1, 2, \dots$) is $r / (u+1)$, so the sequence is non-increasing from $u = \lceil r \rceil - 1$ on and tends to 0. Hence

$$\frac{r^u}{u!} \leq \frac{r^N}{N!} \quad \text{for integers } u \geq N \geq r - 1, \quad \text{and} \quad \frac{r^N}{N!} \rightarrow 0 \quad (N \rightarrow \infty). \quad (3)$$

We use the convention $0^0 = 1$, so that $0^N / N! = 0$ for $N \geq 1$.

3 One universal function for all monomials

Construction A. Choose integers $0 = n_0 < n_1 < n_2 < \dots$ such that for every $j \geq 1$, with $N_j := n_j - n_{j-1}$,

$$(A1) \quad n_j \delta_j \geq 2,$$

$$(A2) \quad N_j \geq \max\{d_{j-1} + 1, j - 1\} \quad \text{and} \quad (d_j + 1) M_j \frac{(j-1)^{N_j}}{N_j!} \leq 2^{-j}.$$

Such integers exist and can be chosen one after the other: if n_{j-1} is fixed, the left-hand side of the inequality in (A2) is non-increasing in N_j for $N_j \geq j - 2$ and tends to 0, by (3), so every sufficiently large n_j satisfies (A1) and (A2). An explicit admissible choice is

$$n_j := \max\{n_{j-1} + N'_j, \lceil 2/\delta_j \rceil\}, \quad N'_j := \max\{d_{j-1} + 1, \lceil 2e(j-1) \rceil, j + \lceil \log_2((d_j + 1)M_j) \rceil\},$$

where e is Euler's number: since $e^N \geq N^N/N!$, one has $(j-1)^N/N! \leq (e(j-1)/N)^N \leq 2^{-N}$ for $N \geq 2e(j-1)$, and then $(d_j + 1)M_j 2^{-N} \leq 2^{-j}$ for $N \geq j + \log_2((d_j + 1)M_j)$.

Let $B_j := \{n_j, n_j + 1, \dots, n_j + d_j\}$ be the j -th block. Since $n_{j+1} - n_j = N_{j+1} \geq d_j + 1$ by (A2), the blocks are pairwise disjoint and B_j lies to the left of B_{j+1} . Define

$$a_{n_j+k} := b_{j,k} \quad (j \geq 1, 0 \leq k \leq d_j), \quad a_m := 0 \quad \left(m \notin \bigcup_{j \geq 1} B_j\right), \quad f_A(z) := \sum_{m \geq 0} a_m \frac{z^m}{m!}.$$

Lemma 3.1. *Let $j \geq 1$, $i > j$ and $0 \leq t \leq j$. Then*

$$\sum_{m \in B_i} |a_m| \frac{t^{m-n_j}}{(m-n_j)!} \leq 2^{-i}. \quad (4)$$

Proof. Let $m \in B_i$ and $u := m - n_j$. Since $n_j \leq n_{i-1}$, we have $u \geq n_i - n_{i-1} = N_i \geq i - 1 \geq j$. By (3) with $r = i - 1 \geq 1$ (note $u \geq N_i \geq i - 1 \geq r - 1$),

$$\frac{t^u}{u!} \leq \frac{j^u}{u!} \leq \frac{(i-1)^u}{u!} \leq \frac{(i-1)^{N_i}}{N_i!}.$$

The block B_i has $d_i + 1$ elements, and $|a_m| \leq M_i$ on B_i by (2). Now (A2) gives (4). $\square$

Proposition 3.2. *The series defining f_A converges on $\mathbb{C}$, so f_A is entire and $f_A^{(m)}(0) = a_m$ for all m . For every $j \geq 1$, $\|f_A^{(n_j)} - g_j\|_j \leq 2^{-j}$. The function f_A belongs to $\text{HC}(D)$.*

Proof. Convergence. Let $R > 0$ and let $j \geq R$ be an integer. For $m \in B_i$ with $i > j$ put $u := m - n_j$; then $u \geq j$, as in the proof of Lemma 3.1, so $m!/u! = (u+1)(u+2)\dots(u+n_j) \geq j^{n_j}$ and $R^m/m! \leq j^m/m! \leq j^u/u!$. By (4) with $t = j$, $\sum_{m > n_j+d_j} |a_m| R^m/m! \leq \sum_{i > j} 2^{-i} < \infty$. Hence $\sum_m |a_m| R^m/m! < \infty$ for every $R > 0$.

Approximation. By Lemma 2.1(a), $f_A^{(n_j)}(z) = \sum_{m \geq n_j} a_m z^{m-n_j}/(m-n_j)!$. The terms with $m \in B_j$ add up to $\sum_{k=0}^{d_j} b_{j,k} z^k/k! = g_j(z)$, the terms with $n_j + d_j < m < n_{j+1}$ vanish, and the remaining terms are those with $m \in B_i$, $i > j$. For $|z| \leq j$, (4) with $t = |z|$ gives $|f_A^{(n_j)}(z) - g_j(z)| \leq \sum_{i > j} 2^{-i} = 2^{-j}$.

Universality. Let $h \in H(\mathbb{C})$, let $K \subset \mathbb{C}$ be compact and $\varepsilon > 0$. Choose $R \geq 1$ with $K \subset \{|z| \leq R\}$ and, by Lemma 2.3, $q \in \mathcal{P}$ with $\|h - q\|_R < \varepsilon/2$. Since $q = g_j$ for infinitely many j , there is such a j with $j \geq R$ and $2^{-j} < \varepsilon/2$. Then $\max_K |f_A^{(n_j)} - h| \leq \|f_A^{(n_j)} - g_j\|_j + \|q - h\|_R < \varepsilon$. So the derivatives of f_A are dense in $H(\mathbb{C})$. $\square$

Proof of Theorem A. By Proposition 3.2, $f_A \in \text{HC}(D)$. Let $c \in \mathbb{C} \setminus \{0\}$, let $d \geq 1$ and $g := cz^d f_A$. By Lemma 2.1(b), $g^{(n)}(0) = cn^d a_{n-d}$ for every $n \geq 0$. If $a_{n-d} \neq 0$, then $n-d = n_j + k$ for some $j \geq 1$ and $0 \leq k \leq d_j$, and n^d is a product of d factors, each at least $n-d+1 \geq 1$, so $n^d \geq n-d+1 = n_j + k + 1$. By (2) and (A1),

$$|g^{(n)}(0)| \geq |c| (n_j + k + 1) |b_{j,k}| > |c| n_j \delta_j \geq 2|c|.$$

Hence $g^{(n)}(0) \in \{0\} \cup \{w : |w| \geq 2|c|\}$ for every $n \geq 0$, and Lemma 2.2 with $w_0 = c$ and $N = 0$ gives $g \notin \text{HC}(D)$. For $c = d = 1$ this says that $(zf_A)^{(n)}(0) = na_{n-1}$ is 0 or of modulus at least 2 for every $n \geq 0$; then $|(zf_A)^{(n)}(0) - 1| \geq 1$, so no derivative of zf_A lies in $\{h : |h(0) - 1| < 1\}$, an open set that contains the constant function 1. $\square$

Remark 3.3. The same reasoning applies to the weighted sequence $(nf_A^{(n)})_{n \geq 1}$. If $n \geq 1$ and $a_n \neq 0$, then $n = n_j + k \in B_j$ and $|na_n| \geq n_j |b_{j,k}| \geq n_j \delta_j \geq 2$ by (2) and (A1). So the values $(nf_A^{(n)})(0) = na_n$ are 0 or of modulus at least 2, the functions $nf_A^{(n)}$ ($n \geq 1$) avoid the open set $\{h : |h(0) - 1| < 1\}$, and $f_A \notin \text{HC}(\{nD^n\})$. In particular f_A does not belong to the residual set $\bigcap_{b \in \mathbb{R}} \text{HC}(\{n^b D^n\})$ of [Cos10, Theorem 1.7].

4 One universal function for all non-constant polynomials

Construction B. Choose integers $0 = n_0 < n_1 < n_2 < \dots$ such that for every $j \geq 1$, with $N_j := n_j - j - n_{j-1}$,

$$(B1) \quad n_j \geq j(M_j/\delta_j + 1),$$

$$(B2) \quad N_j \geq d_{j-1} + j + 1 \quad \text{and} \quad (j + d_j + 1) M_j (2j)^j \frac{(j-1)^{N_j}}{N_j!} \leq 2^{-j}.$$

They exist for the same reason as in Construction A. An explicit admissible choice is $n_j := \max\{n_{j-1} + j + N'_j, \lceil j(M_j/\delta_j + 1) \rceil\}$ with $N'_j := \max\{d_{j-1} + j + 1, \lceil 2e(j-1) \rceil, j + \lceil \log_2((j + d_j + 1)M_j(2j)^j) \rceil\}$.

The j -th zone is $Z_j := \{n_j - j, \dots, n_j + d_j\}$; it consists of the *ramp* $R_j := \{n_j - j, \dots, n_j - 1\}$ and the *block* $B_j := \{n_j, \dots, n_j + d_j\}$. Define

$$a_{n_j-l} := M_j (2j)^l \quad (1 \leq l \leq j), \quad a_{n_j+k} := b_{j,k} \quad (0 \leq k \leq d_j), \quad a_m := 0 \quad \left(m \notin \bigcup_{j \geq 1} Z_j\right),$$

for $j \geq 1$, and $f_B(z) := \sum_{m \geq 0} a_m z^m / m!$. We record two facts about this layout.

(L1) For $j \geq 2$, the integers strictly between $\max Z_{j-1} = n_{j-1} + d_{j-1}$ and $\min Z_j = n_j - j$ are $N_j - d_{j-1} - 1 \geq j$ in number, by (B2); the integers $0, \dots, n_1 - 2$ below Z_1 are $n_1 - 1 = N_1 \geq 2$ in number. Hence the zones are pairwise disjoint and Z_j lies to the left of Z_{j+1} ; moreover $a_m = 0$ for the j integers m directly below Z_j and for the $j+1$ integers m directly above Z_j .

(L2) $a_m \neq 0$ for every $m \in Z_j$; $\delta_j \leq |a_m| \leq M_j$ for $m \in B_j$; $a_{n_j-l} = M_j (2j)^l$ for $1 \leq l \leq j$; and $|a_m| \leq M_j (2j)^j$ for every $m \in Z_j$.

Lemma 4.1. *The function f_B is entire, $\|f_B^{(n_j)} - g_j\|_j \leq 2^{-j}$ for every $j \geq 1$, and $f_B \in \text{HC}(D)$.*

Proof. Let $j \geq 1$, $i > j$, $m \in Z_i$ and $u := m - n_j$. Since $n_j \leq n_{i-1}$, we have $u \geq (n_i - i) - n_{i-1} = N_i \geq i - 1 \geq j$. The zone Z_i has $i + d_i + 1$ elements, and $|a_m| \leq M_i (2i)^i$ on Z_i by (L2). As in the proof of Lemma 3.1, for $0 \leq t \leq j$,

$$\sum_{m \in Z_i} |a_m| \frac{t^{m-n_j}}{(m-n_j)!} \leq (i + d_i + 1) M_i (2i)^i \frac{(i-1)^{N_i}}{N_i!} \leq 2^{-i} \quad (5)$$

by (B2). The ramp R_j lies below n_j , and $a_m = 0$ for $n_j + d_j < m < \min Z_{j+1}$ by (L1); so $f_B^{(n_j)} - g_j$ is the sum of the terms $a_m z^{m-n_j} / (m - n_j)!$ with $m \in Z_i$, $i > j$. The proof of Proposition 3.2 now applies word for word, with (5) in place of (4). $\square$

Proposition 4.2. *Let $p(z) = \sum_{i=0}^d c_i z^i$ with $d \geq 1$ and $c_d \neq 0$, and put*

$$S := \sum_{i=0}^d |c_i|, \quad c_{\min} := \min\{|c_i| : c_i \neq 0\}, \quad j_0 := \max\{d, \lceil (2S + 4)/c_{\min} \rceil\}, \quad n_* := n_{j_0} - j_0.$$

Then for every $n \geq n_*$ the number $s_n := (pf_B)^{(n)}(0)$ is 0 or satisfies $|s_n| \geq S + 2$.

Proof. Fix $n \geq n_*$. By Lemma 2.1(b), $s_n = \sum_{i=0}^d c_i n^i a_{n-i}$. Let $I := \{i \in \{0, \dots, d\} : c_i \neq 0 \text{ and } a_{n-i} \neq 0\}$. If $I = \emptyset$, then $s_n = 0$. Otherwise let $i^* := \max I$ and $m^* := n - i^*$; then $a_{m^*} \neq 0$, so $m^* \geq 0$.

Step 1: the non-zero terms come from one zone Z_j with $j \geq j_0$. We have $m^* \geq n - d \geq n_{j_0} - j_0 - d \geq n_{j_0} - 2j_0$, because $d \leq j_0$. By (L1), $a_m = 0$ for the j_0 integers $n_{j_0} - 2j_0 \leq m < n_{j_0} - j_0$ (for negative m by convention). Since $a_{m^*} \neq 0$, it follows that $m^* \geq n_{j_0} - j_0 = \min Z_{j_0}$, and hence $m^* \in Z_j$ for some $j \geq j_0 \geq d$. Now let $i \in I$ with $i < i^*$. The position $n - i = m^* + (i^* - i)$ satisfies $m^* < n - i \leq m^* + d \leq m^* + j$. Since $m^* \in Z_j$, every integer in $(m^*, m^* + j]$ lies in Z_j or among the $j + 1$ integers directly above Z_j , where a vanishes by (L1); as $a_{n-i} \neq 0$, we get $n - i \in Z_j$.

Step 2: falling factorials. For $0 \leq i < i^*$ we have $n^{i^*} = n^i \cdot \prod_{t=i}^{i^*-1} (n - t)$, and each of the $i^* - i \geq 1$ factors is at least $n - i^* + 1 = m^* + 1 \geq 1$. Also n^{i^*} is a product of i^* factors, each at least $m^* + 1$. Hence

$$n^i \leq \frac{n^{i^*}}{m^* + 1} \leq n^{i^*} \quad (0 \leq i < i^*), \quad n^{i^*} \geq 1, \quad n^{i^*} \geq m^* + 1 \text{ if } i^* \geq 1. \quad (6)$$

The terms of s_n with $i \notin I$ vanish, and $|c_{i^*}| \geq c_{\min}$; so

$$|s_n| \geq c_{\min} n^{i^*} |a_{m^*}| - \sum_{i \in I, i < i^*} |c_i| n^i |a_{n-i}|, \quad (7)$$

where by Step 1 every a_{n-i} in the sum is an entry of Z_j at a position larger than m^* , and $\sum_{i \in I, i < i^*} |c_i| \leq S$. Recall that $j \geq j_0 \geq (2S + 4)/c_{\min}$.

Case R: $m^* \in R_j$, say $m^* = n_j - l$ with $1 \leq l \leq j$. The positions of Z_j larger than m^* are ramp positions $n_j - l'$ with $l' < l$, where $|a_{n_j-l'}| = M_j (2j)^{l'} \leq L := M_j (2j)^{l-1}$, and block positions, where $|a_m| \leq M_j \leq L$ by (L2). Since $|a_{m^*}| = M_j (2j)^l = 2jL$, (7) and (6) give

$$|s_n| \geq n^{i^*} L (2jc_{\min} - S) \geq 2j_0 c_{\min} - S \geq 2(2S + 4) - S = 3S + 8,$$

where we used $2jc_{\min} - S \geq 3S + 8 > 0$, $n^{i^*} \geq 1$ and $L \geq 1$.

Case K: $m^* \in B_j$, say $m^* = n_j + k$ with $0 \leq k \leq d_j$. First, $i^* \geq 1$. Otherwise $i^* = 0$, $m^* = n$, and $d \notin I$; as $c_d \neq 0$, this means $a_{n-d} = 0$. But $n_j - j \leq n - d < n = m^* \leq n_j + d_j$, because $d \leq j$ and $n \geq n_j$, so $n - d \in Z_j$, where all entries are non-zero by (L2), a contradiction. Next, by Step 1 the positions $n - i$ with $i \in I$, $i < i^*$, lie in Z_j and are larger than $m^* \geq n_j$, so they lie in B_j , where $|a_{n-i}| \leq M_j$; moreover $|a_{m^*}| \geq \delta_j$. With the first inequality of (6), (7) gives

$$|s_n| \geq n^{i^*} \left(c_{\min} \delta_j - \frac{SM_j}{m^* + 1} \right).$$

By (B1), $m^* + 1 > n_j \geq jM_j/\delta_j$, so $SM_j/(m^* + 1) < S\delta_j/j \leq S\delta_j c_{\min}/(2S + 4) < c_{\min} \delta_j/2$, and the bracket exceeds $c_{\min} \delta_j/2$. By (6), $n^{i^*} \geq m^* + 1 > jM_j/\delta_j$. Therefore

$$|s_n| > \frac{jM_j}{\delta_j} \cdot \frac{c_{\min} \delta_j}{2} = \frac{jM_j c_{\min}}{2} \geq \frac{j_0 c_{\min}}{2} \geq S + 2,$$

since $M_j \geq 1$. In both cases $|s_n| \geq S + 2$. $\square$

Proof of Theorem B. By Lemma 4.1, $f_B \in \text{HC}(D)$. Let p be a non-constant polynomial, and let $n_*(p)$ be the number n_* of Proposition 4.2. For every $n \geq n_*(p)$, $(pf_B)^{(n)}(0)$ is 0 or of modulus at least $S + 2 > 2$, and Lemma 2.2 with $w_0 = 1$ and $N = n_*(p)$ gives $pf_B \notin \text{HC}(D)$. The function f_B does not depend on p ; only the threshold $n_*(p)$ does. $\square$

5 Remarks

Remark 5.1 (Why ramps). The function f_A need not satisfy the conclusion of Proposition 4.2 for $p(z) = 1 + z$. By Lemma 2.1(b), $((1 + z)f_A)^{(n)}(0) = a_n + na_{n-1}$. If $a_{n_j-1} = 0$, which holds for instance when $N_j \geq d_{j-1} + 2$, this number equals $a_{n_j} = b_{j,0} = g_j(0)$ at $n = n_j$. The constant terms $g_j(0)$ run through all of $\mathbb{Q}(i) \setminus \{0\}$, since every non-zero Gaussian rational is a constant polynomial in $\mathcal{P}$. So if $a_{n_j-1} = 0$ for all j , the values $((1 + z)f_A)^{(n)}(0)$ are dense in $\mathbb{C}$, and Lemma 2.2 cannot be applied to $(1 + z)f_A$; we do not know whether $(1 + z)f_A \in \text{HC}(D)$. For f_B , one term of (1) dominates the others at every $n \geq n_*$: the term with $i = i^*$, which has the leftmost non-zero position and the largest falling factorial. In the ramp it wins by the factor $2j$ between consecutive ramp entries (Case R in the proof of Proposition 4.2), in the block by (B1) (Case K). Without the ramps, and with at least d zeros before every block, one would have $s_{n_j} = c_0 g_j(0)$ at the first position of every block, and these values are dense in $\mathbb{C}$ if $c_0 \neq 0$. In the computations of Section 6 (the first 120 blocks of the size enumeration, with the least admissible n_j), $a_{n_j-1} = 0$ for every j , and the 96 values of $n \leq n_{120} + d_{120}$ for which $((1 + z)f_A)^{(n)}(0)$ is neither 0 nor of modulus at least 2 are all block starts n_j ; for 15 of them the value lies in the disc $|w - 1| < 1$.

Remark 5.2 (The hypotheses). In the proof of Theorem B, the fact that all coefficients of the g_j are non-zero (the reason for working with $\mathcal{P}$ rather than with all polynomials with Gaussian-rational coefficients) is used only in Case K, to show $i^* \geq 1$; Theorem A does not need it. The size of the ramp entries is used only in Case R. That they are non-zero is also used in Case K: if m^* were one of the first d positions of B_j with $i^* = 0$, then $n - d$ would be a ramp position, and without the ramps this case occurs (Remark 5.1). (B1) is used only in Case K, and (B2) in Lemma 4.1 and for (L1). The assumption $d \geq 1$ also enters only in Case K. For a constant $p = c_0 \neq 0$ one has $s_n = c_0 a_n$, and these values are dense in $\mathbb{C}$, as they must be, since $c_0 f_B \in \text{HC}(D)$. The data of f_A and f_B are explicit once (g_j) is fixed: the $b_{j,k}$, M_j and δ_j are (Gaussian) rationals, and the n_j are given by the formulas after (A2) and (B2) or chosen least possible.

6 Computations and reproducibility

No theorem, proposition or lemma of this note depends on a computation. The reproducibility package contains three groups of programs; all of them use exact rational arithmetic unless stated otherwise.

The exact program of this note, `check_note.py` (standard library only; about 14 seconds), uses two explicit enumerations of $\mathcal{P}$: the *size enumeration*, which lists for $s = 1, 2, \dots$ all polynomials of degree at most s with coefficients in $\{(a + bi)/c : a, b, c \in \mathbb{Z}, |a|, |b| \leq s, (a, b) \neq (0, 0), 1 \leq c \leq s\}$ (so every member of $\mathcal{P}$ recurs), and a *mixed enumeration* that starts with 24 fixed polynomials of degrees up to 6 with small coefficients. For the first 120 members of each, with the least admissible n_j , it checks (A1), (A2), (B1), (B2), (L1), (L2) and the explicit choices after (A2) and (B2); Lemma 2.1(b) against repeated termwise differentiation in 300 random instances; (3) and $N! \geq (N/e)^N$ on finite ranges; the values in the proof of Theorem A for z and six other monomials at every n up to the end of the 120-th block, and Remark 3.3; the conclusion of Proposition 4.2, with the exact thresholds, for 13 polynomials whose coefficients have rational moduli (degrees 1 to 5, j_0 from 3 to 58) at every n from n_* to the end of the 120-th zone, and the number of exceptions below n_* (at most 3 for each

polynomial, as Lemma 2.2 allows); the identity of Remark 5.1; that the sums over $j < i \leq 120$ of the left-hand sides of the inequalities in (A2) and (B2) are at most 2^{-j} ; and exact majorants of the sums (4) and (5) for $j \leq 5$, $j < i \leq j + 6$. All 644 checks pass. For the size enumeration, $n_{120} = 23770$ in Construction A and $n_{120} = 54162$ in Construction B.

The program of the first version (original/) checks the same constructions for the size enumeration with 120 and 60 blocks and for the mixed one with 60 blocks; its printed n_j agree with those of `check_note.py`. Its test of Proposition 4.2 checks the weaker conclusion of the first version, that s_n is 0 or of modulus at least 2 (not $S+2$); it replaces S and $c_{\min}$ by rational upper and lower bounds, so its threshold j'_0 is at least j_0 , and the test starts at $n'_* = n_{j'_0} - j'_0 \geq n_*$. The package contains a corrected copy: the line on Remark 5.1 now says that it counts the n with $((1+z)f_A)^{(n)}(0) \notin \{0\} \cup \{|w| \geq 2\}$ (96, 60 and 59 such n in the three runs, all block starts), and the final line “ALL EXACT CHECKS PASSED” is printed only if no check failed. The programs of the three verification runs and their outputs are also included (`verification_run_A/`, `verification_run_B/` and `independent_run_2/`); they are described in the next paragraph.

Verification. The results were first obtained with written proofs and the program in `original/`. Two independent verification runs, both AI-assisted, examined this first written version; each read the source [Cos08] in the publisher’s open PDF, re-derived all proofs line by line and wrote its own programs, and neither read the other’s report. Both confirmed Theorems A and B and found no gap; both also read [Cos10] and [Gar14].

The first run proved Lemma 2.1(b) both by the Leibniz rule and from the Cauchy product. Its exact program uses coefficients with rational moduli and adversarial data (degrees up to 12, coefficients between $1/1000$ and $50 \cdot 12!$): Theorem A with four monomials on 150 blocks; the conclusion of Proposition 4.2 for 43 polynomials on 130 zones in the tightest layout its proof allows, with $|s_n| \geq S+2$ throughout, and for 35 polynomials on data with all $|b_{j,k}| = 1$; small coefficients (j_0 up to 242); and three mutations (ramps removed, ramps of height 1, (B1) dropped), each of which produced counterexamples at some $n \geq n_*$, as it should. It reproduced the outputs of the first-version program byte for byte.

The second run wrote its programs before reading the first-version program: Lemma 2.1(b) against symbolic differentiation (476 random instances); Constructions A and B for 120 blocks of two enumerations of its own (one with coefficients from 10^{-5} to 10^4 and degrees up to 10), with Theorem A for six monomials and Proposition 4.2 for 18 polynomials; single zones Z_j , $j \geq j_0$, with the smallest n_j that (B1) allows, for 8 polynomials, most of them with tiny coefficients (j_0 up to 1 000 002; about 16.4 million windows for each of two random seeds, none failing); five polynomials with irrational coefficients; and nine defective constructions, all detected. Its tests of Proposition 4.2 check the weaker conclusion of the first version, that s_n is 0 or of modulus at least 2. It reproduced the outputs and the printed n_j of the first-version program.

The runs required the following corrections, all of which were made: the counts of exceptional n in Remark 5.1 (the text of the first version gave them wrongly) and the two defects of its program mentioned above; the removal of an unverified theorem number in [BM09]; the description of [GE90, Shk93] as in [Cos08]; that Question 4 of [Cos08] concerns $U_1(\mathbb{D}, 0)$; the citation of [Cos10, Theorem 1.7] with Remark 3.3; Shkarin’s remark on ideals [Shk10] and why it does not answer Question 1.2; the caveat on the monographs [BM09, GEP11]; and the cruder threshold of the first-version program.

A third verification run, also AI-assisted, examined the final text of this note. It downloaded the sources again ([Cos08, Cos10, Gar14] in the publishers’ files and [Shk10] in the arXiv version, all identical to the copies read before) and compared every statement made about them; it re-derived every proof line by line, in particular the parts written or changed for this note: the rational bounds M_j , δ_j , the bound $|s_n| \geq S+2$ of Proposition 4.2 with the factor bound $m^* + 1 > n_j$ in Case K, the

thresholds j_0 and n_* , the independence of f_B from p , and Remark 3.3; and it repeated the literature search. Its own program (`independent_run_2/`; exact arithmetic, except for a 50-digit check of (4) and (5)) checks Lemma 2.1(b) as a symbolic identity for all $d \leq 6$ and $n \leq 22$; builds Constructions A and B for the first 120 members of the size enumeration and of an enumeration $g_j = q_{v(j)+1}$ as in Section 2, and for 100 adversarial members of $\mathcal{P}$ of degrees up to 14 with moduli from 10^{-6} to 10^6 , each with the least admissible n_j , with the explicit choices after (A2) and (B2), and in two tight layouts that keep only the gaps of (L1) and either (B1) or the weaker condition $n_j \geq jM_j/\delta_j$ used in Case K; tests Theorem A and Remark 3.3; tests $|s_n| \geq S+2$ for 13 polynomials at every n from n_* to the end of the last zone on all these layouts, checking each inequality of the proof of Proposition 4.2 on the way, and, on the size enumeration, for four polynomials with irrational moduli $|c_i|$ by interval arithmetic; tests single zones placed at $n_j = \lceil jM_j/\delta_j \rceil$ or slightly later, for six polynomials with j_0 up to 6002; checks (4) and (5) for the size enumeration and all $1 \leq j < i \leq 120$; and reproduces $n_{120} = 23770$ and 54162 and the numbers of Remark 5.1. Three mutations (ramps removed, ramps of size 10^{-6} , (B1) dropped) produce counterexamples at $n \geq n_*$ only in the case of the proof that uses the removed property. The run also executed the programs of the package again from the archive and reproduced their recorded outputs. It found no error in a theorem, proposition, lemma or proof. It required corrections, which were made: the sentence of Remark 5.2 on the use of the ramps; the novelty sentence of the abstract; the statements that the first-version program and the second run test the weaker conclusion; and precision points in the introduction, in Remark 5.1, in the next paragraph and in the reference [MP23]. No proof depends on a computation.

Scope and priority. The note answers Question 1.2 (Question 5 of [Cos08]) in the negative (Theorem A) and shows that one function in $\text{HC}(D)$ defeats all non-constant polynomials (Theorem B). It does not describe the set of $f \in \text{HC}(D)$ with $zf \in \text{HC}(D)$ beyond Theorem 1.1, and it does not decide whether $(1+z)f_A \in \text{HC}(D)$. The question and Theorem 1.1 are due to Costakis [Cos08]. The construction of f_A and f_B is the classical block construction of universal functions with respect to derivatives (compare [Mac52], and [Bir29] for translations) and is not claimed as new; only the use of formula (1) on such functions, with the ramps of Section 4, is ours. Gardiner [Gar14] answered the analogous Questions 1 and 2 of [Cos08] for universal Taylor series in the negative. The argument is elementary and may be known to specialists; it may appear, for instance, as an exercise in the monographs of Bayart and Matheron [BM09] or of Grosse-Erdmann and Peris [GEP11], which were not accessible to us. No printed answer to the question was found in the sources searched: the report [Cos08] and Costakis’s one-page abstract on the web page of the workshop; [Cos10], [Gar14] and [Shk10] (arXiv version) in the relevant parts, and [BGCCSS18] and [CHW20] in their arXiv versions; the works citing [Cos08] that zbMATH and Semantic Scholar list, [MN11] and [MP23], and the citing paper [MM16], of which the abstracts were read; the titles of Costakis’s later papers in zbMATH and arXiv, and of the works citing [Cos10] that zbMATH and Semantic Scholar list; and queries to arXiv, zbMATH, Crossref and Semantic Scholar and nine web searches, made on 18 and 19 October 2026. OpenAlex returned its records of [Cos08] and [Cos10], with three and ten citing works, but not the lists of these works. A search that finds nothing is not a proof of novelty, and no priority is claimed. We did not read [Mac52], [Bir29], [GE90] and [Shk93]; they are cited for background, the last two as in [Cos08].

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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