

Real-Rooted Integer Polynomials Satisfy a g -Theorem: Proof of a Conjecture of Brenti and Welker

Alper Ferudun*

October 17, 2026

Abstract

In his abstract *Conjectures on f -vectors* in Oberwolfach Report 1/2008, V. Welker recorded the following conjecture of F. Brenti and himself (Conjecture 8 there). Let $h(t) = 1 + h_1t + \dots + h_d t^d$ have positive integer coefficients and only real roots. Then $(h_0, \dots, h_d)$ satisfies a g -theorem: it increases strictly up to a first peak h_r and does not increase afterwards, and the differences $g_i = h_i - h_{i-1}$, $1 \leq i \leq r$, satisfy Macaulay's condition $g_{i+1} \leq g_i^{\langle i \rangle}$. We prove this conjecture, with the classical pseudo-power $g^{\langle i \rangle}$; the definition printed in the source contains a slip under which the source's own statement of the g -theorem would fail. No symmetry is assumed. More generally, the conclusion holds for every sequence of positive integers with $h_0 = 1$ and $ih_i^2 \geq (i+1)h_{i-1}h_{i+1}$; by Newton's classical inequalities this class contains the coefficient sequences of real-rooted polynomials. The proof combines the elementary inequality $(k+1)!g_{k+1} \leq (k!g_k)^{(k+1)/k}$ for the differences of such sequences with a lower bound for Macaulay's pseudo-power, $a^{\langle k \rangle} \geq (k!a)^{(k+1)/k}/(k+1)!$ for all positive integers a and k ; both inequalities are asymptotically sharp. The palindromic case, in which the conclusion says that h is the h -vector of a simplicial polytope, was proved by Guo and Kang (2026, unrefereed preprint) by a different method; Theorem 9 of the source (Kubitzke and Nevo) and a theorem of Bell and Skandera also follow. The contribution of this note is the lower bound for the pseudo-power and the resulting proof of the general case. The stronger Kruskal–Katona version, asked by Kubitzke and Welker, remains open. The first conjecture of the corpus record, on real roots for barycentric subdivisions of non-simplicial polytopes, is not treated; as far as we know, it is open. This is an unrefereed note.

1 Introduction

1.1 The conjecture

The g -theorem of Billera and Lee [BL81] and Stanley [Sta80] characterises the h -vectors $(h_0, \dots, h_d)$ of the boundary complexes of simplicial d -polytopes: such a vector is symmetric with $h_0 = 1$, and its differences $g_i = h_i - h_{i-1}$ up to the middle degree are nonnegative and satisfy Macaulay's condition $g_{i+1} \leq g_i^{\langle i \rangle}$, with the pseudo-power $n^{\langle i \rangle}$ of Section 2.1. In his extended abstract [Wel08] in the report of the Oberwolfach workshop *Combinatorics* of 6–12 January 2008 (organised by J. Kahn, L. Lovász and H. J. Prömel), V. Welker states the numerical conditions for arbitrary vectors. An integer vector $(h_0, \dots, h_d) \in \mathbb{N}^{d+1}$ satisfies a g -theorem if

$$1 = h_0 < h_1 < \dots < h_r \geq h_{r+1} \geq \dots \geq h_d \tag{1}$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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for some r and, with $g_0 = h_0 = 1$ and $g_i = h_i - h_{i-1}$ for $1 \leq i \leq r$,

$$g_{i+1} \leq g_i^{(i)} \quad (1 \leq i \leq r-1) \quad (2)$$

[Wel08, p. 70]. The index r is determined by the vector: it is the largest index with $h_0 < h_1 < \dots < h_r$, the *first peak*. No symmetry is required. In this language the g -theorem says that h is the h -vector of the boundary complex of a simplicial d -polytope if and only if $h_i = h_{d-i}$ for all i and h satisfies a g -theorem [Wel08, Theorem 7]. Welker then records the following conjecture, which he attributes to Brenti and himself, citing a paper in preparation that we have not found [Wel08, Conjecture 8].

Conjecture 1.1 (Brenti–Welker). *Let $h(t) = h_0 + h_1 t + \dots + h_d t^d \in \mathbb{N}[t]$ be a polynomial of degree d with $h_0 = 1$ and $h_i \geq 1$ for $1 \leq i \leq d$. If $h(t)$ has only real roots, then $(h_0, \dots, h_d)$ satisfies a g -theorem.*

As evidence, the source mentions computations for small d and its Theorem 9, attributed to Kubitzke and Nevo [KN09], which gives the conclusion for the h -polynomials of barycentric subdivisions of Cohen–Macaulay complexes (Corollary 5.3). The definition of the pseudo-power printed in [Wel08] contains a slip; Remark 2.2 explains why the classical definition, which we use, is the only tenable reading.

Conjecture 1.1 is the second part, called “Conjecture 2” there, of record OWR-1782-017 of the UnsolvedMath collection [ula26], which describes the conclusion as the classical g -theorem conditions with the “Macaulay binomial-expansion inequalities”. The first part of the record, which is also its title statement, is Conjecture 6 of [Wel08]: the h -polynomial of the barycentric subdivision of the boundary complex of every, not necessarily simplicial, d -polytope has only real roots. This note does not treat it. It holds for simplicial polytopes, as [Wel08, p. 70] notes, and as far as we know it is open in general; in 2026 it is still listed as an open question in [FMV24, Question 5.8].

1.2 Results

A finite sequence $h_0, h_1, \dots, h_d$ of positive real numbers is called (ULC) if

$$i h_i^2 \geq (i+1) h_{i-1} h_{i+1} \quad (1 \leq i \leq d-1), \quad (3)$$

that is, if the sequence $(i! h_i)_i$ is log-concave.

Theorem A. *Let $d \geq 1$ and let $h_0 = 1, h_1, \dots, h_d$ be positive integers satisfying (3). Then $(h_0, \dots, h_d)$ satisfies a g -theorem.*

Theorem B. *Conjecture 1.1 is true: every polynomial $h \in \mathbb{N}[t]$ of degree $d \geq 1$ with $h_0 = 1$, $h_i \geq 1$ for $1 \leq i \leq d$, and only real roots satisfies a g -theorem.*

Theorem B follows from Theorem A, because by Newton’s inequalities the coefficients of a real-rooted polynomial with positive coefficients satisfy (3) (Lemma 2.3); the class of (ULC) sequences is strictly larger. The proof of Theorem A has two independent parts, which are connected by the function

$$\Phi_k(x) := \frac{(k! x)^{(k+1)/k}}{(k+1)!} \quad (x \geq 0, k \geq 1).$$

Proposition C. *Let $h_0 = 1, h_1, \dots, h_d$ be positive real numbers satisfying (3), and let $1 \leq k \leq d-1$ with $h_{k-1} < h_k$. Then $g_{k+1} \leq \Phi_k(g_k)$, that is, $(k+1)! g_{k+1} \leq (k! g_k)^{(k+1)/k}$, where $g_i = h_i - h_{i-1}$.*

Lemma D. *For all integers $a \geq 1$ and $k \geq 1$ we have $a^{(k)} \geq \Phi_k(a)$, that is, $((k+1)! a^{(k)})^k \geq (k! a)^{k+1}$.*

Proposition C is a continuous form of Macaulay’s condition and uses no integrality; Lemma D says that the discrete Macaulay bound is never smaller than the continuous one. Both are asymptotically sharp for every k (Remarks 6.1 and 6.2). Section 5 also derives the palindromic case (Corollary 5.2), the source’s Theorem 9 (Corollary 5.3) and the theorem of Bell and Skandera (Corollary 5.5).

1.3 What is known and what is new

The palindromic case. Brenti asked whether every symmetric real-rooted polynomial in $\mathbb{N}[t]$ with $h_0 = 1$ is the h -polynomial of a simplicial polytope [Bre24, Problem 2.9], a question circulated by him since 2004 and checked there for $d \leq 9$ and $h_i \leq 100$. By the g -theorem it is equivalent to the palindromic case of Conjecture 1.1. Guo and Kang [GK26, Theorem 1.1] answered it affirmatively in an unrefereed preprint of September 2026. Their proof passes to the γ -vector of h , applies the theorem of Bell and Skandera to it, and builds a pure multicomplex from ballot complexes; it needs the symmetry of h . Corollary 5.2 gives a second proof of their theorem, by a different method.

Special cases and weaker statements. The source attributes its Theorem 9, on barycentric subdivisions of Cohen–Macaulay complexes, to Kubitzke and Nevo [KN09], who proved with an almost strong Lefschetz element that the g -vector of such a subdivision up to the middle degree is an M -sequence [KN09, Corollary 1.3] and located the peak of its h -vector [KN09, Corollary 4.7] (Remark 5.4). Bell and Skandera [BS07] proved that the coefficient sequence of a real-rooted $1 + a_1t + \cdots + a_rt^r \in \mathbb{N}[t]$ is itself an M -sequence, a weaker statement (Corollary 5.5); we know their theorem from three later papers and have not read [BS07]. Mu and Welker [MW26] study the question of Bell and Skandera whether that sequence is even the f -vector of a simplicial complex. Their argument for the theorem of Bell and Skandera is the closest technique to ours that we know: writing the coefficients of an ultra log-concave polynomial of degree d as $\binom{x_i}{i}$ with real x_i and $x_d \geq d$, they show $x_1 \geq \cdots \geq x_d$ [MW26, Theorem 3.1], the real-rooted case being [MW26, Corollary 3.2(i)]. Proposition C and Lemma D compare a continuous and a discrete form of Macaulay’s bound in a similar spirit, but for the differences.

The Kruskal–Katona strengthening. Kubitzke and Welker asked whether, for a real-rooted $1 + c_1t + \cdots + c_dt^d \in \mathbb{N}[t]$ with $1 < c_1 < \cdots < c_l$, the vector $(1, c_1 - 1, c_2 - c_1, \dots, c_l - c_{l-1})$ is the f -vector of a simplicial complex [KW12, Question 4.4]; this is [Bre24, Problem 2.10]. It is stronger than Conjecture 1.1 and remains open; Remark 6.3 explains why our method does not reach it.

What is new. Newton’s inequalities and Macaulay’s theorem are classical (Section 2). The contribution of this note is Lemma D and the resulting proof of the general, non-palindromic case of Conjecture 1.1 (Theorems A and B). We found no proof of the general case, nor Lemma D, in the literature; the searches are described in the paragraph “Scope and priority” at the end, and no priority is claimed.

2 Preliminaries

2.1 Binomial representations and M -sequences

We use the following classical facts; see [BH98, §4.2].

Lemma 2.1 (binomial representation). *Let $n \geq 1$ and $i \geq 1$ be integers. There are unique integers $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$ such that*

$$n = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}. \quad (4)$$

This is the i -binomial representation of n . The *pseudo-power* of n is

$$n^{(i)} := \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}, \quad \text{and} \quad 0^{(i)} := 0.$$

Two simple consequences will be used. If $1 \leq n \leq i$, then (4) is $n = \binom{i}{i} + \binom{i-1}{i-1} + \cdots + \binom{i-n+1}{i-n+1}$, a sum of n ones, and therefore $n^{(i)} = n$. And $n \leq n^{(i)}$ for all $n \geq 0$, because $\binom{a+1}{l+1} = \binom{a}{l+1} + \binom{a}{l} \geq \binom{a}{l}$.

A sequence $(s_0, s_1, \dots, s_m)$ of nonnegative integers is an M -sequence if $s_0 = 1$ and $s_{i+1} \leq s_i^{\langle i \rangle}$ for $1 \leq i \leq m-1$. By Macaulay's theorem [Mac27], [BH98, Theorem 4.2.10], a function $H: \mathbb{N} \rightarrow \mathbb{N}$ is the Hilbert function of a standard graded algebra over a field if and only if $H(0) = 1$ and $H(i+1) \leq H(i)^{\langle i \rangle}$ for all $i \geq 1$; so a finite sequence is an M -sequence if and only if it becomes such a Hilbert function when it is extended by zeros. Macaulay's theorem is used only in Corollary 5.5.

In this language, a vector $h \in \mathbb{N}^{d+1}$ satisfies a g -theorem if and only if it has the shape (1) and $(g_0, g_1, \dots, g_r)$ is an M -sequence. Since h increases strictly up to h_r , the numbers $g_1, \dots, g_r$ are positive, so all pseudo-powers in (2) are taken of positive integers; for $r \leq 1$ the condition (2) is empty.

Remark 2.2 (the definition printed in the source). The source states the expansion (4) “for numbers $n \geq i \geq 1$ ” and then defines $n^{\langle i \rangle}$ as $\binom{a_i-1}{i-1} + \dots + \binom{a_j-1}{j-1}$ [Wel08, p. 70]. Read literally, this operator gives $n^{\langle 1 \rangle} = 1$ for every n , so every vector satisfying a g -theorem would have $g_2 \leq 1$, and the source's own Theorem 7, the g -theorem, would be false: the boundary of the 4-dimensional cross-polytope has $h = (1, 4, 6, 4, 1)$ and $g = (1, 3, 2)$, while classically $3^{\langle 1 \rangle} = \binom{4}{2} = 6 \geq 2$. The Kruskal–Katona operator $\sum_l \binom{a_l}{l+1}$ cannot be meant either: the cyclic polytope $C(8, 5)$ has $h = (1, 3, 6, 6, 3, 1)$ and $g = (1, 2, 3)$, and $\binom{2}{2} = 1 < 3$. The restriction $n \geq i$ is a second imprecision: the M -sequence $g = (1, 5, 3, 2, 1)$ is the g -vector of a simplicial 8-polytope with $h = (1, 6, 9, 11, 12, 11, 9, 6, 1)$ [BL81], and its condition at $i = 3$ involves $2^{\langle 3 \rangle} = 2$, where $2 < 3$; if the restriction were part of the definition, Theorem 7 would fail again. If instead the conditions with $g_i < i$ were simply dropped, Theorem 7 would fail in the other direction: $h = (1, 4, 5, 10, 5, 4, 1)$ is symmetric and would satisfy a g -theorem, but its g -vector $(1, 3, 1, 5)$ is not an M -sequence, since $1^{\langle 2 \rangle} = 1 < 5$. Of these readings, only the classical definition used here makes the source's Theorem 7 the g -theorem; it is also the reading of the corpus record. Under the literal reading Conjecture 1.1 would be false already for $(1+2t)(1+4t) = 1+6t+8t^2$, with $g = (1, 5, 2)$. The printed operator is the operator κ_i of [MW26, Section 2], which occurs in the equivalent form $\kappa_{i+1}(g_{i+1}) \leq g_i$ of Macaulay's condition [MW26, Theorem 2.1]; in the source it is applied to g_i instead of g_{i+1} .

2.2 Newton's inequalities and the shape of (ULC) sequences

Lemma 2.3. *Let $h(t) = \sum_{i=0}^d h_i t^i \in \mathbb{R}[t]$ have degree $d \geq 1$, positive coefficients and only real roots. Then $(h_0, \dots, h_d)$ satisfies (3).*

Proof. By Newton's inequalities (see [Brä15, Lemma 1.1] for a short proof), the numbers $h_i / \binom{d}{i}$ form a log-concave sequence: $h_i^2 \geq h_{i-1} h_{i+1} \binom{d}{i}^2 / (\binom{d}{i-1} \binom{d}{i+1})$ for $1 \leq i \leq d-1$. Since $\binom{d}{i}^2 / (\binom{d}{i-1} \binom{d}{i+1}) = (1 + \frac{1}{i})(1 + \frac{1}{d-i}) \geq 1 + \frac{1}{i}$ and $h_{i-1} h_{i+1} > 0$, we get $h_i^2 \geq (1 + \frac{1}{i}) h_{i-1} h_{i+1}$, which is (3). $\square$

The factor $1 + \frac{1}{d-i}$ is simply dropped; this is why Theorem A holds for all (ULC) sequences. The class of (ULC) sequences is strictly larger: $(1, 2, 2)$ and $(1, 4, 8)$ satisfy (3) with equality, but $1+2t+2t^2$ and $1+4t+8t^2$ have negative discriminants.

Lemma 2.4 (shape). *Let $d \geq 1$ and let $h_0, \dots, h_d$ be positive real numbers satisfying (3), and put $\rho_i := h_i / h_{i-1}$ and $\lambda_i := i \rho_i$ for $1 \leq i \leq d$. Then $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$ and $\rho_1 > \rho_2 > \dots > \rho_d$. Let r be the number of indices i with $\rho_i > 1$. Then*

$$h_0 < h_1 < \dots < h_r \geq h_{r+1} > h_{r+2} > \dots > h_d.$$

If moreover $h_0 = 1$ and the h_i are integers, then h has the shape (1) with this r , the differences $g_1, \dots, g_r$ are positive integers, and $r = 0$ only if $h = (1, 1)$.

Proof. Dividing (3) by $h_{i-1} h_i$ gives $\lambda_i \geq \lambda_{i+1}$. Hence $\rho_{i+1} = \lambda_{i+1} / (i+1) \leq \lambda_i / (i+1) < \lambda_i / i = \rho_i$. As ρ is strictly decreasing, the indices with $\rho_i > 1$ are $1, \dots, r$, so h increases strictly up to h_r ; moreover

$\rho_{r+1} \leq 1$ and $\rho_i < 1$ for $i \geq r + 2$. For integers this gives $g_i = h_i - h_{i-1} \geq 1$ for $1 \leq i \leq r$. If $r = 0$, then $h_1 \leq h_0 = 1$, so $h_1 = 1$; if $d \geq 2$, (3) at $i = 1$ would give $1 \geq 2h_2 \geq 2$, so $d = 1$. $\square$

3 Proof of Proposition C

Let $h_0 = 1, h_1, \dots, h_d$ be positive reals satisfying (3), let $1 \leq k \leq d - 1$ with $h_{k-1} < h_k$, and let ρ_i, λ_i be as in Lemma 2.4. Put $L := \lambda_k = kh_k/h_{k-1}$. Then $L > k$, because $\rho_k > 1$, and $u := k/L$ lies in $(0, 1)$.

Step (i). Since $h_{k-1} = kh_k/L$, we have $g_k = h_k - h_{k-1} = h_k(L - k)/L$, that is, $h_k = g_k L / (L - k)$.

Step (ii). By Lemma 2.4, $\rho_{k+1} = \lambda_{k+1}/(k+1) \leq L/(k+1)$. Hence, using $L(L - k - 1) = (L - k)(L - 1) - k$,

$$g_{k+1} = h_k(\rho_{k+1} - 1) \leq h_k \frac{L - k - 1}{k + 1} = \frac{g_k}{k + 1} \cdot \frac{L(L - k - 1)}{L - k} = \frac{g_k}{k + 1} \left(L - 1 - \frac{k}{L - k} \right). \quad (5)$$

Step (iii). Since $h_0 = 1$ and $\lambda_i \geq \lambda_k = L$ for $i \leq k$,

$$h_k = \prod_{i=1}^k \rho_i = \frac{1}{k!} \prod_{i=1}^k \lambda_i \geq \frac{L^k}{k!}.$$

With step (i), $k! g_k = k! h_k(L - k)/L \geq L^{k-1}(L - k) = L^k(1 - u)$, so

$$y := (k! g_k)^{1/k} \geq L(1 - u)^{1/k}. \quad (6)$$

Step (iv). For $0 < u < 1$ and $k \geq 1$,

$$(1 - u)^{1/k} \geq 1 - \frac{u}{k(1 - u)}. \quad (7)$$

Indeed, $\ln x \leq x - 1$ with $x = 1/(1 - u)$ gives $\ln(1 - u) \geq -u/(1 - u)$, hence $(1 - u)^{1/k} = \exp(\frac{1}{k} \ln(1 - u)) \geq \exp(-\frac{u}{k(1 - u)}) \geq 1 - \frac{u}{k(1 - u)}$.

Step (v). By (6), (7) and $Lu/k = 1$,

$$y \geq L - \frac{Lu}{k(1 - u)} = L - \frac{1}{1 - u} = L - \frac{L}{L - k} = L - 1 - \frac{k}{L - k}.$$

Together with (5) and $g_k > 0$ this gives

$$g_{k+1} \leq \frac{g_k y}{k + 1} = \frac{g_k (k! g_k)^{1/k}}{k + 1} = \frac{(k! g_k)^{(k+1)/k}}{(k + 1)!} = \Phi_k(g_k).$$

This proves Proposition C. $\square$

The proof uses (3) only at the indices $1, \dots, k$, and it uses $h_0 = 1$; it does not use integrality. Both inequalities used in (7) are strict for $0 < u < 1$, so in fact $g_{k+1} < \Phi_k(g_k)$.

4 Proof of Lemma D

Let $k \geq 1$ and $a \geq 1$, and let $a = \binom{a}{k} + \binom{a}{k-1} + \dots + \binom{a}{j}$ be the k -binomial representation (Lemma 2.1). Put $n := a_k$ and $b := a - \binom{n}{k}$, the sum of the remaining terms (zero if $j = k$). Since the a_l decrease strictly and $a_j \geq j$, we have $n \geq k$ and $a_l \leq n - k + l$ for $j \leq l \leq k$. Two facts follow.

(F1) $0 \leq b \leq \binom{n}{k-1} - 1$. Indeed, $b \leq \sum_{l=1}^{k-1} \binom{n-k+l}{l} = \binom{n}{k-1} - 1$, by the identity $\sum_{l=0}^{k-1} \binom{n-k+l}{l} = \binom{n}{k-1}$.

(F2) If $k \geq 2$, then $a^{(k)} = \binom{n+1}{k+1} + b^{(k-1)}$. Indeed, if $b \geq 1$, the terms $\binom{a_l}{l}$ with $l \leq k-1$ satisfy the conditions of Lemma 2.1 for b and $i = k-1$, so they form the $(k-1)$ -binomial representation of b ; if $b = 0$, then $a^{(k)} = \binom{n+1}{k+1}$ and $0^{(k-1)} = 0$.

We prove Lemma D by induction on k , for all a simultaneously.

The case $k = 1$. Here $a = \binom{a}{1}$, and $a^{(1)} = \binom{a+1}{2} = a(a+1)/2 \geq a^2/2 = \Phi_1(a)$.

The case $k \geq 2$, $n = k$. Then $\binom{k}{k} \leq a < \binom{k+1}{k}$ by (F1), that is, $1 \leq a \leq k$, and $a^{(k)} = a$ (Section 2.1). The inequality $a \geq \Phi_k(a)$ is equivalent to $((k+1)!)^k a^k \geq (k!)^{k+1} a^{k+1}$, that is, to $(k+1)^k \geq k! a$. This holds because $k! a \leq k \cdot k! \leq k \left(\frac{k+1}{2}\right)^k \leq (k+1)^k$, by the inequality of arithmetic and geometric means, $k! \leq \left(\frac{1+2+\dots+k}{k}\right)^k = \left(\frac{k+1}{2}\right)^k$, and by $k \leq 2^k$.

The case $k \geq 2$, $n \geq k+1$. By (F2) and the induction hypothesis applied to b (or $\Phi_{k-1}(0) = 0$ if $b = 0$),

$$a^{(k)} \geq \binom{n+1}{k+1} + \Phi_{k-1}(b).$$

Put $P := n(n-1)\dots(n-k+1) = k!\binom{n}{k}$, $\beta := k!b$, $p := k/(k-1)$ and $q := (k+1)/k$. Then $(k+1)!\binom{n+1}{k+1} = (n+1)P$, $(k+1)!\Phi_{k-1}(b) = (k+1)((k-1)!b)^p = (k+1)(\beta/k)^p$ and $(k+1)!\Phi_k(a) = (k!a)^q = (P+\beta)^q$. So it suffices to prove

$$(n+1)P + (k+1)\left(\frac{\beta}{k}\right)^p \geq (P+\beta)^q \quad \text{for } 0 \leq \beta \leq \beta_{\max} := k!\binom{n}{k-1}, \quad (8)$$

which covers $\beta = k!b$ by (F1). Note that $\beta_{\max} = kP/(n-k+1)$ and that $P' := P + \beta_{\max} = (n+1)P/(n-k+1) = (n+1)n\dots(n-k+2)$.

Step 1 (a chord). The function $x \mapsto x^q$ is convex, so on $[P, P']$ it lies below its chord: $(P+\beta)^q \leq P^q + m\beta$ for $0 \leq \beta \leq \beta_{\max}$, where $m := (P'^q - P^q)/(P' - P)$. By the mean value theorem $m = q\xi^{q-1} = q\xi^{1/k}$ for some $\xi \in (P, P')$, so $0 < m < qP'^{1/k}$.

Step 2 (a minimum). Let $c > 0$ and $F(\beta) := (k+1)(\beta/k)^p - c\beta$ for $\beta \geq 0$. The substitution $\beta = ks^{k-1}$, a bijection of $[0, \infty)$ onto itself for which $(\beta/k)^p = s^k$, turns F into $G(s) := (k+1)s^k - cks^{k-1}$. Since $G'(s) = ks^{k-2}((k+1)s - c(k-1))$, the function G decreases on $[0, s_*]$ and increases on $[s_*, \infty)$, where $s_* := c(k-1)/(k+1)$. Hence, for all $\beta \geq 0$,

$$F(\beta) \geq G(s_*) = s_*^{k-1}((k+1)s_* - ck) = -cs_*^{k-1} = -c^k \left(\frac{k-1}{k+1}\right)^{k-1}. \quad (9)$$

Step 3. By Steps 1 and 2 with $c = m$, for $0 \leq \beta \leq \beta_{\max}$,

$$\begin{aligned} (n+1)P + (k+1)\left(\frac{\beta}{k}\right)^p - (P+\beta)^q &\geq (n+1)P - P^q + F(\beta) \\ &\geq (n+1)P - P^q - m^k \left(\frac{k-1}{k+1}\right)^{k-1} > (n+1)P - P^q - \gamma_k P', \end{aligned}$$

where, since $m^k < q^k P'$,

$$\gamma_k := q^k \left(\frac{k-1}{k+1}\right)^{k-1} = \frac{(k+1)(k-1)^{k-1}}{k^k}.$$

Dividing by P and using $P^q/P = P^{1/k}$ and $P'/P = (n+1)/(n-k+1)$, we see that (8) follows from

$$(n+1) - P^{1/k} \geq \gamma_k \frac{n+1}{n-k+1}. \quad (10)$$

Step 4 (proof of (10)). By the inequality of arithmetic and geometric means, $P^{1/k} \leq \frac{1}{k} \sum_{t=0}^{k-1} (n - t) = n - \frac{k-1}{2}$, so the left side of (10) is at least $\frac{k+1}{2}$. On the right, $\frac{n+1}{n-k+1} = 1 + \frac{k}{n-k+1} \leq \frac{k+2}{2}$ for $n \geq k+1$, and $\gamma_k \leq \frac{3}{4}$: $\gamma_2 = \frac{3}{4}$, and for $k \geq 3$

$$\ln \gamma_k = \ln\left(1 + \frac{1}{k}\right) + (k-1) \ln\left(1 - \frac{1}{k}\right) \leq \frac{1}{k} - \frac{k-1}{k} = -\frac{k-2}{k} \leq -\frac{1}{3},$$

so $\gamma_k \leq e^{-1/3} < \frac{3}{4}$. Hence the right side of (10) is at most $\frac{3}{8}(k+2) \leq \frac{k+1}{2}$, the last inequality being equivalent to $k \geq 2$. This proves (10), hence (8), and Lemma D. $\square$

Steps 1–4 are used only for $n \geq k+1$; for $k = n = 2$, (10) would read $3 - \sqrt{2} \geq \frac{9}{4}$, which is false. In the computations of Section 7 the ratio of the two sides of (10) is smallest, about 1.034, at $(k, n) = (2, 3)$, where (10) reads $\frac{5}{2} \geq \sqrt{6}$.

5 Proofs of Theorems A and B, and consequences

Proof of Theorem A. By Lemma 2.4, h has the shape (1) with its first peak r , and $g_1, \dots, g_r$ are positive integers. Let $1 \leq k \leq r-1$. Then $k \leq d-1$ and $h_{k-1} < h_k$, so Proposition C gives $g_{k+1} \leq \Phi_k(g_k)$, and Lemma D with $a = g_k$ gives $\Phi_k(g_k) \leq g_k^{(k)}$. Hence (2) holds. $\square$

Proof of Theorem B. By Lemma 2.3 the coefficients satisfy (3); apply Theorem A. $\square$

The hypothesis $h_i \geq 1$ of Conjecture 1.1 is automatic:

Lemma 5.1. *Let $h \in \mathbb{R}[t]$ have degree e , nonnegative coefficients, $h(0) = 1$ and only real roots. Then all coefficients $h_0, \dots, h_e$ are positive.*

Proof. For $t \geq 0$ we have $h(t) \geq 1$, so the roots are negative, say $-s_1, \dots, -s_e$ with $s_j > 0$, and $h(t) = \prod_{j=1}^e (1 + t/s_j)$ has positive coefficients up to degree e . $\square$

Corollary 5.2 (the palindromic case; Guo and Kang). *Let $h \in \mathbb{N}[t]$ have degree $d \geq 1$, $h_0 = 1$ and only real roots, and let $h_i = h_{d-i}$ for $0 \leq i \leq d$. Then $(h_0, \dots, h_d)$ is the h -vector of the boundary complex of a simplicial d -polytope.*

Proof. By Lemmas 2.3, 2.4 and 5.1, h attains its maximum exactly at its first peak r , or exactly at r and $r+1$. This set of indices is symmetric about $d/2$, so $r = d/2$ or $r = (d-1)/2$, that is, $r = \lfloor d/2 \rfloor$. By Theorem B, $(g_0, \dots, g_{\lfloor d/2 \rfloor})$ is an M -sequence. By the sufficiency part of the g -theorem [BL81] (the “if” part of [Wel08, Theorem 7]), h is the h -vector of a simplicial d -polytope. $\square$

This is [GK26, Theorem 1.1], where it is proved by a different method (Section 1.3).

Corollary 5.3 (Theorem 9 of the source). *Let Δ be a $(d-1)$ -dimensional simplicial complex with $h_i^\Delta \geq 0$ for $0 \leq i \leq d$, for instance a Cohen–Macaulay complex [KN09, Theorem 3.1]. Then the h -vector of its barycentric subdivision $\text{sd}(\Delta)$ satisfies a g -theorem.*

Proof. By [BW08, Theorem 2.2], $h_j^{\text{sd}(\Delta)} = \sum_{r=0}^d A(d+1, j, r+1) h_r^\Delta$ for $0 \leq j \leq d$, where $A(d+1, j, r+1)$ is the number of permutations of $\{1, \dots, d+1\}$ with j descents and first letter $r+1$. So $h^{\text{sd}(\Delta)}$ has nonnegative integer entries, and $h_0^{\text{sd}(\Delta)} = 1$. By [BW08, Theorem 3.1] (see Theorem 4 of [Wel08]), the polynomial $\sum_j h_j^{\text{sd}(\Delta)} t^{d-j}$ has only real roots, and so has $\sum_j h_j^{\text{sd}(\Delta)} t^j$, since reversing the order of the coefficients preserves this property. Let e be the degree of the latter polynomial. If $e = 0$ there is nothing to prove. Otherwise its coefficients up to degree e are positive (Lemma 5.1), Theorem B applies to it, and appending the zeros $h_{e+1}^{\text{sd}(\Delta)} = \dots = h_d^{\text{sd}(\Delta)} = 0$ affects neither (1) nor (2). $\square$

Remark 5.4 (Kubitzke and Nevo’s Corollary 1.3). For a Cohen–Macaulay complex Δ , Kubitzke and Nevo proved that the g -vector $(g_0, \dots, g_m)$ of $\text{sd}(\Delta)$, with $m = \lfloor d/2 \rfloor$, is an M -sequence [KN09, Corollary 1.3]. Corollary 5.3 alone does not give this, because it concerns the differences up to the first peak r . It does give it together with their result that, for $h^\Delta \geq 0$, the h -vector of $\text{sd}(\Delta)$ has its peak at position $d/2$ if d is even and at $(d-1)/2$ or $(d+1)/2$ if d is odd [KN09, Corollary 4.7]. Indeed, if $r \geq m$, then $(g_0, \dots, g_m)$ is an initial segment of the M -sequence $(g_0, \dots, g_r)$. If $r < m$, then by Lemma 2.4 the maximum is attained only at r and possibly at $r+1$, so the peak is at $m = r+1$ with $h_r = h_{r+1}$; then $g_m = 0$, and $(g_0, \dots, g_r, 0)$ is again an M -sequence.

Corollary 5.5 (the theorem of Bell and Skandera). *Let $h \in \mathbb{N}[t]$ have degree d , $h_0 = 1$ and only real roots. Then $(h_0, \dots, h_d)$ is an M -sequence.*

Proof. We may assume $d \geq 1$. By Lemma 5.1 and Theorem B, h has the shape (1) and $(g_0, \dots, g_r)$ is an M -sequence. By Macaulay’s theorem there is a standard graded algebra A over a field K whose Hilbert function is $g_0, \dots, g_r, 0, 0, \dots$. The polynomial ring $A[y]$, with y of degree 1, is again standard graded, and $\dim_K A[y]_i = \sum_{l=0}^i \dim_K A_l$; so its Hilbert function is $h_0, \dots, h_r, h_r, h_r, \dots$, and Macaulay’s theorem gives $h_{i+1} \leq h_i^{\langle i \rangle}$ for $1 \leq i \leq r-1$. For $r \leq i \leq d-1$ we have $h_{i+1} \leq h_i \leq h_i^{\langle i \rangle}$ by (1) and Section 2.1. $\square$

This is the theorem of Bell and Skandera [BS07] as stated in [KW12, proof of Proposition 4.3] and [GK26, Theorem 1.4], and, in terms of h -vectors of Cohen–Macaulay complexes, in [MW26, Corollary 3.3]; the first two papers cite it as Theorem 3.6 of [BS07], the third as Theorem 4.2 there. We have not read [BS07].

6 Remarks

Remark 6.1 (sharpness of Lemma D). For every fixed k , Lemma D is asymptotically sharp. For $a = \binom{n}{k}$ we have $a^{\langle k \rangle} = \binom{n+1}{k+1}$, hence $a^{\langle k \rangle} / \Phi_k(a) = (n+1)P / P^{(k+1)/k} = (n+1) / P^{1/k}$ with $P = n(n-1) \cdots (n-k+1)$; since $P^{1/k} = n - \frac{k-1}{2} + O(\frac{1}{n})$, the ratio is $1 + \frac{k+1}{2n} + O(n^{-2})$. Inside a block the ratio is smaller: for $k=2$ and $a = \binom{n}{2} + b$ with $b/n \rightarrow x \in [0, 1]$ it is $1 + (3x^2 - 3x + \frac{3}{2})/n + o(\frac{1}{n})$, smallest, $1 + \frac{3}{4n}$, in the middle of the block. In the ranges that were computed, however, the smallest ratio occurs at $k=1$, where $a^{\langle 1 \rangle} / \Phi_1(a) = (a+1)/a$: for every $A \leq 20,000$ the smallest ratio over $1 \leq a \leq A$ and $1 \leq k \leq 40$ is $(A+1)/A$, at $k=1$ and $a=A$ (Section 7). Heuristically, for $k \geq 2$ the relevant n is only of order $a^{1/k}$; we have no proof for all ranges.

Remark 6.2 (sharpness of Proposition C). Proposition C is asymptotically sharp for every k . The real sequence $h_i = L^i / i!$ ($0 \leq i \leq d$) has $\lambda_i = L$ for all i , so it satisfies (3) with equality. It has $g_k = L^{k-1}(L-k)/k!$ and $g_{k+1} = L^k(L-k-1)/(k+1)!$, hence

$$\frac{g_{k+1}}{\Phi_k(g_k)} = \left(\frac{L}{L-k} \right)^{1/k} \frac{L-k-1}{L-k} = 1 - \frac{k+1}{2L^2} + O(L^{-3}) \quad (L \rightarrow \infty).$$

Remark 6.3 (the Kruskal–Katona strengthening). The question of Kubitzke and Welker [KW12, Question 4.4], which is [Bre24, Problem 2.10], asks for more than (2): that $(1, g_1, \dots, g_r)$ be the f -vector of a simplicial complex. Theorem A cannot give this, since under (3) alone it is false: $(1, 4, 8)$ satisfies (3) and a g -theorem, but $g = (1, 3, 4)$ is not an f -vector, because a simplicial complex with three vertices has at most three edges. (The polynomial $1 + 4t + 8t^2$ is not real-rooted.) A proof of the Kruskal–Katona version would have to use more of real-rootedness than (3), for instance the factor $1 + \frac{1}{d-i}$ dropped in Lemma 2.3. The original programs and the first verification run (Section 7) also tested this version on their real-rooted examples and found no counterexample; the question remains open.

7 Computations and reproducibility

All statements of this note are proved without computation; the computations test them on exact instances. The program of this note, `check_note.py` (Python standard library, exact arithmetic; about 12 seconds), tests: the uniqueness of binomial representations ($i \leq 6$, $n \leq 300$, by exhaustive search); Remark 2.2, including Theorem 7 of the source on the h -vectors of 135 cyclic polytopes and cross-polytopes (no violation with the classical pseudo-power, 91 with the printed operator and 91 with the Kruskal–Katona operator); Lemma D for all $k \leq 30$ and $a \leq 2500$, on 67,992 points of the blocks $[(\binom{n}{k}, \binom{n+1}{k})]$ with $k \leq 12$ and on 2000 random pairs with $a < 10^{45}$, together with (F1), (F2), the constants and identities of its proof, (9), and (10) with Step 4 for $k \leq 30$, $n \leq k + 150$; Proposition C with (5)–(7) on 6000 random rational (ULC) sequences; Theorem A and Lemma 2.4 on all 370,612 integer (ULC) sequences with $h_0 = 1$, $h_i \leq 30$ and $2 \leq d \leq 10$; Theorem B and Corollaries 5.2, 5.3 and 5.5 on all real-rooted $(1, h_1, \dots, h_d)$ with $d = 2, 3, 4$ and $h_i \leq 60, 40, 25$ (3008, 7538 and 542 polynomials, by exact Sturm sequences), on $(1 + t)^n$ ($n \leq 200$), the Eulerian and Narayana polynomials ($n \leq 22$), 3000 random products of real-rooted factors (1014 of them palindromic) and 400 h -vectors of barycentric subdivisions computed by [BW08, Theorem 2.2], with the peak position used in Remark 5.4; the examples of Remark 6.3 and the partial-sum argument of Corollary 5.5; and, numerically, (8) and Remarks 6.1 and 6.2. All checks pass. The exhaustive tests of real-rooted polynomials reach only small degrees (the original program found only $(1 + t)^5$ in the boxes of degree 5, 6 and 7 with $h_i \leq 12, 7$ and 5), so the conditions (2) with $k \geq 2$ are exercised mainly by the (ULC) enumerations, the families and the random products.

The programs with which the results were first obtained (`original/`) test Lemma D exactly on 2,379,243 pairs with $k \leq 60$, Conjecture 1.1 on all 11,089 real-rooted polynomials in the boxes of degree 2 to 7 and on 20,000 random products of linear factors, Theorem A on the 370,612 sequences above, and (7)–(10) numerically. The programs of the verification runs, with their recorded outputs, are in `verification_run_A/`, `verification_run_B/` and `independent_run_2/` (see the next paragraph). All recorded outputs of the package were reproduced from copies of the package on 17 October 2026, byte for byte apart from the lines of `independent_run_2/` that report running times; the longest program runs for about six and a half minutes. The file `README.md` of the package lists the command lines and running times.

Verification. The results were first obtained with the proofs of Sections 2 to 5 and the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs line by line and wrote its own programs. The first rendered page 70 of the source as an image, which confirmed the printed signs of Remark 2.2, and checked exactly Lemma D on 990,415 pairs (a, k) , Proposition C with (5) and (6) on 300,000 random rational (ULC) sequences, Theorem A on all 34,721,012 integer (ULC) sequences with $h_i \leq 55$ and $2 \leq d \leq 12$, Conjecture 1.1 on 23,575 real-rooted polynomials of degree up to 120 (certified by its own Sturm sequences or by factorisation), and every identity of Sections 3 and 4. The second also read page 70 as an image; it tested three readings of the pseudo-power against Theorem 7 of the source on 135 polytopes (Remark 2.2), checked Lemma D exactly on 1,201,509 pairs, some with $a < 10^{60}$, Proposition C over the rationals, and Theorem A on every integer (ULC) sequence with entries up to 10^5 by an exhaustive dynamic programme (2.27×10^{10} cases, validated against a direct enumeration); it certified real-rootedness by Hermite’s criterion, re-ran the original programs (identical outputs) and repeated the literature search. Both runs confirmed the statement, the reading of the source and all proofs, and found no mathematical error. All corrections they required were made; they concern the organisers of the workshop, the binomial representation for all $n, i \geq 1$ (Section 2.1, Remark 2.2), the sharpness remarks (Remarks 6.1 and 6.2), the relation to [KN09, Corollary 1.3] (Remark 5.4), the hypotheses of [MW26, Theorem 3.1], the partial status of the corpus record, and the degree range of the exhaustive tests. In this text Step 2 of the proof

of Lemma D uses the substitution $\beta = ks^{k-1}$; the minimum is the one checked by both runs. The program `check_note.py` was written after the two runs, together with this text.

After the note was written, a further independent verification run, also AI-assisted, examined its text, first the parts that the two runs had not seen: Step 2 of the proof of Lemma D, Lemma 5.1, the proofs of Corollaries 5.2 and 5.3, Remark 5.4 and the expansions of Remarks 6.1 and 6.2. It read the source again, with page 70 as an image, and the versions of the cited papers listed below, compared every quoted statement and number with them, and re-derived all proofs line by line. Its own programs (`independent_run_2/`), written before it ran those of the package, check Lemma D exactly on 627,794 pairs (a, k) , some with $a < 10^{80}$, with (F1) and (F2), and (10) exactly for $k \leq 60$, $n \leq k + 400$; Proposition C in its strict form over the rationals; Theorem A and Lemma 2.4 on all 7,038,047 integer (ULC) sequences with $h_0 = 1$, $h_i \leq 45$ and $1 \leq d \leq 12$; Theorem B and Corollary 5.5 on the boxes above, found again by deciding real-rootedness with Descartes' rule of signs and bisection, and, with Corollary 5.2, on 3500 products of real-rooted factors, 1000 of them palindromic; Corollary 5.3, Remark 5.4 and the formula of [BW08, Theorem 2.2] on 974 simplicial complexes, with the f -vectors of the subdivisions counted from chains of faces; and the expansions of Remarks 6.1 and 6.2, symbolically, and the observation at the end of Remark 6.1 for all ranges $a \leq A \leq 20,000$. It ran all programs of the package again from the extracted archive and reproduced their recorded outputs, and it repeated the literature search. It found no error in a proof. It required the statement of the status of the first conjecture of the record with a source (Section 1), two additions to Remark 2.2 (the reading in which conditions with $g_i < i$ are dropped, and the operator κ_i), the hypothesis $d \geq 1$ in Lemma 2.4, the restriction of the end of Remark 6.1 to the computed ranges, the exact ranges of degrees in two counts of this section, two more precise sentences in Section 1, and the record of this run. All corrections were made.

Scope and priority. The note proves Conjecture 8 of [Wel08], the second conjecture of record OWR-1782-017 of [ula26], under the classical definition of the pseudo-power (Remark 2.2), and more generally its analogue for (ULC) sequences (Theorem A). The first conjecture of the record, which is also its title statement, is Conjecture 6 of [Wel08] on the barycentric subdivisions of non-simplicial polytopes; it is not treated, so the record is answered only in part. The conjecture proved here is due to Brenti and Welker, as recorded by Welker [Wel08]. Its palindromic case was proved by Guo and Kang [GK26] (unrefereed preprint, September 2026) by a different method, through the γ -vector and the theorem of Bell and Skandera; Corollary 5.2 gives a second proof of that case. The case of barycentric subdivisions of Cohen–Macaulay complexes is due to Kubitzke and Nevo [KN09], and the M -sequence property of the coefficients themselves to Bell and Skandera [BS07]. Newton's inequalities [Brä15] and Macaulay's theorem [Mac27, BH98] are classical, and the g -theorem is due to Billera and Lee [BL81] and Stanley [Sta80]. The contribution of this note is Lemma D and the resulting proof of the general, non-palindromic case. The stronger Kruskal–Katona version [KW12, Question 4.4], [Bre24, Problem 2.10] remains open (Remark 6.3). No printed answer to the question was found in the sources searched on 17 October 2026: the arXiv (queries to its API, and the texts of [GK26, KN09, KW12, Bre24, MW26, BW08, Brä15] and of further papers on real-rooted h -polynomials), Crossref, OpenAlex and Semantic Scholar (among them the lists of works citing [BW08], [BS07], [KW12], [Bre24] and [GK26]), zbMATH Open, and six web searches. Some of these requests were refused because of rate limits. The verification run of the written note repeated the search in zbMATH Open, among them all its records of 2026 on real-rooted polynomials, in Crossref, and in a seventh web search, read the discussions of the first conjecture of the record in [FMV24], and found nothing new; its queries to the arXiv API and to OpenAlex were refused because of rate limits. A search that finds nothing is not a proof of novelty, and no priority is claimed. What was read: [Wel08] completely, with page 70 also as a rendered image; [GK26] completely; of [KN09] the introduction,

Section 3 and Corollary 4.7; of [KW12] Section 4.1; of [Bre24] Section 2; of [MW26] Sections 1 to 3.1; of [BW08] the introduction and the statements of Sections 2 and 3; of [Brä15] Section 1; of [FMV24] the introduction and Section 5.3. We did not read [BS07], whose publisher’s site refused automated access and of which no other copy was found; what we say about it is taken from [KW12, GK26, MW26]. Nor did we read the survey [MW17] or the classical sources [Mac27, BH98, BL81, Sta80]; the paper “in preparation” to which [Wel08] attributes the conjecture was not found.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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