A Negative Answer to Bäumler's Question on the Number of Spin-Glass Ground States

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Abstract

Consider the Edwards–Anderson spin glass at zero temperature on an infinite, connected, locally finite graph, with i.i.d. absolutely continuous couplings, and let $\mathcal{G}(J)$ be its set of ground states. Bäumler proved that on every locally finite tree, for coupling laws of linear growth, $|\mathcal{G}(J)|$ is almost surely 2 or ∞, and that $|\mathcal{G}(J)| = 2$ if and only if simple random walk on the tree is recurrent. He asked whether $|\mathcal{G}(J)| \in \{2, \infty\}$ holds for all graphs and all distributions of linear growth. We show that it does not. Join each pair of consecutive integers $k, k+1$ by m_k internally disjoint paths of length two. For couplings uniform on $(-1, 1)$, the resulting graph has $|\mathcal{G}(J)| = 4$ almost surely if $\sum_k m_k^{-1/2} < \infty$, and $|\mathcal{G}(J)| = 2$ almost surely otherwise; the first conclusion holds for every absolutely continuous coupling law. We also give a planar graph of maximum degree 4 on which simple random walk is recurrent and $|\mathcal{G}(J)| = 4$ almost surely for couplings uniform on $(0, 1)$, a recurrent example with couplings of both signs, and a recurrent graph on which $|\mathcal{G}(J)|$ is a non-degenerate random variable. The last example answers the other half of the question in the form Bäumler posed it in 2019. The examples show that several implications of the tree theorem, between uniqueness, vanishing maximal flow and recurrence, fail on general graphs. The mechanism, a unique cheapest finite domain wall on a two-ended graph, is elementary. The question remains open for bounded-degree graphs with a symmetric coupling law and for quasi-transitive graphs such as $\mathbb{Z}^d$. This is an unrefereed note.

1 Introduction

Let $G = (V, E)$ be an infinite, connected, locally finite simple graph, and let $J = (J_e)_{e \in E}$ be i.i.d. real couplings with law ν. Throughout, ν is absolutely continuous with respect to Lebesgue measure; the couplings may take both signs. For a finite set $B \subset V$ let ∂B be the set of edges with exactly one endpoint in B. A configuration $\sigma \in \{-1, +1\}^V$ is a *ground state* if

$$\sum_{xy \in \partial B} J_{xy} \sigma_x \sigma_y \;\geq\; 0 \qquad \text{for every finite } B \subset V, \tag{1}$$

that is, if flipping the spins in a finite set never lowers the energy $H(\sigma) = -\sum_{xy \in E} J_{xy} \sigma_x \sigma_y$ [AD14, Bä19]. Let $\mathcal{G}(J)$ be the set of ground states. It is nonempty by compactness, and $\sigma \in \mathcal{G}(J)$ if and only if $-\sigma \in \mathcal{G}(J)$, so $|\mathcal{G}(J)| \geq 2$ is even or infinite [Bä19]. The law ν has *linear growth* if $c\varepsilon \leq \nu((-\varepsilon, \varepsilon)) \leq C\varepsilon$ for some $0 < c < C$ and all small $\varepsilon > 0$.

Bäumler [Bä19, Theorem 2.1] proved that for a locally finite tree and a law of linear growth the following are equivalent: (i) almost surely the two "natural" ground states, which satisfy every edge,

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are the only ground states; (ii) almost surely the maximal flow from a vertex to infinity with capacities $|J_e|$ vanishes; (iii) simple random walk (SRW) on the tree is recurrent. If (i) fails, then $|\mathcal{G}(J)| = \infty$ almost surely. The link between (ii) and (iii) uses the resistance bounds of Lyons, Pemantle and Peres [LPP99]. In the report of his talk at the Oberwolfach mini-workshop “One-sided and Two-sided Stochastic Descriptions” [Bä20, p. 635], Bäumler asked the following.

Problem 1.1 ([Bä20, Problem 1])**.** *“Does $|\mathcal{G}(J)| \in \{2, \infty\}$ hold for all graphs and all distributions of linear growth?”*

His Problem 2 asks, more loosely, whether random walks and ground states are connected on graphs more general than trees. In the last section of [Bä19], the question is posed in a broader form. It asks whether there are a locally finite graph and a law with $\nu((-\varepsilon, \varepsilon)) > 0$ for all $\varepsilon > 0$ such that $|\mathcal{G}(J)|$ is not almost surely constant, or such that $|\mathcal{G}(J)| \in \mathbb{N} \setminus \{2\}$ with positive probability. Problem 1.1 is record OWR-17474-010 of the UnsolvedMath dataset [ula26]. For comparison: on $\mathbb{Z}$ one has $|\mathcal{G}(J)| = 2$. For continuous coupling laws with support $\mathbb{R}$, Arguin and Damron proved that on the half-plane $\mathbb{Z} \times \mathbb{N}$ the number $|\mathcal{G}(J)|$ is almost surely 2 or almost surely ∞, and that on $\mathbb{Z}^d$, $d \geq 2$, it is an almost sure constant whose value is unknown [AD14].

For disconnected graphs the answer is trivially negative: two disjoint copies of $\mathbb{Z}$ carry exactly four ground states, almost surely. All our graphs are connected and locally finite.

Definition 1.2 (Thickened lines)**.** Let $(m_k)_{k\in\mathbb{Z}}$ be positive integers and $(n_k)_{k\in\mathbb{Z}}$ integers ≥ 2. The graph $L(m, n)$ has the *hubs* $k \in \mathbb{Z}$ and, for each k, m_k paths $P_{k,1}, \ldots, P_{k,m_k}$ of length n_k from hub k to hub $k + 1$, internally disjoint from each other and from all other paths. We call these paths *gadget* k. The graph is simple, connected, locally finite (hub k has degree $m_{k-1} + m_k$, all other vertices have degree 2) and two-ended. We write $L(m)$ when $n_k = 2$ for all k.

Theorem 1.3. *Let ν be uniform on $(-1, 1)$. On $L(m)$, almost surely*

$$|\mathcal{G}(J)| = 4 \quad \text{if} \quad \sum_{k\in\mathbb{Z}} m_k^{-1/2} < \infty, \qquad |\mathcal{G}(J)| = 2 \quad \text{otherwise.}$$

SRW on $L(m)$ is transient if and only if $\sum_{k\geq 0} 1/m_k < \infty$ or $\sum_{k<0} 1/m_k < \infty$. Moreover, if $\sum_k m_k^{-1/2} < \infty$, then $|\mathcal{G}(J)| = 4$ almost surely for every *absolutely continuous law ν.*

Corollary 1.4. *Let $m_k = (|k| + 1)^3$. The connected, locally finite graph $L(m)$ has $|\mathcal{G}(J)| = 4$ almost surely, for every absolutely continuous coupling law, in particular for every absolutely continuous law of linear growth. So the answer to Problem 1.1 is negative, and the second alternative of the question in [Bä19, Section 4] occurs. For $m_k = (|k| + 1)^2$ and ν uniform on $(-1, 1)$, $L(m)$ is transient and $|\mathcal{G}(J)| = 2$ almost surely.*

The graph of Corollary 1.4 has unbounded degrees. This is within the scope of the question, since Bäumler's theorem covers all locally finite trees. For a ferromagnetic law of linear growth the degrees can be bounded.

Theorem 1.5. *Let $w(x) = \lceil (|x|+1)^{1/3} \rceil$ and let Γ be the subgraph of $\mathbb{Z}^2$ induced by $\{(x, y) \in \mathbb{Z}^2 : 0 \leq y \leq w(x)\}$. Then Γ is planar, has maximum degree 4, and SRW on Γ is recurrent. If ν is uniform on $(0, 1)$, then $|\mathcal{G}(J)| = 4$ almost surely.*

For ferromagnetic couplings, (1) says that the set of unsatisfied edges cannot be made cheaper by flipping a finite set of spins. So ground states correspond to minimal cuts, and in the plane to geodesics of first-passage percolation on the dual graph with passage times J_e [Weh97]. For random ferromagnets, Wehr [Weh97] proved that $|\mathcal{G}(J)| \in \{2, \infty\}$ on $\mathbb{Z}^d$, $d \geq 2$, and Wehr and Woo [WW98] proved that $|\mathcal{G}(J)| = 2$ on the half-plane (see also [AD14]; we refer to these papers for the precise hypotheses on the coupling law). Theorem 1.5 is consistent with these results because Γ is not quasi-transitive: its width grows with $|x|$, so, unlike the half-plane, it is not even invariant under horizontal translations.

Theorem 1.6. *(a) Let ν be uniform on $(-1, 2)$ and $m_k = \lceil 730 \log(|k|+2) \rceil$. Then $L(m)$ is recurrent, almost surely it contains infinitely many frustrated cycles, and $|\mathcal{G}(J)| = 4$ almost surely.*

(b) Let ν be uniform on $(0, 1)$. Let $m_0 = 3$, $n_0 = 2$, and for $k \neq 0$ let $j = |k| + 1$, $n_k = j^4$ and $m_k = j^4 + j^3 + 1$. Then $L(m, n)$ is recurrent,

$$\mathbb{P}(|\mathcal{G}(J)| = 4) \ \geq \ 47/90 \qquad \textit{and} \qquad \mathbb{P}(|\mathcal{G}(J)| = 2) \ \geq \ 0.105,$$

and $|\mathcal{G}(J)| \in \{2, 4\}$ almost surely. The constant 0.105 comes from a product evaluated in exact rational arithmetic.

All laws here have linear growth: $\nu((-\varepsilon, \varepsilon))$ equals ε for $(-1, 1)$ and $(0, 1)$, and $2\varepsilon/3$ for $(-1, 2)$, when $\varepsilon \leq 1$. Part (b) answers the first alternative in [Bä19, Section 4]: $|\mathcal{G}(J)|$ need not be almost surely constant. It is a linear-growth analogue of Bäumler's tree example with couplings uniform on $(1, 3)$, a law without linear growth, for which $\mathbb{P}(|\mathcal{G}(J)| = 2) = \mathbb{P}(|\mathcal{G}(J)| = 4) = 1/2$ [Bä19, Section 3.1], [Bä20, p. 634].

The mechanism. All examples rest on the same elementary observation. On a two-ended graph whose cross-section grows, the energy cost of a finite domain wall separating the two ends grows with its distance from the origin. Then there is a unique cheapest such wall, and the two configurations with this wall join the two ground states without a wall. On thickened lines the reduction to a one-dimensional chain is exact (Section 2). For ferromagnetic laws this is easy, even on recurrent graphs (Remark 4.1). For a symmetric law, frustration makes the effective coupling of gadget k a sum of m_k symmetric terms, of typical size $m_k^{1/2}$, and the threshold in Theorem 1.3 is $\sum_k m_k^{-1/2} < \infty$ rather than the resistance condition $\sum_k m_k^{-1} < \infty$. The same square-root mechanism, which produces the threshold $\sum_k m_k^{-1/2}$ in Theorem 1.3, appears in a heuristic example of White and Fisher [WF06]: a three-dimensional ferromagnet in which only the couplings across one plane have random signs. In an $L \times L \times L$ box the coupling between the two half-spaces is then of order $\sqrt{L^2} = L$, so boundary conditions can control the two half-spaces separately, and they describe four ground states. Their discussion is heuristic, and their couplings are not identically distributed, so their model is not an example with i.i.d. couplings.

Section 6 collects what the examples say about Problem 2. On general graphs, uniqueness does not imply recurrence, recurrence does not imply uniqueness, and neither uniqueness nor recurrence implies that the maximal flow vanishes. Problem 1.1 remains open for bounded-degree graphs with a symmetric law, and for quasi-transitive graphs, where our mechanism cannot work (Remark 6.2).

2 Reduction to a chain

If two configurations σ, τ differ on a finite set D, local finiteness makes $\Delta(\tau, \sigma) := \sum_{xy \in E} J_{xy}(\sigma_x \sigma_y - \tau_x \tau_y)$ a finite sum, and $\Delta(\tau, \sigma) = 2 \sum_{xy \in \partial D} J_{xy} \sigma_x \sigma_y$. So $\sigma \in \mathcal{G}(J)$ if and only if $\Delta(\tau, \sigma) \geq 0$ for every such τ. For a finite subgraph P write $H_P(\sigma) = -\sum_{xy \in E(P)} J_{xy} \sigma_x \sigma_y$.

Lemma 2.1 (Paths)**.** *Let P be a path $x_0 x_1 \cdots x_n$ with nonzero couplings $J_1, \ldots, J_n$, where J_i sits on $x_{i-1} x_i$ and $\min_i |J_i|$ is attained at a single index. Put*

$$c_P = \sum_i |J_i| - \min_i |J_i|, \qquad Y_P = \operatorname{sgn}(J_1 \cdots J_n) \ \min_i |J_i|.$$

For $s, s' \in \{\pm 1\}$, the minimum of H_P over configurations with $\sigma_{x_0} = s$ and $\sigma_{x_n} = s'$ equals $-c_P - Y_P ss'$, and it is attained by exactly one configuration.

Proof. Given $\sigma_{x_0} = s$, the map $\sigma \mapsto (t_i)_{i \le n} = (\sigma_{x_{i-1}}\sigma_{x_i})_{i \le n}$ is a bijection onto $\{t \in \{\pm 1\}^n : \prod_i t_i = ss'\}$, and $H_P = -\sum_i J_i t_i$. If $ss' = \operatorname{sgn}(J_1 \cdots J_n)$, then $t_i = \operatorname{sgn} J_i$ is admissible; it is the unique minimiser, and $H_P = -\sum_i |J_i| = -c_P - Y_P ss'$. Otherwise every admissible t has an odd, hence positive, number of indices with $t_i \neq \operatorname{sgn} J_i$. So $H_P \ge -\sum_i |J_i| + 2\min_i |J_i| = -c_P - Y_P ss'$, with equality only if $t_i \neq \operatorname{sgn} J_i$ exactly at the minimising index. □

Lemma 2.2 (Chains)**.** *Let $(K_k)_{k \in \mathbb{Z}}$ be nonzero reals with pairwise distinct absolute values. Call $s \in \{\pm 1\}^{\mathbb{Z}}$ a* chain ground state *if $\sum_{k \colon \{k,k+1\} \in \partial A} K_k s_k s_{k+1} \ge 0$ for every finite $A \subset \mathbb{Z}$, and call the bond k* satisfied *if $K_k s_k s_{k+1} > 0$. The chain ground states are the two configurations in which every bond is satisfied, together with, if $\inf_k |K_k|$ is attained at k_*, the two configurations in which k_* is the only unsatisfied bond. So there are $2 + 2 \cdot \mathbf{1}\{\inf_k |K_k|$ is attained$\}$ of them.*

Proof. For finite $A \subset \mathbb{Z}$, ∂A is a finite set of bonds of even size. Conversely, a finite set $T = \{b_1 < \cdots < b_{2r}\}$ of bonds equals ∂A for exactly one finite A, namely the union of the hub intervals $\{b_{2i-1} + 1, \ldots, b_{2i}\}$. Since $K_k s_k s_{k+1} = \pm |K_k|$, s is a chain ground state if and only if, for every finite set T of bonds of even size, the sum of $|K_k|$ over the unsatisfied bonds in T is at most the sum over the satisfied ones. If two bonds are unsatisfied, the set T consisting of these two violates this. If no bond is unsatisfied, the condition holds, and there are two such configurations. If exactly one bond k_0 is unsatisfied, the condition says $|K_{k_0}| \le \sum_{k \in T \setminus \{k_0\}} |K_k|$ for every such T containing k_0. Taking $T = \{k_0, k\}$ shows that this holds if and only if $|K_{k_0}| \le |K_k|$ for all k, that is, $k_0 = k_*$. □

Proposition 2.3. *On $L(m,n)$ suppose that all $J_e \neq 0$, that $\min_{e \in P} |J_e|$ is attained once on every path $P = P_{k,i}$, and that the numbers*

$$K_k := \sum_{i=1}^{m_k} Y_{P_{k,i}} \qquad (k \in \mathbb{Z})$$

are nonzero with pairwise distinct absolute values. Then $\sigma \mapsto \sigma|_{\mathbb{Z}}$ is a bijection from $\mathcal{G}(J)$ onto the chain ground states for (K_k). In particular $|\mathcal{G}(J)| = 2 + 2 \cdot \mathbf{1}\{\inf_k |K_k|$ is attained$\}$. If ν is absolutely continuous, these assumptions hold almost surely, and the K_k are independent.

Proof. Every edge lies on exactly one path $P = P_{k,i}$, whose end spins are σ_k, σ_{k+1}. By Lemma 2.1, $H_P(\sigma) \ge -c_P - Y_P \sigma_k \sigma_{k+1}$, with equality if and only if the interior spins of P are the unique optimal ones for the given end spins; call σ *optimal on P* in that case.

Let $\sigma \in \mathcal{G}(J)$. Changing σ only inside one path P gives $\Delta = H_P(\tau) - H_P(\sigma) \ge 0$, so σ is optimal on every path. Let $A \subset \mathbb{Z}$ be finite, and let τ flip the hubs in A and be optimal on each of the finitely many paths with an end in A; elsewhere $\tau = \sigma$. Then

$$0 \le \Delta(\tau, \sigma) = \sum_P \big(H_P(\tau) - H_P(\sigma)\big) = \sum_k K_k \big(\sigma_k \sigma_{k+1} - \tau_k \tau_{k+1}\big),$$

which is the energy change of the chain under the flip of A. So $\sigma|_{\mathbb{Z}}$ is a chain ground state. Conversely, let s be a chain ground state, and let σ extend s and be optimal on every path. If τ differs from σ on a finite set, then

$$\Delta(\tau, \sigma) = \sum_P \big(H_P(\tau) - H_P(\sigma)\big) \;\ge\; \sum_k K_k \big(\sigma_k \sigma_{k+1} - \tau_k \tau_{k+1}\big) \;\ge\; 0,$$

so $\sigma \in \mathcal{G}(J)$. Since ground states are optimal on every path, the restriction map is injective, and we have shown that it is onto. The count follows from Lemma 2.2.

If ν is absolutely continuous, then almost surely all $|J_e|$ are nonzero and pairwise distinct. Each Y_P has no atoms, since $\mathbb{P}(Y_P = y) \le \sum_{e\in P} \mathbb{P}(|J_e| = |y|) = 0$. The K_k are independent (they depend on disjoint edge sets) and have no atoms, so almost surely $K_k \neq 0$ and $|K_k| \neq |K_l|$ for all $k \neq l$. □

Lemma 2.4. *SRW on $L(m,n)$ is recurrent if and only if $\sum_{k\ge 0} n_k/m_k = \sum_{k<0} n_k/m_k = \infty$.*

Proof. With unit conductances, gadget k has effective resistance n_k/m_k between hubs k and $k+1$, and every hub is a cut vertex. By the series and parallel laws, the effective resistance from hub 0 to the set of hubs $\{-N, N\}$ is the parallel combination of $\sum_{0\le k<N} n_k/m_k$ and $\sum_{-N\le k<0} n_k/m_k$. Letting $N \to \infty$, SRW is recurrent if and only if the limit is infinite [LP16, Chapter 2], that is, if and only if both sums diverge. □

3 Symmetric couplings on thickened lines

Lemma 3.1. *Let J, J' be independent and uniform on $(-1,1)$, let $Y = \operatorname{sgn}(JJ')\min(|J|,|J'|)$, and let K be the sum of $m \ge 1$ independent copies of Y.*

(a) Y has density $1-|y|$ on $(-1,1)$, and K has the law of $U_1 + \cdots + U_{2m} - m$ with U_i i.i.d. uniform on $(0,1)$. K has a continuous density f that is symmetric and nonincreasing in $|x|$, and $\operatorname{Var} K = m/6$.

(b) $\mathbb{P}(|K| \le M) \le 2.36\, M\, m^{-1/2}$ for every $M > 0$.

(c) $\mathbb{P}(|K| < \varepsilon) \ge \frac{3}{4}\,\varepsilon/(\varepsilon + 2\sqrt{m/6})$ for every $\varepsilon > 0$.

Proof. (a) The variables $|J|, |J'|$ are independent and uniform on $(0,1)$, and the signs of J, J' are fair, independent of each other and of $|J|, |J'|$. Hence $\mathbb{P}(\min(|J|,|J'|) > t) = (1-t)^2$ on $[0,1]$, and $\operatorname{sgn}(JJ')$ is a fair sign independent of $\min(|J|,|J'|)$. So Y is symmetric with $\mathbb{P}(|Y| > t) = (1-t)^2$, which is the triangular density $1-|y|$, the density of $U - U' = U + (1-U') - 1$. A sum of at least two independent uniform variables has a continuous log-concave density; here it is symmetric about 0, hence nonincreasing in $|x|$. Finally $\mathbb{E}Y^2 = \int_{-1}^{1} y^2(1-|y|)\,dy = 1/6$.

(b) The characteristic function of $U - \frac{1}{2}$ is $\sin(t/2)/(t/2)$, so K has the nonnegative integrable characteristic function $(\sin(t/2)/(t/2))^{2m}$. Fourier inversion and $t = 2u$ give $f(0) = \frac{2}{\pi}\int_0^\infty (\sin u/u)^{2m} du$. On $[0,\pi]$ the product formula $\sin u/u = \prod_{j\ge 1}(1 - u^2/(j\pi)^2)$ and $\log(1-z) \le -z$ give $\sin u/u \le e^{-u^2/6}$; on $[\pi, \infty)$ we use $|\sin u/u| \le 1/u$. Hence

$$f(0) \le \frac{2}{\pi}\Big(\int_0^\infty e^{-mu^2/3} du + \int_\pi^\infty u^{-2m} du\Big) = \sqrt{\frac{3}{\pi m}} + \frac{2\pi^{-2m}}{2m-1}.$$

The sequence $\sqrt{m}\, 2\pi^{-2m}/(2m-1)$ decreases (consecutive ratios are below $\sqrt{2}/\pi^2$), so $\sqrt{m}\, f(0) \le \sqrt{3/\pi} + 2/\pi^2 < 1.18$. As f is maximal at 0, $\mathbb{P}(|K| \le M) \le 2M f(0)$.

(c) Let $t = 2\sqrt{m/6}$. By Chebyshev's inequality $\mathbb{P}(|K| < t) \ge \frac{3}{4}$. Since f is symmetric and nonincreasing in $|x|$, $\mathbb{P}(|K| < \varepsilon) \ge 2\varepsilon f(\varepsilon)$ and $\mathbb{P}(\varepsilon \le |K| < \varepsilon + t) \le 2t f(\varepsilon)$. Hence $\frac{3}{4} \le \mathbb{P}(|K| < \varepsilon + t) \le (1 + t/\varepsilon)\mathbb{P}(|K| < \varepsilon)$. □

Proof of Theorem 1.3. By Proposition 2.3, almost surely $|\mathcal{G}(J)| = 2 + 2\cdot \mathbf{1}\{\inf_k |K_k| \text{ is attained}\}$, where the K_k are independent and K_k is the sum of m_k copies of the variable Y of Lemma 3.1.

Suppose $\sum_k m_k^{-1/2} < \infty$. For every $M \in \mathbb{N}$, $\sum_k \mathbb{P}(|K_k| \le M) \le 2.36M \sum_k m_k^{-1/2} < \infty$, so by the Borel–Cantelli lemma only finitely many k have $|K_k| \le M$, almost surely. Thus $|K_k| \to \infty$ as $|k| \to \infty$, the infimum of $|K_k|$ is a minimum over the finite set $\{k : |K_k| \le |K_0|\}$, and $|\mathcal{G}(J)| = 4$.

Suppose $\sum_k m_k^{-1/2} = \infty$. For $\varepsilon > 0$, Lemma 3.1(c) and $m_k \geq 1$ give $\mathbb{P}(|K_k| < \varepsilon) \geq c_\varepsilon m_k^{-1/2}$ with $c_\varepsilon = \frac{3}{4}\varepsilon/(\varepsilon + 2/\sqrt{6})$. By the second Borel–Cantelli lemma, almost surely $|K_k| < \varepsilon$ for infinitely many k, for every rational $\varepsilon > 0$. So $\inf_k |K_k| = 0$ is not attained and $|\mathcal{G}(J)| = 2$. The transience criterion is Lemma 2.4 with $n_k = 2$.

Now let ν be any absolutely continuous law and $\sum_k m_k^{-1/2} < \infty$. Let $Q(X;\ell) = \sup_x \mathbb{P}(x \leq X \leq x + \ell)$ be the concentration function. The Kolmogorov–Rogozin inequality [Rog61], see also [Pet75, Chapter III], states that there is an absolute constant A such that for independent $X_1, \ldots, X_m$ and $0 < \lambda \leq \ell$,

$$Q(X_1 + \cdots + X_m; \ell) \;\leq\; A\,\ell \Big(\sum_{i=1}^{m} \lambda^2\big(1 - Q(X_i;\lambda)\big)\Big)^{-1/2}.$$

The variable $Y = Y_P$ of a path of length two has no atoms (Proposition 2.3), so its distribution function is uniformly continuous, and we can fix $\lambda \in (0,1]$ with $Q(Y;\lambda) \leq \frac{1}{2}$. For $M \geq \lambda/2$ we get $\mathbb{P}(|K_k| \leq M) \leq Q(K_k; 2M) \leq 2\sqrt{2}AM\lambda^{-1}m_k^{-1/2}$, and the argument of the second paragraph applies verbatim. $\square$

Remark 3.2. For ν uniform on $(-1,1)$ the criterion $\sum_k m_k^{-1/2} < \infty$ implies $\sum_k m_k^{-1} < \infty$. So on the thickened lines $L(m)$ with this symmetric law, $|\mathcal{G}(J)| = 4$ forces transience, while transience does not force $|\mathcal{G}(J)| = 4$.

4 Recurrent thickened lines

Proof of Theorem 1.6(a). Let J, J' be independent and uniform on $(-1,2)$. The variable $Y = \operatorname{sgn}(JJ')\min(|J|,|J'|)$ takes values in $[-2,2]$, and for $t \geq 0$

$$\mathbb{P}(Y > t) = \mathbb{P}(J > t)^2 + \mathbb{P}(J < -t)^2, \qquad \mathbb{P}(Y < -t) = 2\mathbb{P}(J > t)\mathbb{P}(J < -t),$$

so $\mathbb{P}(Y > t) - \mathbb{P}(Y < -t) = (\mathbb{P}(J > t) - \mathbb{P}(J < -t))^2$. This equals $1/9$ on $[0,1)$ and $(2-t)^2/9$ on $[1,2)$, hence $\mu := \mathbb{E}Y = \frac{1}{9} + \frac{1}{27} = \frac{4}{27}$. By Hoeffding's inequality [Hoe63], for $M \geq 0$,

$$\mathbb{P}(K_k \leq M) \leq \exp\Big(-\frac{(m_k\mu - M)^2}{8m_k}\Big) \leq \exp\Big(\frac{M\mu}{4} - \frac{m_k\mu^2}{8}\Big)$$

when $m_k\mu > M$, and the last bound exceeds 1 otherwise. Since $\mu^2/8 = 2/729$ and $m_k \geq 730\log(|k|+2)$, we get $\mathbb{P}(K_k \leq M) \leq e^{M\mu/4}(|k|+2)^{-1460/729}$, which is summable. By the Borel–Cantelli lemma $K_k \to +\infty$ almost surely, so the infimum of $|K_k|$ is attained, and $|\mathcal{G}(J)| = 4$ by Proposition 2.3. The graph is recurrent by Lemma 2.4. Finally, $\mathbb{P}(Y < 0) = 2 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}$, and the 4-cycle formed by two paths of gadget k is frustrated (the product of its couplings is negative) if and only if their values of Y have opposite signs. By the second Borel–Cantelli lemma, infinitely many gadgets contain such a cycle, almost surely. $\square$

Remark 4.1. For ν uniform on $(0,1)$ and $m_k = \lceil 9\log(|k|+2)\rceil$, each Y is the minimum of two uniforms, with mean $1/3$, and Hoeffding's inequality gives $\mathbb{P}(K_k \leq M) \leq e^{4M/3}(|k|+2)^{-2}$. So again $|\mathcal{G}(J)| = 4$ almost surely on a recurrent graph. For ferromagnetic laws the question is thus easy; the symmetric case of Theorem 1.3 is the substantive one.

Proof of Theorem 1.6(b). For ν uniform on $(0,1)$, a path of length n contributes Y_P, the minimum of n independent uniforms, which has the Beta$(1,n)$ law with mean $1/(n+1)$ and variance $n/((n+1)^2(n+2))$. In particular $K_k > 0$.

Let $k \neq 0$, $j = |k|+1$, $n = j^4$, $r = j^3$ and $m = n+r+1$. Then $\mathbb{E}K_k = m/(n+1) = 1 + r/(n+1)$ and $V_k := \operatorname{Var} K_k = mn/((n+1)^2(n+2)) \leq m/n^2 \leq 3j^{-4}$. By Chebyshev's inequality and the Borel–Cantelli lemma, $K_k - \mathbb{E}K_k \to 0$, so $K_k \to 1$ almost surely. By Cantelli's one-sided inequality,

$$\mathbb{P}(K_k \leq 1) \leq c_k := \frac{V_k}{V_k + (r/(n+1))^2} \leq \frac{mn}{(n+2)r^2} \leq \frac{3}{j^2},$$

which is summable; so almost surely $K_k > 1$ for all but finitely many k.

Let $\mathcal{E} = \{K_k > 1 \text{ for all } k\}$. On $\mathcal{E}$, $\inf_k K_k = \lim_k K_k = 1$ is not attained, so $|\mathcal{G}(J)| = 2$ by Proposition 2.3. Off $\mathcal{E}$, almost surely some $K_k < 1$, and since $K_k \to 1$ the infimum is attained, so $|\mathcal{G}(J)| = 4$.

Gadget 0 is a copy of $K_{2,3}$: $K_0 = Y_1 + Y_2 + Y_3$ with Y_i i.i.d. with density $2(1-y)$ on $(0,1)$. With $\Delta_3 = \{y \in \mathbb{R}^3_{\geq 0} : y_1 + y_2 + y_3 \leq 1\}$ and the Dirichlet integrals $\int_{\Delta_3} \prod_i y_i^{a_i}\, dy = \prod_i a_i!/(3 + \sum_i a_i)!$,

$$\mathbb{P}(K_0 \leq 1) = 8 \int_{\Delta_3} \prod_{i=1}^{3} (1 - y_i)\, dy = 8\Big(\frac{1}{6} - \frac{3}{24} + \frac{3}{120} - \frac{1}{720}\Big) = \frac{47}{90}.$$

Hence $\mathbb{P}(|\mathcal{G}(J)| = 4) \geq 47/90$. By independence, $\mathbb{P}(\mathcal{E}) = \prod_k \mathbb{P}(K_k > 1) \geq \frac{43}{90} \prod_{k \neq 0}(1 - c_k)$, which is positive because $c_k < 1$ and $\sum_k c_k < \infty$. Evaluating the factors with $|k| < 3000$ in exact rational arithmetic, rounding down, and bounding the rest by $\prod_{j>3000}(1 - 3/j^2)^2 \geq 1 - 6/3000$ gives $\mathbb{P}(\mathcal{E}) \geq 0.1059$. Finally, gadget k has resistance $n_k/m_k \to 1$, so the graph is recurrent by Lemma 2.4. □

5 A bounded-degree example

Write $C_x = \{x\} \times \{0, \ldots, w(x)\}$ for the columns of Γ and $H(x) = \min(w(x), w(x+1))$. The function $w \geq 1$ is even and nondecreasing in $|x|$, and there are $H(x)+1$ horizontal edges between C_x and C_{x+1}. Γ is connected and planar with maximum degree 4, and it is recurrent by Rayleigh's monotonicity principle, since SRW on $\mathbb{Z}^2$ is recurrent [LP16, Chapter 2].

Since $J > 0$, we identify σ with $S = \{\sigma = +1\}$; an edge xy has $\sigma_x \sigma_y = -1$ if and only if $xy \in \partial S$. For a set F of edges write $J(F) = \sum_{e \in F} J_e$. Then (1) reads

$$J(\partial B \cap \partial S) \leq J(\partial B \setminus \partial S) \qquad \text{for every finite } B \subset V, \tag{2}$$

and $\partial(S \triangle B) = \partial S \triangle \partial B$. The sets $S = \emptyset, V$ are ground states. Fix a ground state $S \notin \{\emptyset, V\}$. Call a column *positive* if $C_x \subseteq S$, *negative* if $C_x \cap S = \emptyset$, and *mixed* otherwise, and let $\mathcal{P}$ and $\mathcal{N}$ be the sets of positive and negative columns.

Step 1: $\mathcal{P}$ and $\mathcal{N}$ are intervals. Let $x_1 < x_2$ be in $\mathcal{P}$ and let B be the set of vertices outside S in the columns strictly between them. Every neighbour of B outside B lies in S, so $\partial B \subseteq \partial S$, and (2) forces $J(\partial B) = 0$. As Γ is connected and $J > 0$, $B = \emptyset$. The same argument applies to $\mathcal{N}$, since (2) is symmetric under $S \mapsto V \setminus S$.

Step 2: long runs of mixed columns are expensive. Let μ_x be the smallest coupling on the $w(x)$ vertical edges of C_x. Let $a < b$, let W be the union of the columns $a, \ldots, b$, and take $B = S \cap W$. An edge of ∂S with both ends in W lies in $\partial B \cap \partial S$, and every edge of $\partial B \setminus \partial S$ joins W to a column $a-1$ or $b+1$. There are at most $(w(a)+1) + (w(b)+1)$ such edges, each with coupling below 1. A mixed column contains a vertical edge of ∂S. So if all columns in $[a,b]$ are mixed, (2) gives $\sum_{x=a}^{b} \mu_x < w(a) + w(b) + 2$.

Step 3: no long mixed runs far out. The μ_x are independent, take values in $[0,1]$ and have mean $1/(w(x)+1)$. For $a \geq 1$ let $E_a = \sum_{x=a}^{2a} \mathbb{E}\mu_x \geq (a+1)/(w(2a)+1)$. Hoeffding's inequality gives $\mathbb{P}(\sum_{x=a}^{2a} \mu_x \leq E_a/2) \leq \exp(-(a+1)/(2(w(2a)+1)^2))$, which is summable in a. For $a \geq 10^6$ put

$q=(a+1)^{1/3}\geq 100$. Then $w(a)\leq q+1$ and $w(2a)\leq 2^{1/3}q+1\leq 1.26q+1$, so $w(2a)+1\leq 1.28q$ and $E_a/2\geq q^3/(2.56\,q)\geq 39q$, while $w(a)+w(2a)+2\leq 2.26q+4\leq 2.3q$. By the Borel–Cantelli lemma, almost surely there is A_0 such that for all $a\geq A_0$ we have $\sum_{x=a}^{2a}\mu_x>E_a/2>w(a)+w(2a)+2$, and the same for the windows $[-2a,-a]$. By Step 2, for every ground state S, no such window consists of mixed columns only.

Step 4: S separates the ends. By Step 3, $\mathcal{P}\cup\mathcal{N}$ is unbounded above and below. Being intervals, one of $\mathcal{P},\mathcal{N}$ contains a ray $[c,\infty)$ and one contains a ray $(-\infty,c']$. They are not the same set, since otherwise it would be all of $\mathbb{Z}$ and $S\in\{\emptyset,V\}$. Let $\mathcal{S}$ be the family of sets $S\subseteq V$ that contain all columns $x\leq c'$ and no vertex of the columns $x\geq c$, for some $c'<c$. We have shown that $S\in\mathcal{S}$ or $V\setminus S\in\mathcal{S}$. Any two members of $\mathcal{S}$ differ on a finite set, and every $S\in\mathcal{S}$ has a finite boundary. By (2) and $\partial(S\triangle B)=\partial S\triangle\partial B$, a set $S\in\mathcal{S}$ is a ground state if and only if it minimises $J(\partial S)$ over $\mathcal{S}$. Since $\sigma\mapsto-\sigma$ corresponds to $S\mapsto V\setminus S$, it remains to prove that almost surely this minimiser exists and is unique.

Step 5: uniqueness. If $S\neq S'$ are in $\mathcal{S}$, then $S\triangle S'$ is finite and nonempty, so $\partial S\triangle\partial S'=\partial(S\triangle S')\neq\emptyset$. Almost surely, distinct finite edge sets have distinct weights, since there are countably many pairs and ν has no atoms. So there is at most one minimiser.

Step 6: existence. Call $S\in\mathcal{S}$ a *bond* if S and $V\setminus S$ both induce connected subgraphs. For every $S\in\mathcal{S}$ there is a bond $S^*\in\mathcal{S}$ with $\partial S^*\subseteq\partial S$. Indeed, let L be the component of $\Gamma[S]$ containing the columns $x\leq c'$, let Q be the component of $\Gamma[V\setminus L]$ containing the columns $x\geq c$, and put $S^*=V\setminus Q$. Then $S^*\in\mathcal{S}$, $\partial S^*=\partial Q\subseteq\partial L\subseteq\partial S$, $\Gamma[Q]$ is connected, and $\Gamma[S^*]$ is connected, because every other component of $\Gamma[V\setminus L]$ has a neighbour in L. We now describe the boundaries of bonds. The bounded faces of Γ are the unit squares $[x,x+1]\times[y,y+1]$ with $0\leq y<H(x)$. There are two unbounded faces: B below the line $y=0$, and T above Γ. Every edge lies on two distinct faces: a vertical edge in C_x lies on a square, because $w(x)=H(x)$ for $x\geq 0$ and $w(x)=H(x-1)$ for $x\leq 0$. So Γ has no bridges.

Lemma 5.1. *Let $S\in\mathcal{S}$ be a bond. There are distinct faces $f_0=\mathsf{B},f_1,\dots,f_{k-1},f_k=\mathsf{T}$, where $f_1,\dots,f_{k-1}$ are unit squares, such that $\partial S=\{e_1,\dots,e_k\}$ and e_i lies on both f_{i-1} and f_i.*

Proof. Choose $R>\max(|c|,|c'|)$ and let Γ_R be the subgraph induced by the columns $-R,\dots,R$. Then $\partial S\subseteq E(\Gamma_R)$. The sets $S\cap V(\Gamma_R)$ and $V(\Gamma_R)\setminus S$ induce connected subgraphs of Γ_R: a path in $\Gamma[S]$ can leave Γ_R only through column $-R$, which lies in S, so every excursion can be replaced by a path inside that column, and similarly for $V\setminus S$. So ∂S is a minimal nonempty cut of the connected plane graph Γ_R, and by plane duality [Die17, Section 4.6] the dual edges of ∂S form a cycle in the dual multigraph of Γ_R. The faces of Γ_R are the unit squares with $-R\leq x<R$ and the outer face O. The bottom row from $(-R,0)$ to $(R,0)$ and the path along the upper boundary from $(-R,w(-R))$ to $(R,w(R))$ are edge-disjoint, start in S and end outside S. So each contains an edge of ∂S, and these edges lie on O. No other edge of ∂S lies on O, and no edge of ∂S is a bridge of Γ_R (each lies on a unit square of Γ_R), so none is a dual loop. A cycle passes through the dual vertex O at most once. Hence exactly two edges of ∂S lie on O, one on the bottom row, which lies on B in Γ, and one on the upper boundary, which lies on T. Deleting O from the cycle leaves the required sequence of distinct squares. □

For an edge e let $\operatorname{col}(e)$ be the set of columns containing an endpoint of e. For $x\geq 1$ put $L_x=\min\{w(\lceil x/2\rceil-1)+1,\lfloor x/2\rfloor+2\}$, and put $L_x=L_{|x|}$ for $x\leq -1$.

Lemma 5.2. *Let $x\neq 0$, and let $S\in\mathcal{S}$ be a bond such that $x\in\operatorname{col}(e)$ for some $e\in\partial S$. Then $|\partial S|\geq L_x$.*

Proof. By the reflection $(x,y)\mapsto(-x,y)$, which exchanges the roles of S and $V\setminus S$, we may assume $x\geq 1$. Put $x_0=\lceil x/2\rceil-1\geq 0$. If no edge of ∂S has an endpoint in a column $\leq x_0$, then the connected subgraph induced by the columns $\le x_0$ contains no edge of ∂S and meets S (in the columns $\le \min(c', x_0)$), so it lies in S. Since w is nondecreasing on $[0,\infty)$, for $0 \le y \le w(x_0)$ the rows $(x_0, y), (x_0+1, y), \dots$ are edge-disjoint rays in Γ; each starts in S and eventually leaves S, so $|\partial S| \ge w(x_0)+1$. Otherwise some $e_b \in \partial S$ has an endpoint in a column $\le x_0$, and some e_a has an endpoint in column x (notation of Lemma 5.1). If $a = b$, then $x - x_0 \le 1$, so $x = 1$, and $|\partial S| \ge 2 = L_1$ because $e_1 \ne e_k$. Let $a \ne b$. For $1 \le i < k$ the edges e_i, e_{i+1} lie on the square f_i, whose vertices lie in two consecutive columns. By induction on $|i-j|$, $|u - v| \le |i-j|$ whenever $i \ne j$, $u \in \mathrm{col}(e_i)$ and $v \in \mathrm{col}(e_j)$. Hence $|a-b| \ge x - x_0$ and $|\partial S| = k \ge x - x_0 + 1 = \lfloor x/2 \rfloor + 2$. □

Lemma 5.3. *Let $x \ne 0$, $\ell = \ell_x = \lceil L_x/2 \rceil$ and $\delta = 1/(6e)$. The probability that some bond $S \in \mathcal{S}$ with $x \in \mathrm{col}(e)$ for some $e \in \partial S$ has $J(\partial S) \le \delta\ell$ is at most $(3w(x)+2)\,2^{-\ell}$.*

Proof. Let S be such a bond, with $e_1, \dots, e_k$ as in Lemma 5.1 and $x \in \mathrm{col}(e_i)$. The sequences $(e_i, e_{i+1}, \dots, e_k)$ and $(e_i, e_{i-1}, \dots, e_1)$ have lengths adding up to $k+1$. By Lemma 5.2 one of them has at least $\lceil k/2 \rceil \ge \ell$ terms. Its first ℓ terms $g_1, \dots, g_\ell$ are distinct edges of ∂S, and consecutive terms lie on a common unit square. For $t \ge 2$ this square is the face on the other side of g_t from the square containing g_{t-1} and g_t. The walk never passes through T or B, because $f_1, \dots, f_{k-1}$ are squares. Hence the sequence is determined by g_1, by one of at most 2 squares on g_1, and by one of at most 3 further edges of the current square at each later step. So there are at most $2 \cdot 3^{\ell-1}$ sequences for each g_1. At most $3w(x)+2$ edges have an endpoint in C_x. For ℓ distinct edges, $\mathbb{P}(\sum_t J_{g_t} \le \delta\ell) \le (\delta\ell)^\ell/\ell! \le (e\delta)^\ell = 6^{-\ell}$, the volume of a simplex. Since $J(\partial S) \ge \sum_t J_{g_t}$, a union bound gives at most $(3w(x)+2) \cdot 2 \cdot 3^{\ell-1} \cdot 6^{-\ell} \le (3w(x)+2)2^{-\ell}$. □

We now finish Step 6. Since $w(\lceil x/2 \rceil - 1) \ge (x/2)^{1/3}$, we have $\ell_x \ge L_x/2 \ge (|x|/2)^{1/3}/2$, while $w(x) \le (|x|+1)^{1/3}+1$. So $\sum_{x \ne 0}(3w(x)+2)2^{-\ell_x} < \infty$. By Lemma 5.3 and the Borel–Cantelli lemma, almost surely there is X such that for $|x| \ge X$ every bond $S \in \mathcal{S}$ with $x \in \mathrm{col}(e)$ for some $e \in \partial S$ has $J(\partial S) > \delta\ell_x$. Since $\ell_x \to \infty$, for every M the bonds with $J(\partial S) \le M$ have their boundaries inside a fixed finite set of edges, and a set in $\mathcal{S}$ is determined by its boundary. So there are finitely many of them. Fix a bond $S_0 \in \mathcal{S}$ (for example the union of the columns $x \le 0$) and let $S_{\min}$ minimise $J(\partial\,\cdot)$ over the finite nonempty set of bonds with $J(\partial S) \le J(\partial S_0)$. For every $S \in \mathcal{S}$ we have $J(\partial S) \ge J(\partial S^*) \ge J(\partial S_{\min})$. So the minimum over $\mathcal{S}$ exists. With Steps 4 and 5, $|\mathcal{G}(J)| = 4$ almost surely. □

6 Random walks and maximal flows

Table 1 collects the examples. For the maximal flow we use the cutset form of [Bä19, (3)]: $\mathrm{MF}(J) = \inf\{\sum_{e\in\Pi} |J_e|\}$ over cutsets Π, sets of edges meeting every infinite self-avoiding path from a fixed vertex.

graph	law of J_e	degrees	SRW	$\lvert\mathcal{G}(J)\rvert$	reference
$L(m)$, $m_k = (\lvert k\rvert+1)^3$	any abs. continuous	unbounded	transient	4 a.s.	Corollary 1.4
$L(m)$, $m_k = (\lvert k\rvert+1)^2$	uniform on $(-1,1)$	unbounded	transient	2 a.s.	Corollary 1.4
$L(m)$, $m_k = \lvert k\rvert+1$	uniform on $(-1,1)$	unbounded	recurrent	2 a.s., MF > 0	Proposition 6.1
strip Γ	uniform on $(0,1)$	≤ 4, planar	recurrent	4 a.s.	Theorem 1.5
$L(m)$, $m_k \approx 730\log\lvert k\rvert$	uniform on $(-1,2)$	unbounded	recurrent	4 a.s.	Theorem 1.6(a)
$L(m,n)$ of Theorem 1.6(b)	uniform on $(0,1)$	unbounded	recurrent	2 or 4	Theorem 1.6(b)

Table 1: The examples. All laws in rows 2–6 have linear growth. Row 1 holds for every absolutely continuous law, and not every such law has linear growth.

Proposition 6.1. *Let $m_k = |k| + 1$ and ν uniform on $(-1, 1)$. Then $L(m)$ is recurrent, $|\mathcal{G}(J)| = 2$ almost surely, and* $\mathrm{MF}(J) > 0$ *almost surely.*

Proof. Recurrence follows from Lemma 2.4, and $|\mathcal{G}(J)| = 2$ from Theorem 1.3, since $\sum_k (|k|+1)^{-1/2} = \infty$. Let Π be a cutset for hub 0. If for every $k \geq 0$ some path of gadget k avoided Π, these paths would form an infinite self-avoiding path from hub 0 avoiding Π. So Π meets every path of some gadget $k \geq 0$, and similarly of some gadget $k < 0$. Hence $\sum_{e \in \Pi} |J_e| \geq \inf_{k \geq 0} A_k + \inf_{k<0} A_k$, where $A_k = \sum_i \min_{e \in P_{k,i}} |J_e|$. Each A_k is positive and is a sum of m_k independent variables in $[0, 1]$ with mean $1/3$. By Hoeffding's inequality $\mathbb{P}(A_k < 1) \leq \exp(-2(m_k/3 - 1)^2/m_k)$ for $m_k \geq 3$, which is summable. So almost surely $A_k \geq 1$ for all but finitely many k, and both infima are positive. □

On trees, $|\mathcal{G}(J)| = 2$, $\mathrm{MF}(J) = 0$ and recurrence are equivalent [Bä19]. On connected locally finite graphs, with laws of linear growth, the examples show the following. Uniqueness ($|\mathcal{G}(J)| = 2$) does not imply recurrence (Corollary 1.4). Recurrence does not imply uniqueness (Theorems 1.5 and 1.6); our examples of this use non-symmetric laws. Neither recurrence nor uniqueness implies $\mathrm{MF}(J) = 0$ (Proposition 6.1). In the symmetric family of Theorem 1.3 the relevant quantity is $\sum_k m_k^{-1/2}$, which is not a random-walk quantity.

Remark 6.2 (Quasi-transitive graphs; a sketch)*.* The following heuristic, which we do not use elsewhere, suggests that our mechanism cannot work on quasi-transitive graphs. Let G be infinite and quasi-transitive, and suppose $|\mathcal{G}(J)| < \infty$. Let $D(J)$ be the union of the sets $\partial\{\sigma \neq \tau\}$ over the pairs $\sigma, \tau \in \mathcal{G}(J)$ for which this set is finite. Then $D(J)$ is a finite edge set, covariant under automorphisms. For $N \in \mathbb{N}$ let $D_N = D$ if $|D| \leq N$ and $D_N = \emptyset$ otherwise. By invariance, $\mathbb{P}(e \in D_N)$ is constant on each of the finitely many edge orbits, and each orbit is infinite, while $\mathbb{E}|D_N| \leq N$. So $\mathbb{P}(e \in D_N) = 0$ for all e and N, and $D = \emptyset$ almost surely: distinct ground states could not differ by finite domain walls. A rigorous version needs the measurability of $\mathcal{G}(J)$ and of $D(J)$, which we have not checked; for the measurability of $\mathcal{G}(J)$ see [AD14, Section 2].

Open questions.

1. Problem 1.1 for bounded-degree graphs and a symmetric law of linear growth, such as the uniform law on $(-1, 1)$. Our proof of Theorem 1.5 uses $J > 0$ throughout.

2. Problem 1.1 for quasi-transitive graphs, including $\mathbb{Z}^d$. On the half-plane the answer is positive for continuous coupling laws with support $\mathbb{R}$ [AD14]. Only such laws are covered: the half-plane with, for example, couplings uniform on $(-1, 1)$ is not.

3. Can $|\mathcal{G}(J)| > 2$ hold with positive probability on a recurrent graph for a symmetric law of linear growth? A proof that recurrence forces $|\mathcal{G}(J)| = 2$ for such laws would in particular give $|\mathcal{G}(J)| = 2$ on $\mathbb{Z}^2$, which is expected but open [AD14]. Newman and Stein [NS25] recently proved, for the coupling laws considered there, that on $\mathbb{Z}^2$ the metastate generated by periodic boundary conditions is supported on a single pair of spin-reversed ground states; this does not determine $|\mathcal{G}(J)|$.

Verification. The claims were checked as follows. The proofs do not depend on these computations, except for the numerical value 0.105 in Theorem 1.6(b), a finite product evaluated in exact rational arithmetic whose positivity is proved analytically.

1. Programs by the finder (Python, standard library, exact rational arithmetic unless noted) check Lemma 2.1 on 3000 random paths of length at most 5; Proposition 2.3 on small windows of thickened lines with frozen boundary hubs, including literal enumeration of all flips for the

smallest windows; and Lemma 2.2 by enumerating all subset flips for up to 10 bonds. No errors were found. They compare the bound on $f(0)$ in the proof of Lemma 3.1(b) with the exact Irwin–Hall density for $m \leq 80$ (the ratio is at most 0.9991), and Lemma 3.1(c) with the exact distribution function for $m \leq 60$, and they compute $\mathbb{E}Y = 4/27$ in Theorem 1.6(a). As evidence only, they compute minimal end-separating cuts of the strip in windows $[-R, R]$, $R \leq 1599$, cross-checked against brute force and integer max-flow, and simulate the chains of Theorem 1.3.

2. Audit programs written for this note check Lemma 5.2 for $1 \leq x < 3000$ by breadth-first search in the dual graph, with T and B allowed only as end points; verify Lemma 5.1 exhaustively on the windows $[-2, 2]$ and $[-3, 3]$ (all 24 and 100 bonds have the stated form); and compute $\mathbb{P}(K_0 \leq 1) = 47/90$ and the bound $\mathbb{P}(\mathcal{E}) \geq 0.1059$ of Theorem 1.6(b). There, a Monte Carlo estimate (4000 samples) gives $\mathbb{P}(|\mathcal{G}(J)| = 4) = 0.54 \pm 0.01$, and a numerical convolution (item 3) gives $\mathbb{P}(|\mathcal{G}(J)| = 4) \approx 0.5555$. They also show that the inequality of Step 3 already holds for $528 \leq a \leq 10^6$, which the proof does not need.

3. Two independent verifiers checked the version of this note that preceded the audit of item 2: the first checked all proofs by hand, and the second those of Sections 2 and 3. Both reran the finder's programs, which gave identical output. With their own programs they compared Lemma 3.1(b) with the exact Irwin–Hall density for $m \leq 150$; checked the ground-state structure on windows of $L(m)$, $m_k = (|k|+1)^3$, by transfer matrices that enumerate the middle spins directly, without Lemma 2.1; and checked a slightly weaker form of Lemma 5.2 for $x < 1200$ by a constrained dual search. An unconstrained search, which lets walks pass through T or B, fails; this is why Lemma 5.3 counts walks through squares only. The parts revised after their check, namely gadget $0 = K_{2,3}$ with the constants 47/90 and 0.105 in Theorem 1.6(b) and Lemmas 5.1 to 5.3 as now stated, were rechecked by the audit programs of item 2 and by a second referee. This referee checked all proofs line by line and, with independent programs, found 24, 100 and 374 bonds in the windows $[-2, 2]$, $[-3, 3]$ and $[-4, 4]$, all of the form of Lemma 5.1; checked Lemma 5.2 for $1 \leq x \leq 3000$; obtained $\mathbb{P}(\mathcal{E}) \geq 0.105965$ in exact rational arithmetic and $\mathbb{P}(K_0 \leq 1) = 47/90$ by two methods; and evaluated $\mathbb{P}(|\mathcal{G}(J)| = 4) \approx 0.5555$ in Theorem 1.6(b) numerically.

Scope and priority. Corollary 1.4 answers Problem 1.1 as posed in [Bä20]. Together with Theorem 1.6(b) it answers both alternatives of the broader form in [Bä19, Section 4], with connected, locally finite graphs. The counterexample of Corollary 1.4 works for every absolutely continuous law. Its degrees are unbounded. With bounded degrees we have a counterexample only for a ferromagnetic law of linear growth (Theorem 1.5). The bounded-degree case with a symmetric law and the quasi-transitive case remain open. The underlying mechanism is elementary and experts may regard it as natural. Its square-root form, which gives the threshold $\sum_k m_k^{-1/2}$ in Theorem 1.3, appears in the heuristic example of White and Fisher [WF06]: a three-dimensional ferromagnet with random-sign couplings across one plane, whose half-spaces are coupled with strength of order $\sqrt{L^2} = L$ and which has four ground states. Their couplings are not identically distributed, and they give no proof. For random ferromagnets on $\mathbb{Z}^d$, $d \geq 2$, and on the half-plane, Theorem 1.5 is consistent with the results of Wehr [Weh97] and of Wehr and Woo [WW98], since Γ is not quasi-transitive and, unlike the half-plane, not even invariant under horizontal translations. We searched arXiv, Crossref, OpenAlex and zbMATH, and the verifiers also used Semantic Scholar and a web search. OpenAlex lists two records citing [Bä19], the arXiv and journal versions of [AH20], on disorder chaos on $\mathbb{Z}^d$. Other work on uniqueness of ground states, such as [Ito21, NS25], concerns different questions. We found no answer to Problem 1.1 and no graph with i.i.d. couplings of linear growth and finitely many but more than two ground states. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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