Small Counterexamples to the All-n Form of the Exact Codegree Conjecture for Tight Hamiltonian Cycles

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Abstract

Rödl, Ruciński and Szemerédi stated the following conjecture, which they attribute to Katona and Kierstead: every k-uniform hypergraph on $n \geq k+1 \geq 4$ vertices in which every $(k-1)$-set lies in at least $\lfloor (n-k+3)/2 \rfloor$ edges has a tight Hamiltonian cycle. They proved it for $k=3$ and all sufficiently large n. We observe that the statement, as written for all $n \geq k+1$, fails for small n. The smallest counterexample is a 3-graph on 7 vertices: an apex joined to all pairs of a 6-set, together with the ten faces of the hemi-icosahedron on that set. It has minimum codegree $3 = \lfloor 7/2 \rfloor$ but no tight Hamiltonian cycle, and the proof is two lines. A variant of the construction gives counterexamples for $k=4$ and $k=5$ on $k+4$ vertices. A computer search finds two more, on 9 vertices for $k=3$ and on 10 vertices for $k=5$; the second meets the bound $(n-k+3)/2$ even without the floor. These are exceptions for small n only, and the asymptotic statement is not affected. This is an unrefereed note.

1 Introduction

Let H be a k-uniform hypergraph (k-graph) on a vertex set V with $|V| = n$. The *codegree* of a $(k-1)$-set $S \subseteq V$ is the number of edges containing S, and $\delta_{k-1}(H)$ denotes the minimum codegree. A *tight Hamiltonian cycle* is a cyclic order $v_1, \dots, v_n$ of V in which every k cyclically consecutive vertices form an edge.

Katona and Kierstead [KK99] showed that minimum codegree $(1 - \frac{1}{2k})n + O(1)$ forces a tight Hamiltonian cycle. For $n \geq k^2+1$ they also constructed k-graphs with minimum codegree at least $\lfloor (n-k+1)/2 \rfloor$ and no tight Hamiltonian cycle; see [RR10, Construction 2.1]. Rödl and Ruciński write that Katona and Kierstead implicitly conjectured this lower bound to be the correct threshold [RR10]. The conjecture was stated formally by Rödl, Ruciński and Szemerédi.

Conjecture 1.1 ([RRS11, Conjecture 1.1]; [KLP08, p. 46])**.** *Let H be a k-graph on $n \geq k+1 \geq 4$ vertices. If $\delta_{k-1}(H) \geq \lfloor (n-k+3)/2 \rfloor$, then H has a tight Hamiltonian cycle.*[1]

The same statement, with no range of n, appears in [ABW06, p. 2928]. The all-n form is repeated in later work, for example [LL21, Conjecture 1.1], and [LY26] states the bound $(n-k+3)/2$ without the floor. Rödl, Ruciński and Szemerédi proved an approximate version for every k [RRS08], and the exact statement for $k=3$ and all sufficiently large n [RRS11]. For every fixed $k \geq 3$ and all sufficiently large n, the exact statement has been announced by Letzter, Lang, Ranganathan and Sanhueza-Matamala, as reported in [Zha26].

Conjecture 1.1 quantifies over all $n \geq k+1$. We show that it fails for some small values of n.

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[1] In [KLP08], and in the 2009 preprint of [RRS11], the index range of the wrap-around edges in the definition of a cycle contains a misprint. The intended cyclic definition is the one used here.

Theorem 1.2. (a) *For $k \in \{3,4,5\}$ and $n = k+4$ there is a k-graph H_k on n vertices with $\delta_{k-1}(H_k) = 3 = \lfloor (n-k+3)/2 \rfloor$ and no tight Hamiltonian cycle.*

(b) *There is a 3-graph on 9 vertices with minimum codegree $4 = \lfloor 9/2 \rfloor$ and no tight Hamiltonian cycle.*

(c) *There is a 5-graph on 10 vertices with minimum codegree $4 = (10-5+3)/2$ and no tight Hamiltonian cycle.*

So Conjecture 1.1 fails for $(k,n) \in \{(3,7),(3,9),(4,8),(5,9),(5,10)\}$, and part (c) also refutes the version without the floor. An exhaustive check shows that every k-graph on at most 6 vertices satisfying the bound has a tight Hamiltonian cycle, so H_3 has the least possible number of vertices.

2 The construction

Let $W = \{1,\dots,6\}$ and let D' be the set of the ten faces of the hemi-icosahedron, the 6-vertex triangulation of the real projective plane:

$$D' = \{123,\ 124,\ 135,\ 146,\ 156,\ 236,\ 245,\ 256,\ 345,\ 346\}.$$

Let D be the set of the other ten triples of W.

Lemma 2.1. (F0) *Every pair of W lies in exactly two triples of D' and in exactly two triples of D.*

(F1) *A triple T lies in D' if and only if $W \setminus T$ lies in D. In particular, neither D' nor D contains two disjoint triples.*

Proof. Both are direct checks on the list above. (F0) says that D' is a 2-$(6,3,2)$ design. Since each pair of W lies in four triples of W, (F0) for D follows. For (F1), the complements of the ten triples of D' are 456, 356, 246, 235, 234, 145, 136, 134, 126 and 125, and none of them is in D'. Two disjoint triples of W are complements of each other, so they cannot both lie in D', or both in D. □

For $k \in \{3,4,5\}$ let P be a set of $k-2$ further vertices, the *apexes*, and let $V = W \cup P$, so that $n = k+4$. Let

$$E(H_k) = \left\{ e \in \binom{V}{k} : |e \cap W| \neq 3 \right\} \ \cup\ \left\{ e \in \binom{V}{k} : e \cap W \in D' \right\}.$$

For $k=3$ there is one apex a, and H_3 consists of the 15 triples through a and the 10 triples of D'.

Proposition 2.2. $\delta_{k-1}(H_k) = 3 = \lfloor (n-k+3)/2 \rfloor$ *for $k = 3,4,5$.*

Proof. Let S be a $(k-1)$-set and let $j = |S \cap W|$, so $1 \le j \le 6$. The set S misses $j-1$ apexes and $6-j$ points of W. Adding an apex to S gives a k-set meeting W in j points, and adding a point of W gives one meeting W in $j+1$ points.

- If $j \notin \{2,3\}$, every extension of S is an edge, so S has codegree 5.
- If $j = 2$, the extension by the missing apex is an edge, and by (F0) exactly two of the four extensions by points of W are edges. So S has codegree 3.
- If $j = 3$, the three extensions by points of W meet W in four points and are edges, so S has codegree at least 3.

Sets with $j=2$ exist, so the minimum is 3, and $\lfloor (n-k+3)/2 \rfloor = \lfloor 7/2 \rfloor = 3$. □

For $k = 3$ the absence of a tight Hamiltonian cycle is immediate. In a cyclic order $a, w_1, \ldots, w_6$ the windows $w_1w_2w_3$ and $w_4w_5w_6$ avoid the apex, so both would have to be faces in D'. But they are disjoint, contradicting (F1). For $k = 4, 5$ we use the complementary windows. Fix a cyclic order of V. The complement of a window of k consecutive vertices is a window of 4 consecutive vertices, which we call a 4-*window*.

Lemma 2.3. *A cyclic order of V is a tight Hamiltonian cycle of H_k if and only if every 4-window containing exactly one apex has its three points of W in D.*

Proof. Let e be a k-window and I the complementary 4-window. Then $|e \cap W| = 3$ exactly when $|I \cap W| = 3$, that is, when I contains exactly one apex. In that case $e \cap W = W \setminus (I \cap W)$, which lies in D' if and only if $I \cap W$ lies in D, by (F1). $\square$

Let $w_1, \ldots, w_6$ be the points of W in the cyclic order, with indices modulo 6. Let $g_i \geq 0$ be the number of apexes between w_i and w_{i+1}, so $\sum_i g_i = k - 2$. Put $\tau_i = \{w_i, w_{i+1}, w_{i+2}\}$; the triples τ_i and τ_{i+3} are complementary in W. Call τ_i *forced* if some 4-window with exactly one apex has τ_i as its set of points of W.

Lemma 2.4. *Let $s_i = g_i + g_{i+1}$ and $b_i = g_{i-1} + g_{i+2}$.*

(i) *τ_i is forced if and only if $s_i = 1$, or $s_i = 0$ and $b_i \geq 1$.*

(ii) *If $1 \leq \sum_i g_i \leq 3$, then τ_i and τ_{i+3} are both forced for some i.*

Proof. (i) A window whose points of W are exactly τ_i contains the arc from w_i to w_{i+2}. This arc has $3 + s_i$ vertices, s_i of them apexes. If $s_i \geq 2$ no such 4-window exists. If $s_i = 1$ the arc itself is one. If $s_i = 0$, the 4-window is the arc together with the vertex just before w_i or just after w_{i+2}. It has exactly one apex if and only if that vertex is an apex, and this is possible exactly when $b_i \geq 1$.

(ii) Let $m = \sum_i g_i$. The gaps flanking τ_i also flank τ_{i+3}, so $b_i = b_{i+3}$ and $s_i + s_{i+3} + b_i = m$. Moreover $b_1 + b_2 + b_3 = m \geq 1$. Choose i with b_i maximal, so $b_i \geq 1$. If $s_i + s_{i+3} \leq 1$, then (i) shows that τ_i and τ_{i+3} are both forced. Otherwise $m - b_i \geq 2$, which forces $m = 3$ and $b_1 = b_2 = b_3 = 1$. Then each pair of opposite gaps $\{g_j, g_{j+3}\}$ contains exactly one apex, so the three occupied gaps are either three consecutive gaps or three alternate gaps. In the first case, say $g_1 = g_2 = g_3 = 1$, we have $s_3 = s_6 = 1$. In the second case $s_j = 1$ for every j. Either way (i) gives a forced complementary pair. $\square$

Proof of Theorem 1.2(a). By Proposition 2.2 it remains to show that H_k has no tight Hamiltonian cycle. Take any cyclic order of V. By Lemma 2.4(ii) some complementary triples τ_i and τ_{i+3} are both forced. By Lemma 2.3, if the order were a tight Hamiltonian cycle, both would lie in D, contradicting (F1). $\square$

The argument has been checked by computer in three ways. The criterion in Lemma 2.4(i) was compared with the actual 4-windows for all $6 + 21 + 56 = 83$ gap vectors, and (ii) was verified for each of them. All 360, 2520 and 20160 cyclic orders of V (up to rotation and reflection) were tested directly for $k = 3, 4, 5$. None is a tight Hamiltonian cycle.

Proof of Theorem 1.2(b) *and* (c). Both hypergraphs were found by a SAT search, and their edge lists are in the source archive. The 3-graph on $\{0, \ldots, 8\}$ has 50 edges. Its pair-degrees are 4 for 33 pairs and 6 for 3 pairs. The 5-graph on $\{0, \ldots, 9\}$ has 191 edges, and its 210 codegrees are 4, 5 and 6, taken 130, 45 and 35 times. Three independently written programs show that neither graph has a tight Hamiltonian cycle. They use exhaustive enumeration of the 8!, resp. 9!, vertex orders starting at a fixed vertex, and depth-first extension of tight paths. $\square$

We have no structural description of these two examples. A second, independent search found another counterexample at $(5,10)$, with 186 edges.

Remark 2.5. (1) Add to a k-graph G a new vertex together with all k-sets containing it. The result has a tight Hamiltonian cycle exactly when G has a tight Hamiltonian path. The lower-bound construction of Katona and Kierstead is of this form [RR10, Construction 2.1]. What is new in H_3 is the base: D' has pair-degree 2 and no tight Hamiltonian path, since the first and last triples of such a path would be disjoint.

(2) H_3 is edge-maximal: adding any of its 10 non-edges creates a tight Hamiltonian cycle. Its automorphism group has order 60, fixes the apex, and is isomorphic to A_5, the automorphism group of the hemi-icosahedron.

(3) On 7 vertices there are exactly 2709 labelled counterexamples to Conjecture 1.1 for $k = 3$. They fall into 6 isomorphism classes, with 22, 23, 23, 23, 24 and 25 edges, and H_3 is the only one with 25 edges. This classification relies on a SAT solver.

3 Small cases and scope

Table 1 lists small cases of Conjecture 1.1. Two independent SAT searches with different encodings and solvers were run. One used CaDiCaL with a sequential-counter encoding of the codegree condition, and added the cycle-exclusion clauses either all at once or lazily. The other used Glucose with a direct encoding and all cycle-exclusion clauses. Every counterexample found was re-checked by brute force. The two searches agree on every entry. The entries under "no counterexample" are unsatisfiability results of the solvers, without proof certificates.

k	counterexample for $n =$	no counterexample for $n =$	undecided
3	7, 9	4, 5, 6, 8	10
4	8	5, 6, 7, 9	10
5	9, 10	6, 7, 8	–
6	–	7, 8, 9	10
7	–	8, 9, 10	–

Table 1: Conjecture 1.1 for $n \leq 10$. "Undecided" means that both searches were stopped without an answer, the second after at least 3000 seconds.

One of the two searches also tested the neighbouring thresholds. At each of the five pairs where Conjecture 1.1 fails, raising the codegree bound by one makes the search unsatisfiable, so every failure is by exactly one unit. In particular, for $k = 3$ and $n \leq 9$, pair-degree greater than $n/2$ forces a tight Hamiltonian cycle. All the examples have $n \leq k^2$, while the lower-bound construction of Katona and Kierstead is stated for $n \geq k^2 + 1$. Below that range it can fail: for $(k, n) = (5, 8)$ and $(5, 9)$ their hypergraph has a tight Hamiltonian cycle.

Two questions remain. Does the all-n statement fail for some $k \geq 6$? Does it fail for some $n > k^2$? The construction of Section 2 does not extend to $k \geq 6$: with four or more apexes they can be spread out so that no complementary pair is forced. For $k = 6$ the analogous 10-vertex hypergraph does have tight Hamiltonian cycles.

Verification. The following exact computations, with scripts in the source archive, were used.

1. The finder's script checks (F0) and (F1), the codegrees of H_3, H_4 and H_5, all 83 gap vectors, and all cyclic orders for the five examples. It also checks exhaustively that every 3-graph on $n \leq 6$ vertices meeting the bound has a tight Hamiltonian cycle.

2. An independent referee's code, written from scratch, reproduces all of this with two cycle counters, cross-checked against a dynamic programme. It extends the exhaustive check for $n \leq 6$ to every k, runs the second SAT census, classifies the 7-vertex counterexamples, and tests the thresholds and readings discussed above.

3. Two short programs by the author re-check (F0), (F1), Lemma 2.3, both parts of Lemma 2.4 for all gap vectors, the codegrees, all cyclic orders of H_3, H_4 and H_5, and the two computer-found examples. They also check that H_3 is edge-maximal, that H_6 has a tight Hamiltonian cycle, and the statement about $(5, 8)$ and $(5, 9)$.

Scope and priority. Theorem 1.2 refutes Conjecture 1.1 as stated, for all $n \geq k + 1 \geq 4$. This is the form written in [RRS11, KLP08, LL21], and in [ABW06] with no range of n. The failures are small-n boundary effects. For $k = 3$ there are only finitely many of them, by [RRS11]. The asymptotic statement is the substance of the conjecture, and it is not affected. We searched arXiv, Crossref, OpenAlex, Semantic Scholar and zbMATH in September 2026, including the reviews of the works that zbMATH lists as citing [KK99]. We also read the surveys and later papers cited above. We found no earlier report of small counterexamples or of the hemi-icosahedron example. The paper [KK99] itself was not read; we relied on the account in [RR10]. [RRS11] was read in its 2009 preprint version. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples, to write and run the verification programs, to referee the arguments, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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